数理解析研究所講究録
第 390 巻 1980 年 81-92



# Steady Motions Exhibited by Duffing's Equation

## - A picture book of regular and chaotic motions -

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Abstract. Various types of steady states take place in the system exhibited by Duffing's equation. Among them harmonic, higher harmonic and subharmonic motions are popularly known. Then ultrasubharmonic motions of different orders are fairly known. However chaotic motions are scarcely known. By using analog and digital computers, this report makes a survey of the whole aspect of steady motions exhibited by Duffing's equation.

1. Introduction. Duffing's equation appears in various physical and engineering problems. It is one of the simplest and the most important nonlinear differential equations. The aim of this report is to give the whole aspect of steady states exhibited by the equation. Throughout this paper the term steady state or steady motion means physical state which continues infinitely after the transient has vanished.

There are various types of steady motions exhibited by Duffing's equation. Among them deterministic or regular motions are generally known, e.g., harmonic, higher harmonic, and subharmonic motions. However, owing to the perfectly deterministic nature of the equation, any reference has not been made to the possibility of the existence of chaotic motions for a long time. The occurrence of chaotic motions was originally studied by the author [1, 2, 3, 4]. Holmes has also observed chaotic behavior in analog computer solutions of Duffing's equation [5, 6]. Further Moon has performed experiment for the forced vibrations of a buckled beam and showed the existence of chaotic motions [7, 8].

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The purpose of this report is to make a survey of the steady motions exhibited by Duffing's equation which takes the form

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + x^3 = B\cos t \tag{1}$$

Since the solution of Eq. (1) cannot be obtained analytically, we have relied on analog and digital computers. Thus computer solutions are examined and summarized in this report. Therefore, from the mathematical point of view, they may raise new questions, yet it is of value and interest to introduce them to many researchers in various fields.

2. Preliminaries.

2.1 Discrete dynamical system. Equation (1) is rewritten as

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -ky - x^3 + B\cos t \tag{2}$$

Let us here introduce a diffeomorphism on the xy plane into itself by using the solutions of Eqs. (2). Let $x = x(t, x_0, y_0)$, $y = y(t, x_0, y_0)$ be a solution of equation (2) which starts from a point $p_0 = (x_0, y_0)$ at $t = 0$. Let $p_1 = (x_1, y_1)$ be the location of the solution at the instant of $t = 2\pi$, i.e., $x_1 = x(2\pi, x_0, y_0)$, $y_1 = y(2\pi, x_0, y_0)$; then a $C^\infty$-diffeomorphism

$$f_\lambda : R^2 \to R^2, \quad p_0 \mapsto p_1, \quad \lambda = (k, B) \tag{3}$$

of the xy plane into itself is defined.

A periodic solution of Eqs. (2) is represented by a fixed or n-periodic point of $f_\lambda$, i.e., $p = f_\lambda^n(p)$, $(n \in Z^+)$. A fixed point p is characterized by the eigenvalues $m_1$, $m_2$ of $Df_\lambda(p)$, the derivative of $f_\lambda$ evaluated at the point. A simple fixed or periodic point is classified into: (i) completely stable fixed or periodic point or sink (S), (ii) completely unstable point or source (U), (iii) directly unstable point or saddle (D) and (iv) inversely unstable point or saddle (I).

The steady motion exhibited by Eqs. (2) is represented by an attractor of the diffeomorphism $f_\lambda$. If an attractor is composed of a single periodic group, the corresponding motion turns out to be periodic and hence deterministic or regular. But if it is composed of a closed, invariant set of $f_\lambda$ containing infinitely many unstable periodic groups, chaotic motion appears resulting from the small uncertain factors in the real system.

2.2 Chaotically transitional processes and strange attractors. A periodic motion is represented by asymptotically stable periodic solution. It corresponds to a sink of the diffeomorphism $f_\lambda$ having a wide basin as compared with random noise in the real system. On the other hand a chaotic motion is represented by a bundle of solutions in the txy space which is asymptotically orbitally stable and contains infinitely many unstable periodic solutions. The representative point of the physical state wanders chaotically among the solutions of this bundle under the influence of small uncertain factors in the real system. Considering this nature, we have called the phenomenon chaotically transitional process [1].

