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Various aspects of unipotent group actions in algebraic geometry

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§1. Unipotent group actions on complete varieties

1.1. Let k be an algebraically closed field of characteristic zero. Let G be a connected algebraic group defined over k. Assume that G acts non-trivially on an algebraic variety V,

$$\sigma : G \times V \longrightarrow V .$$

Then we have the canonical Lie algebra homomorphism

$$\sigma_* : \mathfrak{g} := \mathrm{Lie}(G) \longrightarrow H^0(V, \Theta_V),$$

where $\Theta_V := (\Omega^1_{V/k})^*$. If V is smooth over k, Θ_V is a locally free O_V-Module associated with the tangent bundle T_V. For every element τ of $\mathfrak{g}$, $\sigma_*(\tau)$ is thus a holomorphic (tangent) vector field of V.

Now assume that V is a nonsingular projective variety defined over k. Let X be a holomorphic vector field on V such that $X \not\equiv 0$. A point P of V is said to be a zero of X if $X(P) = 0$; the set of all zeros of X is denoted by

Zero(X), which is a closed subvariety of X. Let $P \in$ Zero(X). Then we can consider the Lie derivative L_X;

$$L_X : T_{V,P} \longrightarrow T_{V,P} \quad ; \quad L_X(Y) = [X,Y] .$$

X is said to be <u>generic</u> at P (or X has <u>a simple zero</u> at P) if L_X is nondegenerate on $T_{V,P}$; X is said to be <u>generic</u> (or X has <u>only simple zeros</u>) if L_X is nondegenerate for every zero P of X. If X has a simple zero at P, we can consider the eigenvalues $\theta_1(P),\dots,\theta_n(P)$ of L_X, where n = dim V. The existence of holomorphic vector fields (or actions of algebraic groups) on V imposes some restrictions on the topology and the numerical characters of V. We shall quote some of the known results.

1.2. Let V be a nonsingular projective variety defined over k and let X be a holomorphic vector field on V such that $X \not\equiv 0$. Let Z:= Zero(X). Define the contraction operator i_X as follows:

$$i_X : \Omega_X^p \longrightarrow \Omega_X^{p-1}$$

$$i_X(f dx_1 \wedge \dots \wedge dx_p) = f\left(\sum_{i=1}^{p} (-1)^{i-1} X(x_i) dx_1 \wedge \dots \wedge \overset{\vee}{dx_i} \wedge \dots \wedge dx_p\right).$$

The definition is well-defined, and if ω^p is an element of $H^0(V,\Omega_V^p)$ then $i_X(\omega^p) \in H^0(V,\Omega_V^{p-1})$. Let

$$\mathfrak{h}^1 := \{ X \in H^0(V,\Theta_V) \mid i_X : H^0(V,\Omega_V^1) \longrightarrow H^0(V,O_V) \text{ is the zero map} \}.$$

Then $\mathfrak{h}^1$ is a Lie subalgebra of $H^0(V,\Theta_V)$.

1.2.1. THEOREM. *With the above notations, we have:*

(1) (Kobayashi [7]). *If* $0 \leq \dim Z < n := \dim V$, *then* $P_m(V) = 0$ *for every* $m > 0$. *Hence* $\kappa(V) = -\infty$.

(2) (Carrell-Lieberman [1]). *Assume that* $Z \neq \phi$. *Then*

$$h^{p,q} = \dim_k H^q(V, \Omega_V^p) = 0 \quad \text{whenever} \quad |p-q| > \dim_k Z .$$

(3) (Carrell-Lieberman [1]). *Every element* X *of* $\mathfrak{h}^1$ *has zeros. Hence, if* $h^{1,0}(V) = \dim H^o(V, \Omega_V^1) = 0$ *then every holomorphic vector field has zero. Hence, if* V *has a holomorphic vector field without zero,* $h^{1,0}(V) > 0$.

