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CHEVALLEY ALGEBRAS AND CHEVALLEY GROUPS

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1. Introduction In this note, we introduce and discuss Chevalley algebras over commutative rings R with identity, describe their arithmetic structure in the classical (i.e., non-Kac-Moody) cases, and relate that to the normal structure of Chevalley groups over R. In Section 3, we also describe recent work of Garland which leads to Chevalley algebras and groups associated with Kac-Moody Lie algebras. In several places, we discuss open questions and conjectures. The rest of this section is devoted to notational preliminaries.

Let L be a finite-dimensional simple Lie algebra over the complex field, H an m-dimensional Cartan subalgebra, Φ the set of roots of L relative to H, and $\Pi = \{r_1, r_2, \ldots, r_m\}$ a simple system of roots. For $r \in \Phi$, let L_r be the corresponding root space. Chevalley [5] established the following basic fact.

1.1 Theorem There is a basis $B = \{\bar{e}_r \mid r \in \Phi\} \cup \{\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_m\}$, where $\bar{e}_r \in L_r$, $\bar{h}_i \in H$, such that

(i) $[\bar{h}_i, \bar{h}_j] = 0$ for all i and j,

(ii) $[\bar{e}_r, \bar{e}_{-r}] = \bar{h}_r$, a certain [26, Lemma 1] integral linear combination of $\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_m$.

(iii) If $r + s \neq 0$, then $[\bar{e}_r, \bar{e}_s] = \pm N_{rs}\, \bar{e}_{r+s}$, where N_{rs} is 0 if $r + s \notin \Phi$, and otherwise is $p + 1$, where

p is the largest integer such that $s - pr \in \Phi$.

(iv) $[\bar{h}_r, \bar{e}_s] = \frac{2(s,r)}{(r,r)} \bar{e}_s = s(\bar{h}_r)\, \bar{e}_s$, where $(\, , \,)$ is the Killing form on the dual H^* of H . We note that the Cartan integer $c(r,s) = \frac{2(s,r)}{(r,r)} = p - q$, where q is the largest integer such that $s + qr \in \Phi$.

Denote by L_Z the free abelian group on B . This is the Chevalley lattice of L corresponding to B and is closed under Lie products. Let R be a commutative ring with identity.

1.2 Definition The Chevalley algebra of L over R is

$$L_R = R \otimes_Z L_Z \ .$$

This is uniquely determined up to isomorphism by L [9, pp. 47- 48] .

Let H_Z be the free abelian group on $\{h_1, h_2, \ldots, h_m\}$. Then we denote $R \otimes_Z H_Z$ by H_R . Similarly, if E_Z is the free abelian group on $\{\bar{e}_r \mid r \in \Phi\}$, then E_R stands for $R \otimes_Z E_Z$. Note that H_R is a subalgebra of L_R , but E_R is only an R-submodule.

2. Classical Chevalley algebras

Results on the arithmetic structure of Lie algebras of Chevalley type tend to take the form of sandwich relations (cf. Equations (1) - (4) below). We consider in this section first the ideal structure of Chevalley algebras, and then the nature of orders in L when the underlying ground ring is an integral domain.

Even though L is simple over the complex field, L_R is not in general simple. For an ideal J of R for instance, we can from the projection homomorphism $f_J : R \to R/J$, produce a homomorphism from L_R

onto $L_{R/J}$ with kernel JL_R , which we can identify with L_J . There are then ideals of L_R corresponding to ideals of the ring R . A natural question then arises.

<u>Question 1</u> To what extent is the ideal structure of L_R determined by that of R ?

This question is answered in [10] and [27] , to which the reader is referred for proofs of the first two results below. Assume that 2 and 3 are not zero divisors in R , and if L is of type A_m assume further that $m + 1$ is not a multiple of the characteristic of R , or a 0-divisor.

<u>2.1 Theorem</u> Suppose that $I \not\subseteq H_R$. Then there is an ideal J of R and a positive integer n such that

(1) $$n\,J\,L_R \subseteq I \subseteq J\,L_R \ .$$

Here, n is a product of divisors of $\det C$, where $C = (c_{ij}) = (c(r_i, r_j))$, and powers of $k = (\ell, \ell)/(s, s)$ where ℓ is a long root and s is a short root of L .

For fields of prime characteristic, Question 1 has also been answered by Hogeweij [8] , who determines all ideals of L_R even in case R is of characteristic 2 or 3 or in case its characteristic divides $m + 1$ in type A_m . Using Theorem 2.1 , one can obtain the following characterization of the circumstances under which all ideals of L_R arise from those of the ring R , again retaining the assumptions on 2, 3, and $m + 1$.

<u>2.2 Theorem</u> Every ideal I of L_R has the form JL_R for J an ideal of R if and only if k and $\det C$ are invertible in R .