The set of points on the xy plane consisting of the cross section of the bundle at $t = 2n\pi$ $(n \in Z)$ is called a strange attractor. We have emphasized that the strange attractor is identical with a closure of unstable manifolds of a saddle of the diffeomorphism $f_\lambda$.

## 3. Experimental results on the steady motions.

3.1 kB chart for different types of steady states. In the forced oscillatory system exhibited by Eq. (1), various types of steady states are sustained depending on the system parameters $\lambda = (k, B)$ as well as on the initial conditions. Figure 1 shows the regions on the kB plane in which different steady motions are observed. These regions are obtained by using analog and digiral computers. The Roman numerals I, II, II', II", III and IV characterize harmonic motions. The fractions m/n (m = 1, 3, 4, 5, 6, 7, 11 and n = 2, 3) indicate the regions in which subharmonic or ultrasubharmonic motions of order m/n are sustained. An ultrasubharmonic motion of order m/n is a periodic motion whose principal frequency is m/n times the frequency of external force. Chaotic motions take place in the shaded regions. In the area hatched by

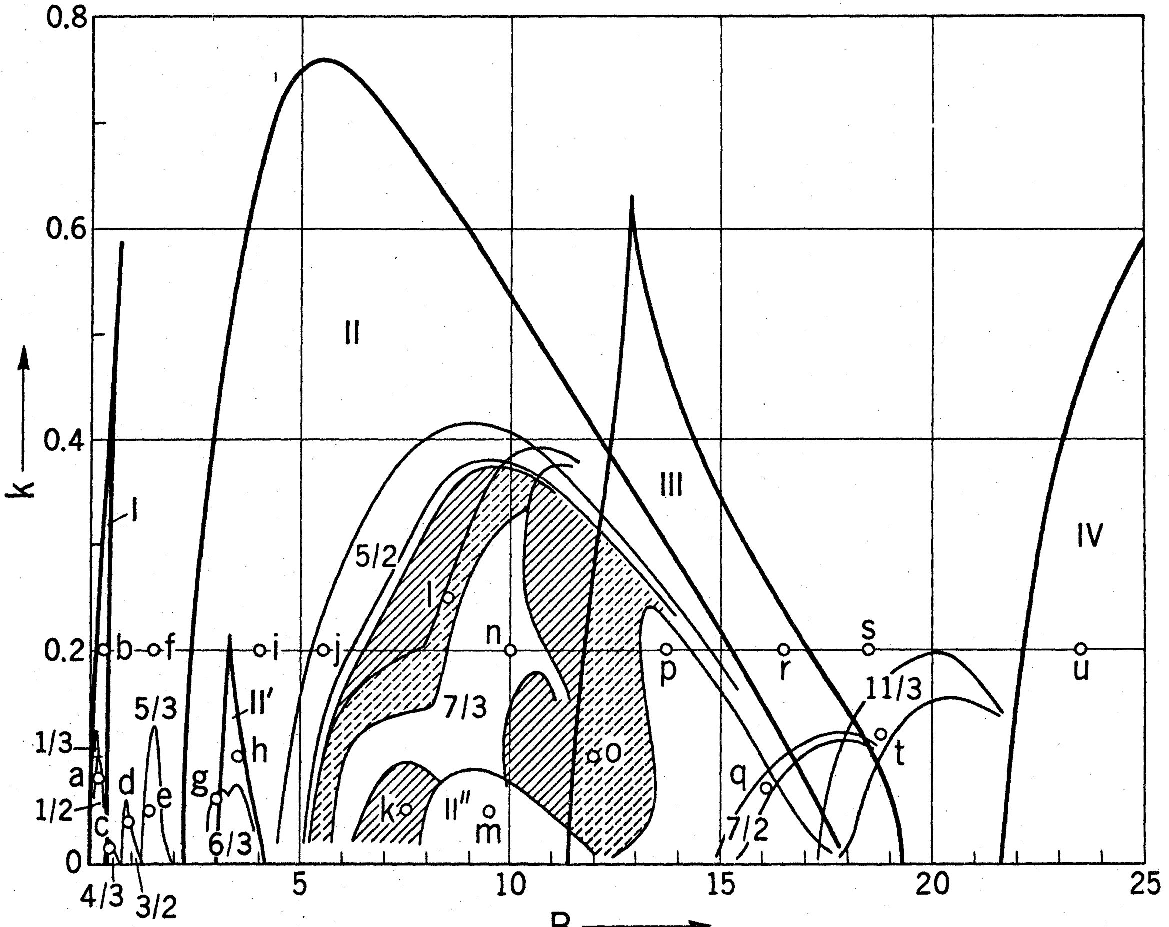



Fig. 1. Regions of different steady states for the system exhibited by Eq. (1).

full lines, chaotic motion occurs uniquely, while in the area hatched by dotted lines, two different steady states take place, i.e., chaotic and regular motions. Which one occurs depends on the initial conditions.