1.2.2. COROLLARY. *Assume that* $\dim V = 2$ *and* V *has a holomorphic vector field* X *with* $\dim \mathrm{Zero}(X) = 0$. *Then* V *is rational.*

Proof. The assumption $\dim \mathrm{Zero}(X) = 0$ implies $h^{1,0}(V) = 0$. Since $X \neq 0$, we have $P_m(V) = 0$ for every $m > 0$. Hence V is rational by Castelnuovo's criterion of rationality.

1.2.3. THEOREM. *Let* $k = \mathbb{C}$. *Assume that* V *has a holomorphic vector field* X *possessing only simple zeros. For a point* P *of* $\mathrm{Zero}(X)$, *let* $\theta_1(P), \ldots, \theta_n(P)$ *be the eigenvalues of* L_X. *Assume that* $\mathrm{Re}\theta_i(P) \neq 0$ *for* $1 \leq i \leq n$ *and every point* $P \in \mathrm{Zero}(X)$. *Then the Betti numbers of* V *are given as follows:*

$$b_{2p}(V) = \#\{P \in \mathrm{Zero}(X) \mid \#\{j \mid \mathrm{Re}\theta_j(P) > 0,\ 1 \leq j \leq n\} = p\}$$

$$b_{2p+1}(V) = 0, \text{ (cf. Carrell-Lieberman [1]).}$$

1.3. *Examples.*

(1) Let G be a semi-simple algebraic group, let P be

a parabolic subgroup of G, let T be a maximal torus with $T \subset P$ and let $V := G/P$. Let t be a regular element of infinite order in T such that there exists a one-dimensional subtorus S of T passing through t. Let S act on V via left translations of G. Let X be a holomorphic vector field on V defined by the canonical Lie algebra homomorphism

$$\sigma_* : \mathfrak{s} := \mathrm{Lie}(S) \longrightarrow H^0(V, \Theta_V) .$$

<u>Then</u> Zero(X) <u>is a finite set and</u> X <u>has only simple zeros</u>.

<u>Proof</u>. We claim that:

(gP) is a fixed point of $S \Longrightarrow g^{-1}tg \in P \Longrightarrow g \in N(T)P$.

Indeed, S is the closure of $\{t^m \mid m \in \mathbb{Z}\}$, and hence (gP) is a fixed point if and only if $g^{-1}tg \in P$. Then $t \in gPg^{-1}$. Since t is a regular element, $T \subset gPg^{-1}$. Hence $g^{-1}Tg \subset P$. Therefore $g^{-1}Tg = p^{-1}Tp$ for some element $p \in P$. Hence $gp^{-1} \in N(T)$. Since $\#(N(T)P/P) < +\infty$, there are only finitely many fixed points of S on V. Let (gP) be a fixed point of S. Let $\mathfrak{g}$ and $\mathfrak{p}$ be the Lie algebras of G and P, respectively. Now, $T_{V,(gP)}$ is identified with $\mathfrak{g}/\mathfrak{p}$ via $\ell_{g,*} : \mathfrak{g}/\mathfrak{p} \xrightarrow{\sim} T_{V,(gP)}$. Then the Lie derivative L_X on $T_{V,(gP)}$ is identified with

$$\begin{matrix} Y \pmod{\mathfrak{p}} \\ \text{with } Y \in \mathfrak{g} \end{matrix} \longmapsto \mathrm{Ad}(g^{-1}tg)(Y) \bmod \mathfrak{p} .$$

Noting that $g^{-1}tg \in P$, we know that L_X is non-degenerate at (gP).