<u>Question 2</u> What is the situation over a general commutative ring R

with identity in which 2 or 3 or $m+1$ may be a zero-divisor?

Chevalley algebras have also been used to study orders in split simple Lie algebras L over a field F which is the field of fractions of an integral domain D. Such algebras have a Chevalley basis over F, and we can identify L with $F \otimes_Z L_Z$. The results below generalize the principal theorems found in the Ph.D. dissertation of M. Harvey Hyman [16]. For a more complete discussion, consult [14]. We first give the basic definition.

2.3 Definition An order in L is a lattice (i.e., a finitely generated D-module whose F-span is L) X which is closed under multiplication.

We can then regard X as a Lie algebra over D. The Chevalley algebra L_D is, of course, a natural order to consider in L, and is called in this context the Chevalley order. In the remainder of this section, X stands for an arbitrary order in L.

2.4 Theorem If $X \supseteq L_D$, then there is an integer n as in Theorem 2.1 such that

(2) $$n\,J\,L_D \subseteq X \subseteq L_D ,$$

where J is the smallest D-submodule of F such that $JL_D \supseteq X$. If D is Noetherian, then J is a fractional ideal.

Observe that J is well-defined, since we have $X \subseteq J'L_D$ for the D-submodule J' of F generated by 1 and all coefficients of elements of X expressed as F-linear combinations of the Chevalley basis elements.

Let $\bar{D}$ denote the integral closure of D in F and $L_D' = E_D \oplus H_D'$, where H_D' is the lattice of coroots,

$$H_D' = \{h \in H \mid r(h) \in D \text{ for all } r \in \Phi\} .$$

We have the following result.

<u>2.5 Theorem</u> (a) Suppose rank L is at least 2 and 2 has an inverse in D in case L is of type B_m or C_m. Let D be a Noetherian domain. Then for any order $X \supseteq L_D$,

(3) $$L_D \subseteq X \subseteq L_{\bar{D}}' .$$

(b) If D is integrally closed and Noetherian, (e.g., a Dedekind domain), then for n as in Theorem 2.1,

(4) $$n L_D' \subseteq X \subseteq L_D'$$

These results describe essentially the nature of orders which contain a certain fixed order L_D. It is perhaps worth noting that, even in the case of a Dedekind domain, infinite descending chains of orders are easily produced. If, for example, $a \in D$ is not invertible, then the chain

$$L_D \supseteq a L_D \supseteq a^2 L_D \supseteq a^3 L_D \supseteq \cdots$$

is an infinite descending chain of orders. It seems to be appropriate then to study orders which contain a fixed order such as L_D. Such orders were referred to by Hyman as comprising the <u>superstructure</u> of the order L_D. One can ask the following question, whose answer one would expect to be related to the ideal structure of L_R' (cf. [11]).

<u>Question 3</u> What is the superstructure of the order L_D' ?

<u>3. Kac-Moody Lie algebras and Chevalley algebras</u> We continue the notation of preceding sections. Garland [6] considers Kac-Moody Lie algebras $L_{\tilde{C}}$ associated with an $m+1$-by-$m+1$ <u>affine</u> Cartan matrix $\tilde{C}$ obtained from a classical Cartan matrix C , and shows that over the

complex field such algebras have an integral basis closely related to the Chevalley basis for L. We discuss this from the more general viewpoint of Moody [20] first, and then specialize to the affine case to state Garland's theorem on Chevalley bases and pose two questions which arise naturally from his construction.

We begin with an n-by-n <u>generalized Cartan matrix</u> (GCM) $A = (a_{ij})$, that is, a matrix of integers such that for all i and j,

$$a_{ij} \leq 0 \text{ if } i \neq j,$$
$$a_{ii} = 2 \text{ for all } i = 1, 2, \ldots, n, \text{ and}$$
$$a_{ij} = 0 \text{ if and only if } a_{ji} = 0 .$$

Let K be any field of characteristic zero. Let $L_1 = L_1(A)$ be the Lie algebra defined by a set $\{h_i, e_i, f_i\}_{i=1}^{n}$ of $3n$ generators with defining relations

$$(5) \quad \begin{cases} [h_i, h_j] = 0, & \text{for all } i \text{ and } j, \\ [e_i, f_j] = \delta_{ij} h_i, & \text{for all } i \text{ and } j, \\ [h_i, e_j] = a_{ij} e_j, & \text{for all } i \text{ and } j, \\ [h_i, f_j] = -a_{ij} f_j, & \text{for all } i \text{ and } j, \\ (\operatorname{ad} e_i)^{-a_{ij}+1}(e_j) = 0 = (\operatorname{ad} f_i)^{-a_{ij}+1}(f_j) & \text{for } i \neq j, \end{cases}$$

$i, j = 1, 2, \ldots, n$. Thus, L_1 is the quotient of the free Lie algebra on these $3n$ generators factored by the ideal generated by the elements obtained by rewriting each equation as an expression equated to zero.