Ultrasubharmonic motions of higher orders ($n = 4, 5, \ldots$) can occur naturally in the system, but they are omitted in Fig. 1. Further we should like to add that, though we performed the experiment carefully and repeatedly, the chart is far from perfect. In particular, the regions which lie between B = 5 and 15 are regarded as very serious problems. Details are doubtful and further investigations will be required.

3.2 <u>A collection of steady motions</u>. In order to illustrate the regions of kB chart, we choose a set of parameters $\lambda = (k, B)$ from every region. The location of these parameters are indicated by alphabets from a to u in Fig. 1. Figure 2 shows the trajectories of the steady motions on these points. Almost all steady motions which will occur for these parameters are supposed to be collected. In the figure, positions of the representative point at the instant $t = 2n\pi$ ($n \in Z^+$) are marked x. So the marks x on the periodic trajectories are the completely stable fixed or periodic points of the diffeomorphism $f_\lambda$. The three cases (k), ($l_1$) and ($o_1$) show chaotic motions, in which the trajectories are drawn after the transients have vanished and hence the marks x appear on the strange attractors.

The periodic motion whose trajectory is symmetric about the origin is expanded into Fourier series consisting of odd order harmonics only. While the motion whose trajectory is unsymmetric about the origin is accompanied by even order harmonics in addition to the odd order ones. In such a case, as we see in Fig. 2 there exist a pair of trajectories symmetric to each other.

3.3 <u>Chaotically transitional processes</u>. The outlines of the strange attractors for the cases (k), ($l_1$) and ($o_1$) are shown in Fig. 3. They are plotted after the transients have vanished. As mentioned before,