(2) Let $V = \mathbb{P}^n_k$ with homogeneous coordinates $(x_0, x_1, \ldots, x_n)$. Let $\alpha_0, \ldots, \alpha_n$ be pairwise prime integers such that $\alpha_0 + \ldots + \alpha_n$

$= 0$. Let G_m act on V via

$$t(x_0,x_1,\dots,x_n) = (t^{\alpha_0}x_0,t^{\alpha_1}x_1,\dots,t^{\alpha_n}x_n).$$

Then the fixed points of G_m on $\mathbb{P}^n$ are O_i's, where $O_i = (0,\dots,0,\overset{i}{\check{1}},0,\dots,0)$. Let $u_j = x_j/x_i$ and $\xi_j = \frac{\partial}{\partial u_j}$ for $0 \leq j \leq n$ and $j \neq i$. Then we have

$$T_{\mathbb{P}^n,O_i} = \sum_{\substack{j=0\\ j\neq i}}^{n} k\xi_j \quad \text{and} \quad L_X(\xi_j) = (\alpha_j-\alpha_i)\xi_j.$$

Instead, consider the following action of G_a on $\mathbb{P}^n_k$,

$$G_a = \{\exp(tA) \mid t \in k,\ A = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \in M_{n+1}(k)\}$$

$$t\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \exp(tA)\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Then $O := (1,0,\dots,0)$ is the unique fixed point of G_a. The holomorphic vector field X on $\mathbb{P}^n$ defined by this action has the following Lie derivative L_X on $T_{\mathbb{P}^n,O}$;

$$u_j = x_j/x_0,\ \xi_j = \frac{\partial}{\partial u_j} \quad (1 \leq j \leq n),$$

$$T_{\mathbb{P}^n,O} = \sum_{j=1}^{n} k\xi_j,$$

$$L_X(\xi_j) = 0 \ \text{ if } \ j = 1; = -\xi_{j-1} \ \text{ if } \ j > 1 .$$

Hence the zero of X at O is not simple.

1.4. Now, we shall be mainly interested in the unipotent group actions on complete algebraic varieties. A main problem is the Carrell conjecture, which we shall state below.

Let G be a unipotent algebraic group defined over k. We shall summarize some of the known results on unipotent group actions.

1.4.1. THEOREM. (1) [Borel fixed point theorem](cf. Fogarty [2], Horrocks [6]). *If a connected solvable affine algebraic group G acts on a complete algebraic variety V then the fixed point locus V^G is nonempty. If G is unipotent, V^G is connected if and only if V is connected.*

(2) *Let G be a connected affine algebraic group. Then G is unipotent if and only if, for any connected complete variety V on which G acts, V^G is connected* (cf. Fogarty [2]).

(3) *Let G and V be the same as in* (2) *above. Then the canonical inclusion $\iota : V^G \hookrightarrow V$ induces an equivalence between the categories of etale coverings $Et(V)$ and $Et(V^G)$. In particular, the inclusion ι yields an isomorphism of algebraic fundamental groups,*

$$\iota_* : \pi_1(V^G)_{alg} \xrightarrow{\sim} \pi_1(V)_{alg} ,$$

(cf. Horrocks [6]).

1.4.2. We also recall the following result:

THEOREM (Matsumura [8]). *Assume that V is a nonsingular complete variety. Then the group of all birational automorphisms of V, $Bir(V)$, contains an affine algebraic group of positive dimension if and only if V is birationally equivalent to $\mathbf{P}^1 \times W$ ($V \sim \mathbf{P}^1 \times W$ as notation), where W is a complete variety. Thus, if the Kodaira dimension $\kappa(V) \geq 0$, $Bir(V)$ cannot contain any affine algebraic group.*

1.4.3. Now, we consider the following:

CARRELL CONJECTURE. Assume that a connected unipotent group G acts on a nonsingular projective variety V in such a way that the fixed point locus V^G consists of a single point. Then V is rational.

1.4.4. A partial solution of the above conjecture is this:

THEOREM. Let G and V be the same as in the Carrell conjecture. Then we have:

(1) If $\dim V \leq 2$, the Carrell conjecture is affirmative.

(2) If $\dim V = 3$, V is one of the following:

(i) V is rational,

(ii) $V \sim \mathbf{P}^1 \times W$, where W is a nonsingular projective surface with $\kappa(W) \geq 1$ and $p_g = q = 0$. Moreover, W is simply connected.