For an n-tuple $(k_1, k_2, \ldots, k_n)$ of integers, we define subspaces $L_1(k_1, k_2, \ldots, k_n)$ as follows. $L_1(0, 0, \ldots, 0) = H(A) =$ the abelian subalgebra of L_1 spanned by $\{h_1, h_2, \ldots, h_n\}$. If $(k_1, k_2, \ldots, k_n)$ consists of nonnegative (resp., nonpositive) integers, then $L_1(k_1, k_2, \ldots, k_n)$ is the subspace of L_1 spanned by all products $[e_{i1}, [e_{i2}, \ldots [e_{i,r-1},$

$e_{ir}]\cdots]]$ (respectively, $[f_{i1}, [f_{i2}, \cdots [f_{i,r-1}, f_{ir}]\cdots]]$), where e_j (resp., f_j) occurs $|k_j|$ times. For all other n-tuples, $L_1(k_1, k_2, \ldots, k_n)$ is defined to be 0 . Each of these subspaces is seen to be finite dimensional, and L_1 is the sum of all the $L_1(k_1, k_2, \ldots, k_n)$ over all members of Z^n . This gives us a Z^n - gradation of L_1. There is a unique graded ideal R_1 maximal among all graded ideals which intersect the span of $\{h_i, e_i, f_i\}_{i=1}^{n}$ only in zero.

<u>3.1 Definition</u> The <u>Kac-Moody Lie algebra</u> L_A is L_1/R_1 .

Notice that if A is a classical Cartan matrix and K is the complex field, then $R_1 = 0$ and $L_A = L_C$ is a classical simple Lie algebra.

We denote the images of h_i, e_i , f_i , H(A) , and $L_1(k_1, k_2, \ldots, k_n)$ by h_i , e_i , f_i , H_A , and $L(k_1, k_2, \ldots, k_n)$ respectively. We define $D_i : L_A \to L_A$ for each $i = 1, 2, \ldots, n$, to be multiplication by the scalar k_i on $L(k_1, k_2, \ldots, k_n)$. This is then a derivation of L_A . Let D_0 be the n-dimensional subspace of commuting derivations spanned by $D_1, D_2, \ldots, D_n$. Let D be a subspace of D_0 and form the semi-direct product algebra $L^e = D \times L_A$ with component-wise addition and multiplication by scalars, and Lie product $[d + \ell, d' + \ell'] = [d, d'] + (d(\ell') - d'(\ell) + [\ell, \ell'])$. Let $H_A^e = D \times H \subseteq L_A^e$, an abelian subalgebra which acts via scalar multiplication on $L(k_1, k_2, \ldots, k_n)$. We further define $a_1, a_2, \ldots, a_n \in (H^e)^*$ by

(6) $$[h, e_i] = a_i(h)\, e_i , \quad \text{for } h \in H_A^e , \quad i = 1, 2, \ldots, n.$$

Thus $a_j(h_i) = a_{ij}$, $i, j = 1, 2, \ldots, n$. Henceforth we assume that D is so chosen that $\{a_1, a_2, \ldots, a_n\}$ is a linearly independent set. This is possible since, for instance, $D = D_0$ will serve, although it is often convenient to use a smaller such D . Observe that $a_i(D_j) = \delta_{ij}$ for i, j ranging between 1 and n . We can now define the roots of L_A.

3.2 Definition Let $a \in (H^e)^*$. Then $L^a = \{x \in L_A \mid [h, x] = a(h)x$ for all $h \in H_A^e\}$. A root of L_A relative to H_A^e is a member a of $(H^e)^*$ for which $L^a \neq 0$. The set of all roots is denoted by $\Delta = \Delta(A)$. The positive roots $\Delta_+ = \Delta_+(A)$ consist of all roots which are non-negative integral linear combinations of $a_1, a_2, \ldots, a_n$. The negative roots $\Delta_- = \Delta_-(A)$ are defined to be the negatives of the positive roots.

Notice that $L_A^0 = H_A$ and $L = H_A \oplus \sum_{a \in \Delta_+} L^a \oplus \sum_{a \in \Delta_-} L^a$.

3.3 Definition The GCM A is symmetrizable if there exist positive rational numbers $q_1, q_2, \ldots, q_n$ such that diag $(q_1, q_2, \ldots, q_n)\, A$ is a symmetric matrix.

Henceforth, we assume that A is symmetrizable. Then we can define a symmetric bilinear form on the subspace of $(H^e)^*$ spanned by Δ by setting

$$(a_i, a_j) = q_i a_{ij} ,$$

for $i, j = 1, 2, \ldots, n$. Then $q_i = (a_i, a_i)/2$ and we set

$$h_i' = \tfrac{1}{2}(a_i, a_i)\, h_i \in H ,$$

for $i = 1, 2, \ldots, n$. For $\phi = \sum_{i=1}^{n} x_i a_i$, we also define

$$h' = \sum_{i=1}^{n} x_i h_i' ,$$

and use this to transfer (,) to H by defining $(h_i', h_j') = (a_i, a_j)$, for $i, j = 1, 2, \ldots, n$, and then $(h_a', h_b') = (a, b)$ for any a and b in the span of Δ.