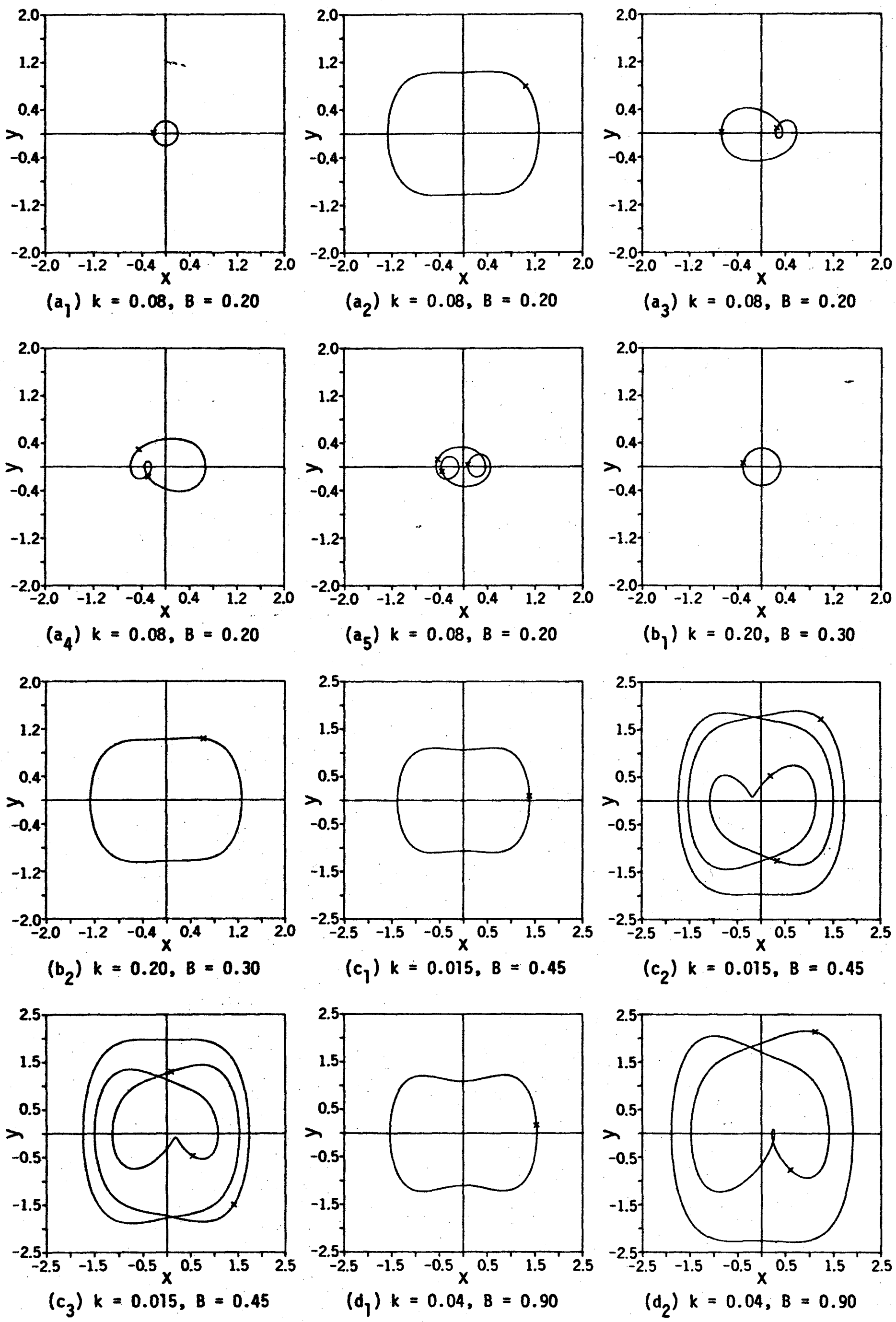



Fig. 2. Trajectories of various types of steady motions.