Proof. Without loss of generality, we may assume that the action of G is effective, i.e., the canonical homomorphism $G \longrightarrow \mathrm{Aut}\,(V)$ is injective.

(1) The case $\dim V = 1$ is obvious by virtue of Matsumura's theorem. Suppose that $\dim V = 2$. If $\dim G = 1$, the action of G on V gives rise to a holomorphic vector field X on V such that $\mathrm{Zero}(X) = V^G$, which consists of a single point. Then, by virtue of Corollary 1.2.2, V is rational. Assume that $\dim G = 2$. Then G is commutative, i.e., $G \cong G_1 \times G_2$ with $G_1 \cong G_2 \cong G_a$. By virtue of Matsumura's theorem, $V \sim \mathbf{P}^1 \times C$, where C is a complete nonsingular model of $k(V^G)$. Then G_2 acts on C effectively. Hence $C \cong \mathbb{P}^1$, and $V \sim \mathbb{P}^1 \times \mathbb{P}^1$.

Namely, V is rational.

(2) Assume that $\dim V = 3$. Consider first the case where $\dim G = 3$. Then G has a central series of subgroups

$$G \supset G_1 \supset G_2 \supset (1) ,$$

such that $G/G_1 \cong G_1/G_2 \cong G_2 \cong G_a$. By Matsumura's theorem, $V \sim \mathbb{P}^1 \times W$, where W is a nonsingular projective surface such that W is a complete nonsingular model of $k(V^{G_2})$ and the unipotent group G/G_2 acts effectively on W. By virtue of the above case where $\dim V = \dim G = 2$, we conclude that W $\mathbb{P}^1 \times \mathbb{P}^1$; note that we did not use in the proof the assumption $V^G = \{\text{single point}\}$. Hence $V \sim \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Suppose next that $\dim G = 2$. By a similar reasoning as above, we know that $V \sim \mathbb{P}^1 \times \mathbb{P}^1 \times C$, where C is a nonsingular complete curve. Since V^G consists of a single point, we know that $\pi_1(V)_{alg} = (0)$ (cf. Theorem 1.4.1, (3)). Since $\pi_1(V)_{alg} \cong \pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \times C)_{alg}$, we know that $\pi_1(C)_{alg} = (0)$, i.e., C is simply connected. This implies that $C \cong \mathbb{P}^1$. Hence V is rational. Suppose finally that $\dim G = 1$. By virtue of Matsumura's theorem, $V \sim \mathbb{P}^1 \times W$, where W is a complete nonsingular model of $k(V^G)$. We may assume that W is relatively minimal. Let p_1 and p_2 be the canonical projections from $\mathbb{P}^1 \times W$ to $\mathbb{P}^1$ and W, respectively. Then we have,

$$\Omega^1_{\mathbb{P}^1 \times W} \cong p_1^* \Omega^1_{\mathbb{P}^1} + p_2^* \Omega^1_W$$

$$\Omega^2_{\mathbb{P}^1 \times W} \cong p_1^* \Omega^1_{\mathbb{P}^1} \wedge p_2^* \Omega^1_W + p_2^* \Omega^2_W .$$

By virtue of Theorem 1.2.1, (1), we have $h^{i,0}(V) = 0$ for

$i = 1, 2$, because the action of G yields a holomorphic vector field X on V with $\mathrm{Zero}(X) = V^G = \{\text{single point}\}$. Since $h^{i,0}(V)$ is a birational invariant (cf. Griffiths-Harris [16; p.494]), we know that $h^{1,0}(W) = h^{2,0}(W) = 0$. Hence $p_g = q = 0$ for W. Moreover, since $\pi_1(V)_{alg} = (0)$, we know that $\pi_1(W)_{alg} = (0)$. Namely, W is simply connected. If W is rational, V is rational. Suppose that W is not rational. If $\kappa(W) = 0$ then $p_g = q = 0$ implies that W is an Enriques surface, which is, however, not simply connected. Hence $\kappa(W) \geq 1$. Q.E.D.