For $i = 1, 2, \ldots, n$, we define the Weyl reflection $w_i : (H^e)^* \to (H^e)^*$ by

$$w_i(a) = a - a(h_i)\, a_i .$$

Thus, in particular, from (6) we see that $w_i(a_j) = a_j - a_{ij} a_i$ for $i, j = 1, 2, \ldots, n$. The <u>Weyl group</u> W of L_A is the subgroup of $\mathrm{Aut}\,(H^e)^*$ generated by all the w_i. We define the set $\Delta_R(B)$ of <u>real roots</u> to be $W(\{r_1, r_2, \ldots, r_n\})$, and the set of <u>imaginary roots</u> $\Delta_I(B)$ to consist of all roots which are not real.

Now suppose that A is a classical m-by-m Cartan matrix C. We take $D = 0$, so that $H^e = H$, and $L_C{}^e = L_C$ is a classical Lie algebra over K. Our form $(\ ,\ )$ on $H_C{}^*$ is just the usual transferred Killing form from L. Using our notation Φ for the set of roots of L_C, the set Π of simple roots determines the positive roots $\Phi_+(C)$. Let $r_0 \in \Phi_+(C)$ be the highest root. We set $r_{m+1} = -r_0$, and form the <u>affine Cartan matrix</u> $\tilde{C}$ where $\tilde{c}_{ij} = 2(r_i, r_j)/(r_i, r_i)$, $i, j = 1, 2, \ldots, m+1$. Then $\tilde{C}$ is a symmetrizable generalized Cartan matrix with associated Kac-Moody Lie algebra $L_{\tilde{C}}$.

Next let $K[t, t^{-1}]$ be the ring of Laurent polynomials over K. We define the infinite dimensional Laurent polynomial Lie algebra

$$\tilde{L} = K[t, t^{-1}] \otimes_K L_C ,$$

with Lie product $[f \otimes x,\ g \otimes y] = fg \otimes [x, y]$ for $f, g \in K[t, t^{-1}]$ and $x, y \in L_C$. Note that from the decomposition of L_C into H_C and the sum of the root spaces L^r, we obtain

$$\tilde{L} = K[t, t^{-1}] \otimes_K H_C \oplus \textstyle\sum_{r \in \Phi} L^r \oplus \sum_{n \in Z^+ \cup Z^-} t^n \otimes_K L_C .$$

Now to avoid ambiguity, we write e_i^*, f_i^*, h_i^* for e_i, f_i, h_i in L_C, $i = 1, 2, \ldots, n$ and h_r^* for h_r in H_C. For r_0, choose $e_0^* \in L^{r_0}$ and $f_0^* \in L^{-r_0}$ so that $[e_0^*, f_0^*] = 2h_{r_0}{}^* / (r_0, r_0)$. The following theorem of Kac [17] and Moody [21] helps to describe the set of roots of

$L_{\tilde{C}}$. In our next result, we identify $1 \otimes x$ in $\tilde{L}$ with x in L_C .

3.4 Theorem There is a unique monomorphism $\tilde{\omega} : L_{\tilde{C}} \to \tilde{L}$ such that $\tilde{\omega}(e_i) = e_i^*$, $\tilde{\omega}(f_i) = f_i^*$, $\tilde{\omega}(h_i) = h_i^*$, $i = 1, 2, \ldots, m$, $\tilde{\omega}(e_{m+1}) = t \otimes f_0^*$, $\tilde{\omega}(f_{m+1}) = t^{-1} \otimes e_0^*$, and $\tilde{\omega}(h_{m+1}) = 2\, h^*_{-r_0}/(r_0, r_0)$. The kernel of $\tilde{\omega}$ is the one-dimensional center of $L_{\tilde{C}}$ and is spanned by $h_\iota^* = \sum_{i=1}^{n} k_i h_i^* + h_{m+1}^*$, where $r_0 = \sum_{i=1}^{m} k_i r_i$.