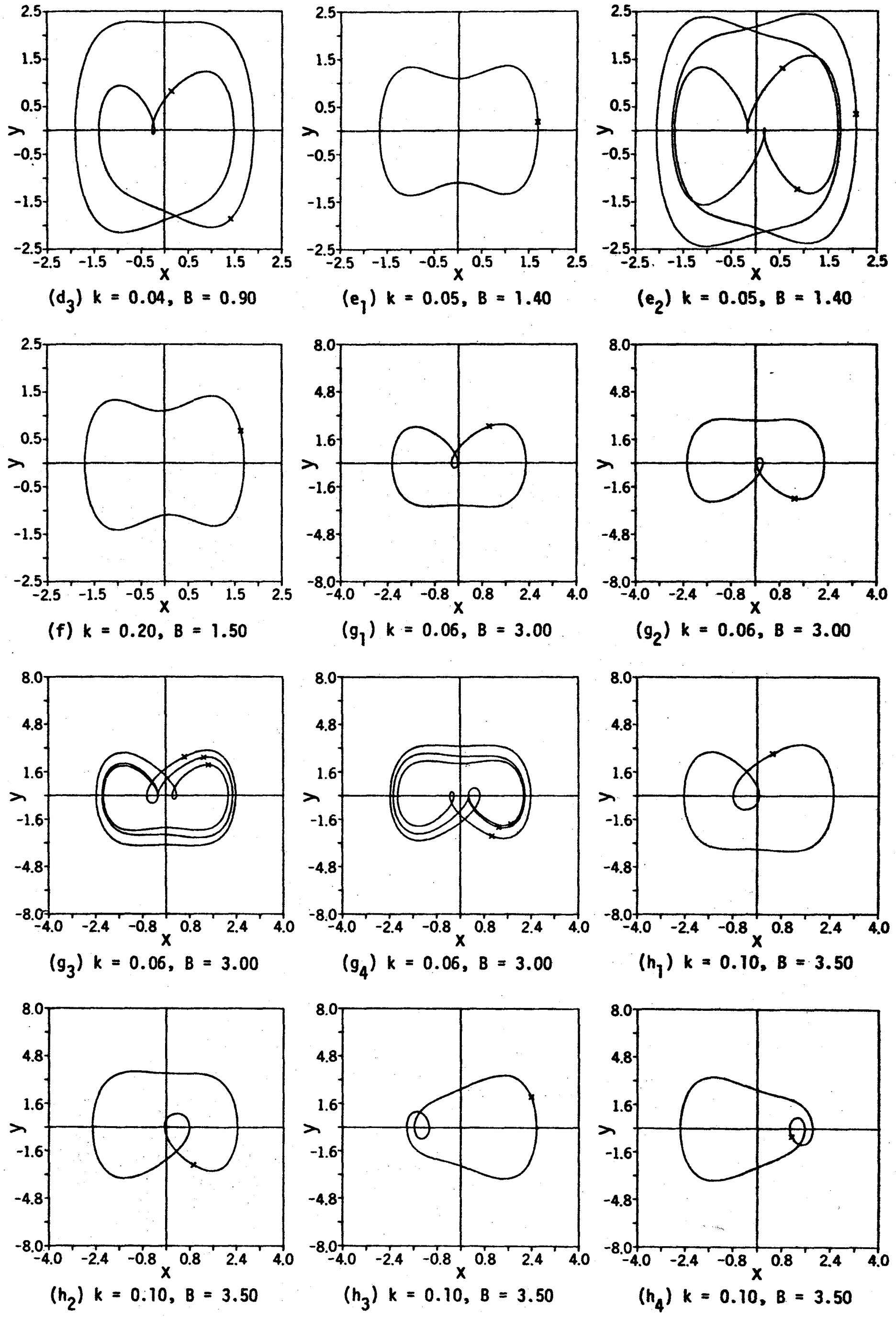