1.5. We shall give the following result on the existence of a G_a-action.

LEMMA (cf. [9; p. 35]). *Let W be a variety defined over k and let $\pi : V \longrightarrow W$ be a $\mathbf{P}^1$-bundle over W. If there exists a nontrivial G_a-action on V whose orbits are contained in fibers of the projection π, then the fixed point locus V^G contains a cross-section S of π. Then there exists a locally free $\mathcal{O}_W$-Module E of rank 2 such that E is an extension of $\mathcal{O}_W$ by an invertible sheaf L on W, $V \cong \mathbf{P}(E)$, and S is the cross-section corresponding to L. Moreover, we have $H^0(W,L^{-1}) \neq 0$. Conversely, if $H^0(W,L^{-1}) \neq 0$, there exists a G_a-action on V along fibers of π.*

Proof. Let V_1^G be the union of irreducible components of V^G of codimension 1, and consider V_1^G as a reduced effective divisor on V. Since G_a acts on V along fibers of π, each general fiber contains one and only one fixed point. Hence

$(V_1^G \cdot \ell) = 1$, where ℓ is a general fiber of π. This implies that V^G contains only one irreducible component S, which is a cross-section of π. Let $L = O_S(S)$ and let $E = \pi_* O_V(S)$. Then we have an exact sequence,

$$0 \longrightarrow O_W \longrightarrow E \longrightarrow L \longrightarrow 0 .$$

By construction, S is the cross-section corresponding to L. The remaining part is proved in [9; p. 35]. Q.E.D.

§ 2. Unipotent group actions on affine varieties

2.1. Let k be an algebraically closed field of characteristic ≥ 0. Recall the following very well-known results:

2.1.1. THEOREM (Nagata [13], Haboush [5]). *Let R be a finitely generated k-algebra and let G be a connected reductive algebraic group. Assume that G acts on R as k-automorphisms of R in such a way that:*

For every $f \in R$, a k-submodule $\sum_{g \in G} f^g k$ of R is a finite k-module; then we say that G acts rationally on R.

Let R^G be the subring consisting of G-invariant elements in R. Then R^G is finitely generated over k.

2.1.2. THEOREM (Nagata [14]). *There exists a unipotent algebraic group G acting rationally on a polynomial ring $R := k[x_1,\ldots,x_n]$ such that R^G is not finitely generated over k.*

The writer believes that there should exist a rational action of the additive group G_a on a polynomial ring $R = k[x_1,\ldots,x_n]$ such that R^{G_a} is not finitely generated over k. If there is such an action, we must have $n \geqq 4$ by virtue of Zariski's theorem (cf. Nagata [14]), and the action is not linear by virtue of the following result of Weitzenböck:

2.1.3. THEOREM (cf. Seshadri [15]). Let there be given a linear action of G_a on a polynomial ring $R = k[x_1,\ldots,x_n]$, where $\mathrm{char}(k) = 0$. Then R^{G_a} is finitely generated over k.

2.2. For the sake of simplicity, we assume that $\mathrm{char}(k) = 0$.

THEOREM. Assume that G_a acts non-trivially on a polynomial ring $R = k[x_1,\ldots,x_n]$, where $n \leqq 3$. Let A be the G_a-invariant subring of R. Then we have:

(1) A is a finitely generated over k, and A is a unique factorization domain.

(2) If either $n \leqq 2$ or A is regular then A is a polynomial ring over k.

Proof. For the proof of the assertion (1) and the case $n \leqq 2$ in the assertion (2), see Miyanishi [9;§§ 1, 3 of Chap.I]. We shall prove the assertion (2) in the case where $n = 3$ and A is regular.