We define $D_{m+1} : L_C \to L_C$ to be the $(m+1)$-st degree derivation, and define D to be the one-dimensional subspace of D_0 spanned by D_{m+1} . It is easy to check that $\{a_1, a_2, \ldots, a_{m+1}\}$ in the resulting $(H^e)^*$ is then a linearly independent set [6, p. 487] . Note that $\tilde{\omega}$ isomorphically maps

$$(7) \qquad \sum_{a \in \Delta_+(\tilde{C})} L^a \to \sum_{r \in \Phi_+} L^r \oplus \sum_{n \in Z^+ \cup Z^-} t^n \otimes L_C$$

and similarly for $\sum_{a \in \Delta_-(C)} L^a$. We thus identify the two sides of (7) . For $r \in \Phi$, $r = \sum_{i=1}^{m} n_i r_i$, $n_i \in Z^+$ or $n_i \in Z^-$ for all i , we define $a(r) \in (H^e)^*$ by the formula $a(r) = \sum_{i=1}^{m} n_i a_i$. We define the Lie algebra derivation $\bar{D}_0 : L \to L$ by $\bar{D}_0(t^n \otimes x) = n t^n \otimes x$ for $n \in Z$ and $x \in L_C$. Then [6, p. 487] $\bar{D}_0 \circ \tilde{\omega} = \tilde{\omega} \circ D_{m+1}$. Setting

$$\iota = \sum_{i=1}^{m} k_i a_i + a_{m+1} \in (H^e)^* ,$$

it follows from Theorem 3.4 that

$$\Delta_+(C) = \{a(r)\}_{r \in \Phi_+} \cup \{a(r) + n\iota\}_{r \in \Phi,\ n \in Z^+} \cup \{n\iota\}_{n \in Z^+}$$

3.5 Proposition (Kac [17, p. 287]) Let A be a GCM . Then the root $a \in \Delta_I(A)$ if and only if ja is a root for all integers $j \neq 0$.

It now follows that $\Delta_I(\tilde{C}) = \{n\iota\}_{n \in Z - \{0\}}$ and $\Delta_R(\tilde{C}) = \{a(r) + n\iota\}_{n \in Z, r \in \Phi}$. Using our identification (7) above, the root spaces L^a of $L_{\tilde{C}}$ are therefore $L^a = t^n \otimes L^r$ (where $a = n\iota + a(r)$, $r \in \Phi$ and $n \in Z$) and $L^a = t^n \otimes H_C$ (where $n \in Z - \{0\}$ and $a = n\iota$).

Next suppose that K is the complex field. We take $q_i = (r_i, r_i)/2$, $i = 1, 2, \ldots, m+1$, so that $q_i > 0$ for each i. Then $\operatorname{diag}(q_1, q_2, \ldots, q_{m+1})\tilde{C}$ is a symmetric matrix with ij-entry (r_i, r_j) for $i, j = 1, 2, \ldots, m+1$. Notice then that with this choice of q_i, $(a_i, a_j) = (r_i, r_j)$, for $i, j = 1, 2, \ldots, m+1$, and hence for $a \in \Delta(\tilde{C})$, we have $2(a, a_i)/(a_i, a_i) \in Z$ for $i = 1, 2, \ldots, m+1$. For each real root $a = a(r) + n\iota$, we define $e_a \in L^a = t^n \otimes L^r$ by $e_a = t^n \otimes \bar{e}_r$, where $\bar{e}_r$ is as in Theorem 1.1. For each imaginary root $n\iota$, we define for $i = 1, 2, \ldots, m$ and each nonzero integer n, $e_i(n) \in L^{n\iota} = t^n \otimes H_A$ by $e_i(n) = t^n \otimes \bar{h}_i$. Note that $\{e_i(n)\}_{i=1}^{m}$ is a basis for $L^{n\iota} = t^n \otimes H_C$ for each $n \in Z$, and that $\{h_1, h_2, \ldots, h_{m+1}\}$ is a basis for $H_{\tilde{C}}$.

3.6 Definition The set $\tilde{B} = \{h_i\}_{i=1}^{m+1} \cup \{e_a\}_{a \in \Delta_R(\tilde{C})} \cup \{e_i(n)\}_{i=1, n \in Z}^{m}$ is called a Chevalley basis for $L_{\tilde{C}}$.

We now are in a position to state Theorem 4.12 of [6], which serves to explain and justify the terminology in the preceding definition.

3.7 Theorem $\tilde{B}$ is an integral basis for $L_{\tilde{C}}$. In fact, the equations of structure are

(i) $[e_a, e_b] = \pm (p+1) e_{a+b}$ if $a + b \neq 0$, where $a = a(r) + n\iota$, $b = b(s) + j\iota$, $r, s \in \Phi$, $n, j \in Z$, and p as in Theorem 1.1 .

(ii) $[e_a, e_{-a}] = h_a$, an integral linear combination of h_1, $h_2, \ldots, h_{m+1}$ for all $a \in \Delta_R(\tilde{C})$.

(iii) If $a = a(r) + n\iota$, $b = a(-r) + j\iota$, $r \in \Phi$, and $n + j = \ell \neq 0$, then $[e_a, e_b] = t^\ell \otimes h_r$ is an integral linear combination of $e_1(\ell), \ldots, e_m(\ell)$.