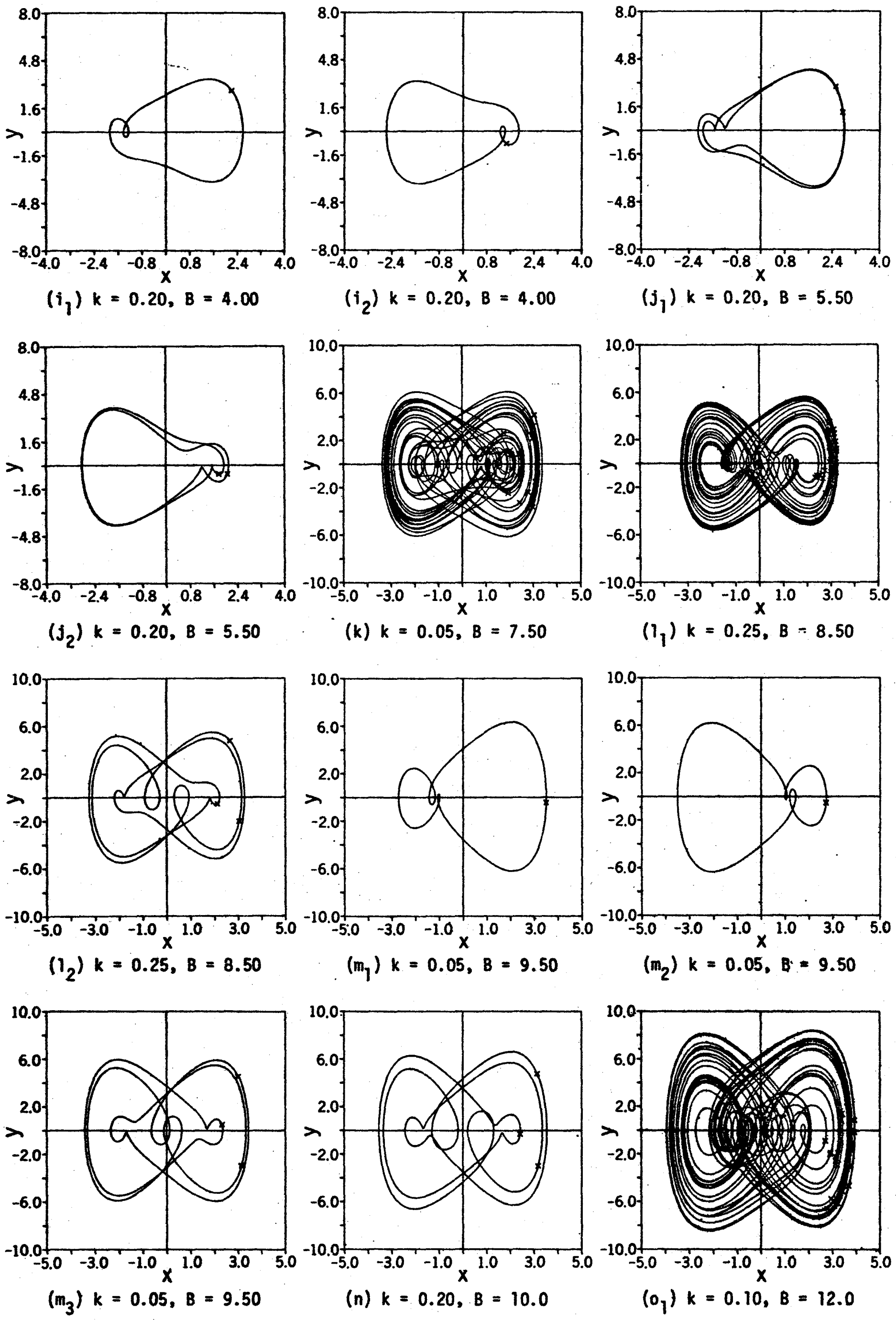



Fig. 2. Continued.

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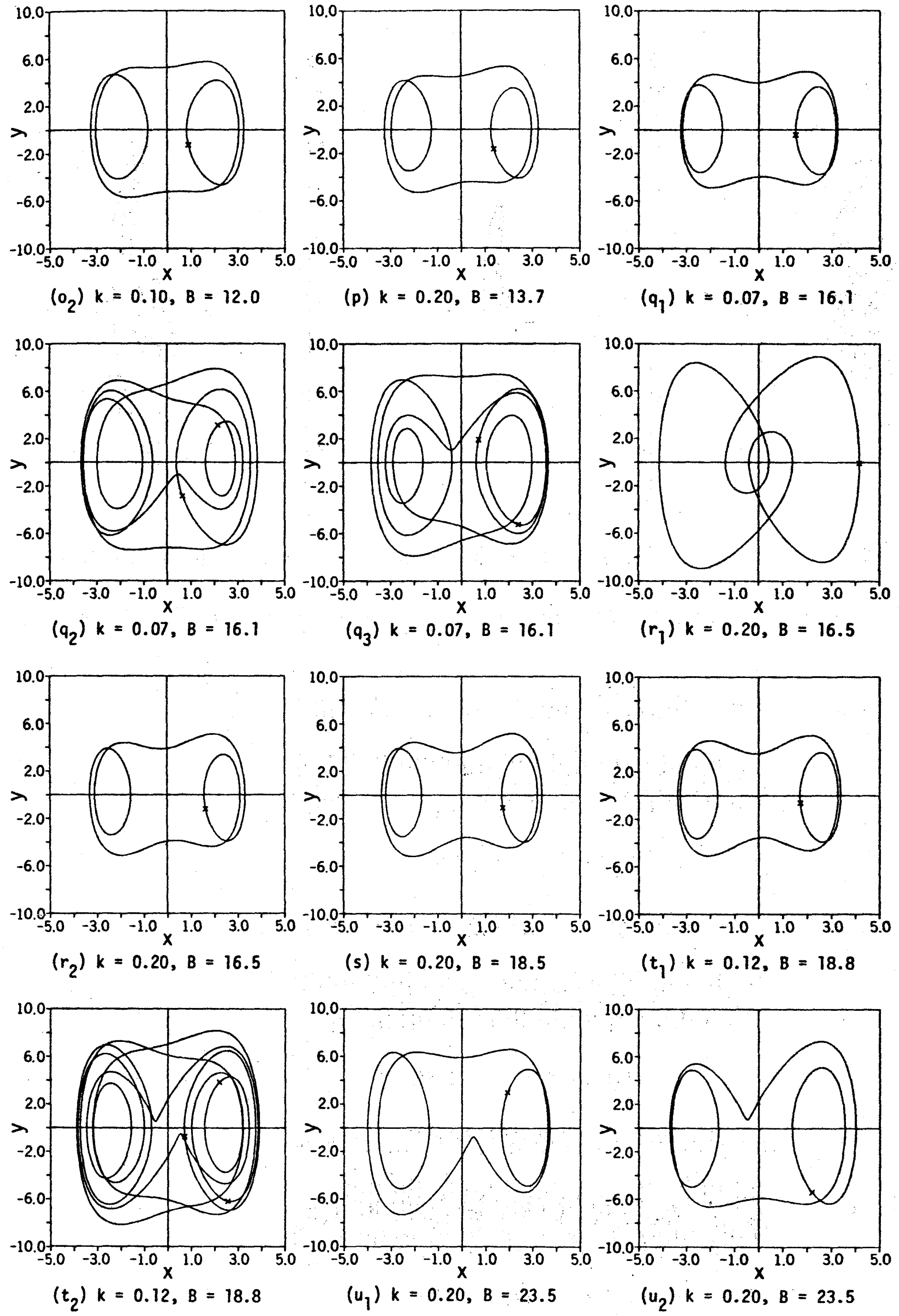


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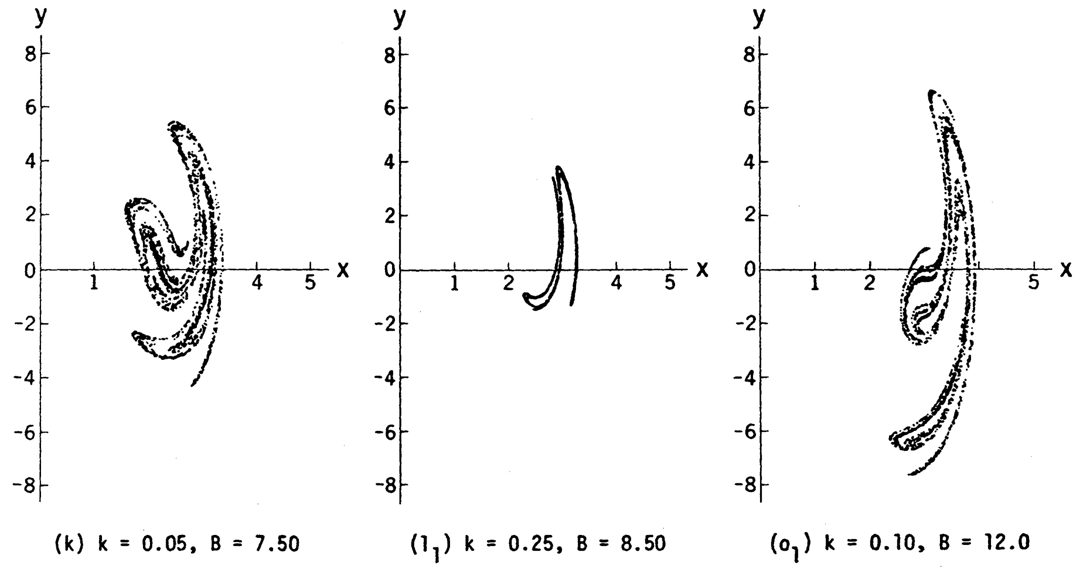


Fig. 3. Strange attractors for the chaotically transitional processes.