(iv) $[e_i(n), e_j(-n)] = n\, c_{ij}\, 2h_\iota^* / (r_j, r_j)$ is an integral linear combination of $h_1, h_2, \ldots, h_{m+1}$

(v) $[h_i, e_a] = (2(a, a_i)/(a_i, a_i))\, e_a$ for $a \in \Delta_R(\tilde{C})$, $i = 1, 2, \ldots, m+1$.

(vi) $[e_i(n), e_a] = (2(r, r_i)/(r_i, r_i))\, e_{a+n\iota}$, where $i = 1, 2, \ldots, m$, $n \neq 0$ is in Z and $a = a(r) + j\iota$, $r \in \Phi$, $j \in Z$.

All other products of elements in B are zero.

This result raises immediately the following question, in view of the results set forth in Section 2.

Question 4 If the free abelian group $(L_{\tilde{C}})_Z$ is formed, and for a commutative ring R with identity 1 , the Kac-Moody Chevalley algebra $(L_{\tilde{C}})_R = R \otimes_Z (L_{\tilde{C}})_Z$ is constructed, then what is the ideal structure of $(L_{\tilde{C}})_R$, and in particular, how is it related to the ideal structure of the Chevalley algebra $L_{R[t, t^{-1}]}$?

This question is currently under investigation by the author and J. Morita. Theorem 3.4 has been generalized to a form which appears useful in considering Question 4.

Garland [7] goes on to construct groups of automorphisms analagous to Chevalley groups of classical simple Lie algebras. In [6], he already constructs (Theorem 5.8) a Z-form $U_Z(\tilde{C})$ of the universal enveloping algebra $U(L_{\tilde{C}})$ of $L_{\tilde{C}}$ which is analogous to U_Z in [19]. He then proves (Theorem 11.3) the existence of a $V_Z{}^{\lambda}$ which is invariant under $U_Z(\tilde{C})$. He is able (Lemma 10.4) to exponentiate scalar multiples of e_a, $a \in \Delta_R(C)$, the Chevalley basis elements of Theorem 3.7 above, and obtains automorphisms of V^{λ}, and can thus define Chevalley groups for $L_{\tilde{C}}$ over a commutative ring R with identity. This brings up another question.

<u>Question 5</u> What is the normal structure of these groups G and how does it relate to the ideal structure of $(L_{\tilde{C}})_R$ and of R ?

Garland [6, p. 495] remarks that the groups G have an infinite dimensional completion $\hat{G}$ which is a central extension of a Chevalley group with rational points in the field of formal Laurent series. Thus there may be some relation between Question 5 and recent work of Morita [22] on Chevalley groups over rings of Laurent polynomials. For a general idea of why one would expect some relationship between ideal structure of the Chevalley algebras $(L_{\tilde{C}})_R$ and the normal structure of the Chevalley groups G, refer to the next section.

<u>4. Chevalley groups over rings</u> We continue the notation of Sections 1 and 2, but remove the bars from the Chevalley basis elements in Theorem 1.1. Let U be the universal enveloping algebra of L. Let U_Z be the Z-algebra generated by all $e_r{}^m/m!$, $r \in \Phi$, $m \in Z^+ \cup \{0\}$. Then [19, 26] under the adjoint representation, each generator of U_Z preserves L_Z. If ρ is a faithful finite dimensional representation of L on a

complex vector space V , then there is [26, p. 17] a lattice M invariant under U_Z . Let $\bar{L}_Z$ be the part of L which preserves M . (If $\rho = \mathrm{ad}$, then $\bar{L}_Z = L_Z$ of Section 1.) We can form the Chevalley algebra $L_R = R \otimes_Z \bar{L}_Z$ as before, and let $\exp(te_r)$, $t \in R$, $r \in \Phi$, act on $R \otimes_Z M$ in the natural [26, p. 21] way, and we label the resulting automorphism $x_r(t)$. The group $E_\rho(\Phi, R)$ generated by all such $x_r(t)$ is the <u>elementary subgroup</u> of the Chevalley group $G_\rho(\Phi, R)$ of L over R . The latter group consists of the points in R of a Chevalley-Demazure group scheme [4, §5] associated with L and ρ , which depends only on Φ and the lattice of weights of ρ . When this is the lattice of fundamental weights, $G_\rho(\Phi, \ )$ is universal, and $G_\rho(\Phi, \mathbf{C})$ is simply connected [26, p. 89] over the complex field $\mathbf{C}$. In this case, $G_\rho(\Phi, R) = E_\rho(\Phi, R)$ when R is a field, local or even semi-local ring, or a Euclidean ring.

While $G_\rho(\Phi, K)$ is simple over a field K in almost all cases when $\rho = \mathrm{ad}$, $G_\rho(\Phi, R)$ has normal subgroups arising from the ideal structure of R in a way that is reminiscent of the ideals of L_R . For notational simplicity, let us fix ρ and Φ and delete them from our notation for the groups. Let $f_J : G(R) \to G(R/J)$ be the natural epimorphism induced by reduction of the ring R modulo the ideal J . Then $G(R, J) = \mathrm{Ker}\, f_J$ is of course a normal subgroup of $G(R)$, as is $f_J^{-1}(\mathrm{Center}\ G(R/J)) = G^*(R, J)$.

<u>4.1 Definition</u> A <u>congruence subgroup</u> of $G(R)$ is a subgroup N such that

(8) $$G(R, J) \subseteq N \subseteq G^*(R, J) .$$

Note the resemblance between (8) and (1). Also note that if N is any congruence subgroup of $G(R)$, then N is necessarily a normal subgroup. For letting (X, Y) stand for the group generated by all commutators $(x, y) = xyx^{-1}y^{-1}$ for $x \in X$ and $y \in Y$, we have $(N, G) \subseteq (N, G^*(R, J) \subseteq G(R, J) \subseteq N$ for any congruence subgroup N. Hence in particular, we have $G(R, J) \supseteq (G^*(R, J), G^*(R, J))$, which finishes the proof of the following basic result.

4.2 Corollary Every congruence subgroup is normal, and the factor group $G^*(R, J)/ G(R, J)$ is an abelian group.

The study of normal subgroups of $G(R)$ has focused on congruence subgroups. We begin our description of the present status of this study by stating the following result of E. Abe [1].

4.3 Theorem Suppose $G(\Phi, \mathbf{C})$ is simple and simply connected as a Lie group. Let R be a local ring such that $R/M \neq GF(3)$ and char $R/M \neq 2$ if L is of type A_1, B_n, C_n, F_4. Suppose also that $R/M \neq GF(2)$ or $GF(3)$ in types B_2 or G_2. (Here M is the maximal ideal of R.) Then $G(R) = E(R)$ and the only normal subgroups of $G(R)$ are congruence subgroups.

4.4 Theorem [12, 13]. Let R be any commutative ring with identity, with 2 and 3 not zero divisors in R and $n + 1$ not a zero divisor if L is of type A_n. Let $\rho =$ ad. Then corresponding to an ideal I of L_R there is a normal subgroup G_I of $E(R)$ generated by $x_r(t)$ such that $te_r \in I$ and by all iterated conjugates of $x_r(t)$ by elements of the form $x_{-r}(u_1)$, $x_r(u_2)$, $x_{-r}(u_3)$, etc. This G_I is the normal closure in $E(R)$ of the subgroup generated by all $x_r(t)$ such that $te_r \in I$. If L is not of type C_n, then $G_I = G_{I'}$ if and only if $I \cap E_R = I' \cap E_R$.

The next theorem actually holds more generally [3], but for simplicity of statement, we restrict ourselves to the following version.

4.5 Theorem (Abe - Suzuki) Let G be simple and simply connected as a complex Lie group, and have rank at least two. Let R be a Noetherian ring or a direct product of fields. Let $\mathrm{Spm}(R) = \{ M \mid M$ is a maximal ideal of $R \}$. If Φ is of type B_2 or G_2, assume for all $M \in \mathrm{Spm}(R)$ that $R/M \neq GF(2)$. If Φ is of type B_n, C_n, or F_4, suppose for all $M \in \mathrm{Spm}(R)$ that char $R/M \neq 2$, and if Φ is of type G_2, suppose that char $R/M \neq 3$. Let $G_0(R)$ be the subgroup of $G(R)$ generated by all $x_r(t)$ for $t \in R$, $r \in \Phi$ and by all $h(\chi) = \mathrm{diag}(\chi(\lambda_1), \ldots, \chi(\lambda_n))$ for a certain [1, pp. 475-476] set $\{\lambda_1, \ldots, \lambda_n\}$ which generates the additive abelian group P generated by the weights of ρ, and arbitrary $\chi \in \mathrm{Hom}(P, \mathbf{C}^*)$. Then any subgroup of $G_0(R)$ which is normalized by $E(R)$ (in particular, any normal subgroup of $E(R)$) satisfies for a unique ideal J

$$E(R, J) \subseteq N \subseteq G^*(R, J),$$

where $E(R, J)$ is the normal closure in $E(R)$ of all $x_r(t)$, $t \in R$, $r \in \Phi$. That is, $E(R, J) = G_I$, where $I = JL_R$.

4.6 Theorem [13] Suppose Φ has a single root length and rank at least two, with $\rho = \mathrm{ad}$. Then $x_r(t)$ has normal closure G_I, where $I = JL_R$ for J the principal ideal in R generated by t. The same result holds if $\Phi = B_n$, $n \geq 3$, or F_4 if r is a long root.

Comparing the preceding result with [18, Satz 3], the following question is suggested.

Question 6 Is $E(R, J) = \mathrm{Ker}\, f_J \mid_{E(R)}$, at least in the single root length cases?