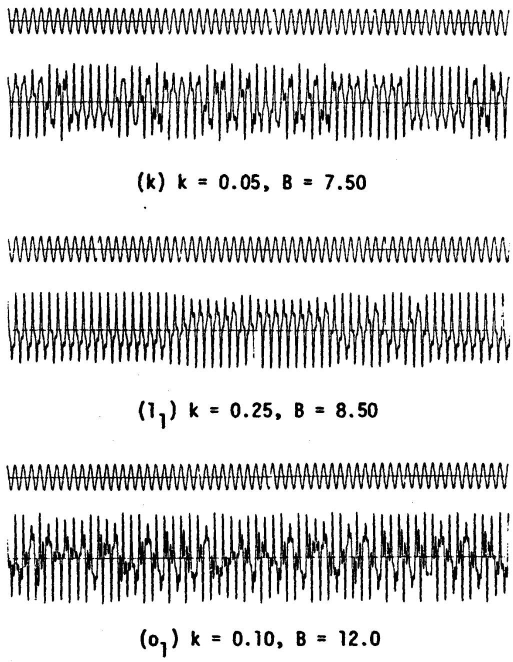


Fig. 4. Waveforms of the chaotically transitional processes.

the strange attractors are identical with the closures of unstable manifolds of some saddles of $f_\lambda$. The waveforms which are the realizations of these chaotically transitional processes are given in Fig. 4. The global phase plane structure of $f_\lambda$ for the case $(o_1)$ is shown in Ref. [2]. The transition of the strange attractors and average power spectra are also reported in Ref. [3].

3.4 Remarks on the experimental results. Here we briefly summarize the experimental results obtained in the preceding sections.

(1) Periodic motions with period $2\pi$, i.e., harmonic or higher harmonic ones, occur almost everywhere except in the regions surrounded by 5/2 harmonic region. In particular, two types of $2\pi$ periodic motions are observed in the regions I, II, II" , III and IV, and four types of them in the region II'. On the boundaries of I, II' and III, jump phenomenon takes place. In other words, SD coalescence (SD extinction or generation) occurs on them, while on the boundaries II and IV, SI branching occurs.

(2) As B increases, the order of ultrasubharmonics grows larger. Ultrasubharmonic regions of all orders except 5/2 and 7/3 lie within the limits of k less than 0.2.

(3) Ultrasubharmonic regions of order 5/2 and 7/3 participate closely in the chaotic regions [3]. Though ultrasubharmonic motions of higher orders appear in the chaotic regions, they are omitted in Fig. 1. Similar circumstances are discussed in detail in Ref. [4]. The main results and remaining unsolved problems are also summarized there relating to chaotically transitional processes exhibited by Duffing's equation.

4. Conclusion. By using analog and digital computers, the whole aspect of steady motions exhibited by Duffing's equation has been surveyed experimentally. It is hoped that the results will be applied to various physical problems and will deserve attention as material for mathematical study.

In conclusion I should like to express my profound gratitude to Professor Chikasa Uenosono of Kyoto University, the President of the Institute of Electrical Engineers of Japan, for his constant support and encouragement during the preparation of this report.

I am particularly indebted to Professor Philip Holmes of Cornell University who gave me the opportunity to present this work at the conference. I have also benefited greatly from conversation with Professor Francis Moon of Cornell University.

This work has been carried out in part under the Collaborating Research Program at the Institute of Plasma Physics, Nagoya University. The author wishes to express his sincere thanks to the staffs of the Institute.

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