E. Abe has been able to answer Question 6 affirmatively in case the exact sequence $0 \to J \to R \to R/J \to 0$ splits. It would be interesting to find other conditions on R which provide a positive answer. Swan [29] states without proof that Question 6 has a positive answer in the case of the stable group E(R) for any commutative ring with identity. Silvester [24] makes a similar statement in the nonstable case. (In both these claims, $\Phi = A_n$.)

4.7 Theorem [13] Under the hypotheses of Theorem 4.6 (first part), the normal closure of a product $x_r(t_1)\,x_s(t_2)$ where $r \neq s$ in E(R) is G_I where $I = J\,L_R$ for J the ideal in R generated by t_1 and t_2 .

Theorem 4.7 also holds for products of three root elements, but in general no such fact is known. This leads to the following question.

Question 7 Under what hypotheses on R and Φ can Theorem 4.7 be extended ?

Suslin [28] has shown that $E(R) \trianglelefteq G(R)$ for $\Phi = A_n$, $n \geq 2$, and in fact has shown that $E(R, J) \trianglelefteq G(R)$ in that case. He even showed normality in $GL_n(R)$, a still larger group. This raises another natural question.

Question 8 Under what hypotheses on R and Φ is $E(R) \trianglelefteq G(R)$ and $E(R, J) \trianglelefteq G(R)$?

This question relates directly to the K_1-functor on Chevalley groups of Stein [25] . Let rank $\Phi \geq 2$. St (Φ, R) stands for the group generated by $x_r(t)$, $t \in R$, $r \in \Phi$, subject to the relations

(R1) $x_r(t)\,x_s(u) = x_r(t + u)$

(R2) $(x_r(t), x_s(u)) = \prod_{ir + js \in \Phi} x_{ir+js}(c_{ijrs}\, t^i u^j)$.

where $r + s \in \Phi$ and the product is taken in some fixed order , with $c_{ijrs} \in Z$ for all $ir + js \in \Phi$.

Since the relations (R1) and (R2) hold , we have a mapping π: St(R) $\to$ G(R) whose image is the elementary subgroup E(R) . By definition, the group $K_2(\Phi, R)$ is the kernel of the map π , and $K(\Phi, R) = \text{Cok}\ \pi$ = G(R)/E(R) as a homogeneous space. Question 8 can therefore be rephrased in the language of algebraic K-theory as follows.

Question 8' Under what conditions on Φ and R is $K_1(\Phi, R)$ a group ?

In [25] , Stein gives the following partial answer to this form of the question.

4.8 Theorem Let Φ be one of the types A_n, B_n, C_n, D_n , or G_2 . Let R be a ring whose maximal ideal space is Noetherian of finite dimension d . Suppose also that if Φ is of type A_n , then $n \geq d + 1$; if Φ is of type B_n , then $n \geq d + 2$; if Φ is of type C_n , then $n \geq (d + 2)/2$; if Φ is of type D_n , then $n \geq d + 2$; and if Φ is of type G_2 , then $d \leq 1$. Then $E(R, J) \trianglelefteq G(\Phi, R)$.

Finally, Silvester [23] considered the ring R = K<X> freely generated over K by a set X of noncommuting indeterminates. For $\Phi = A_n$, he considered GE(R), the subgroup of $GL_n(R)$ generated by all $x_r(t)$ and all $h_i(z) = w_{r_i}(z)\, w_{r_i}(-1)$, i = 1 , 2, ..., n , where $w_r(u) = x_r(u)\, x_{-r}(-u^{-1})\, x_r(u)$. In addition to this group , he also considered GEU(R) , the group generated by symbols $x_r(t)$ and $h_i(z)$ with the usual relations [23, p. 37] in GE(R) . He showed that the natural homomorphism $f : GEU(R) \to GE(R)$ is an isomorphism, in which circumstance R is said to be universal for f . Consideration of the analogous notions for general Φ leads naturally to the following question which is currently under joint investigation by the author and E. Abe.

Question 9 Under what conditions on R and Φ is f an isomorphism for more general root systems ? In particular, is it an isomorphism in the case of $R = K[X]$, the free commutative K-algebra generated by a set X of indeterminates, at least in the single root length systems ?

Silvester's results for the case $R = K<X>$ was a major tool in showing that $K_2(A_n, K<X>) = K_2(A_n, K)$. Thus the answer to Question 9 bears directly upon the question of computing $K_2(\Phi, K[X])$ which is raised by E. Abe [2] in his article in these Proceedings.

In a similar vein, if R is a commutative ring with identity, then for $\Phi = A_n$, Center $E_n(R) = \Omega_n$, the group of n-th roots of 1 in R [15]. Passing to the stable group, Center $E(R) = 1$. For more general Φ , we can pose the following question.

Question 10 What is the center of the group $E(\Phi_n, R)$? What is the center of $E_\rho(R)$, the direct limit of $E_\rho(\Phi_n, R)$ for the classical Φ_n ?

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