(i) By virtue of Zariski's theorem [14; p. 52], A is finitely generated over k. Moreover, A is a UFD and the set A^* of all invertible elements of A is $k^* := k-(0)$. Let $Y := \mathrm{Spec}(R)$, let $X := \mathrm{Spec}(A)$ and let $\pi : Y \longrightarrow X$ be the dominant morphism induced by the injection $A \hookrightarrow R$. We shall prove that the logarithmic Kodaira dimension of X has value $\bar{\kappa}(X) = -\infty$. Then we can apply the following characterization of the affine plane (cf. Miyanishi-Sugie [12] and Fujita [3] as well as the papers of Iitaka's given in the references of these papers):

Let $X = \mathrm{Spec}(A)$ be a nonsingular affine surface. Then $X \cong \mathbf{A}_k^2$ if and only if A is a UFD, $A^* = k^*$ and $\bar{\kappa}(X) = -\infty$.

(ii) We claim that $\pi : Y \longrightarrow X$ <u>is a faithfully flat, equi-dimensional morphism of dimension</u> 1.

We shall first show that π is surjective. Suppose π is not surjective. Then there exists a maximal ideal $\underline{m}$ of A such that $\underline{m}R = R$. Let $(\underline{O}, t\underline{O})$ be a discrete valuation ring of the quotient field K of A such that $\underline{O}$ dominates $A_{\underline{m}}$. Let $R' := R \otimes_A \underline{O}$, which is identified with a subring of the field $L := k(x_1, x_2, x_3)$. Let Δ be a locally nilpotent derivation on R associated with the given G_a-action on Y (cf. [9; § 1, Chap. I]). Then Δ extends naturally to a locally nilpotent $\underline{O}$-derivation in R', and $\underline{O}$ is the ring of Δ-invariants in R', i.e., $\underline{O} = \{r \in R'; \Delta(r) = 0\}$. By assumption, we have $tR' = R'$, where t is a uniformisant of $\underline{O}$. Hence $tr = 1$ for some element $r \in R'$. Then $t\Delta(r) = 0$, whence $r \in \underline{O}$. This is a contradiction. Thus π is surjective.

Secondly, we shall show that every irreducible component of a fiber of π has dimension 1. Note that general fibers of π are isomorphic to $\mathbf{A}_k^1$ (cf. [9; § 1, Chap. I]). Hence each irreducible component of a fiber has dimension ≥ 1. Suppose that an irreducible component T of a fiber $\pi^*(P)$ (with $P \in X$) has dimension 2. Since R is a UFD, there exists an irreducible element $a \in R$ such that $T = \mathrm{Spec}(R/aR)$. Since T is G_a-stable, a is G_a-invariant, i.e., $a \in A$. Let $C :=$

Spec(A/aA). Since A is a UFD, C is an irreducible curve on X and $\pi^{-1}(C) = T \subset \pi^{-1}(P)$. This is a contradiction because π is surjective. Thus π is an equi-dimensional morphism of dimension 1.

Finally, we shall show that R is flat over A. Let $\underline{q}$ be a prime ideal of R and let $\underline{p} = \underline{q} \cap A$. Then $R_{\underline{q}}$ dominates $A_{\underline{p}}$. Since $A_{\underline{p}}$ is regular and $R_{\underline{q}}$ is Cohen-Macaulay, $R_{\underline{q}}$ is flat over $A_{\underline{p}}$ (cf. EGA [4; IV,15.4.2]). Hence π is faithfully flat.

(iii) Let $U := \{P \in X;\ \pi^{*}(P)$ is irreducible and reduced$\}$. Then, by virtue of [9; Th.4.1.1,p.46], $W := \pi^{-1}(U)$ is an $\mathbb{A}^1$-bundle over U. We claim that $\bar{\kappa}(X) = -\infty$.

Let H be a hyperplane in $Y = \mathbb{A}^3_k$ such that $H \cap W \neq \phi$. Suppose $\bar{\kappa}(X) \geqq 0$. Let C be an irreducible curve on H. Consider a morphism:

$$\varphi : C \times \mathbb{A}^1_k \hookrightarrow H \times \mathbb{A}^1_k = Y \xrightarrow{\ \pi\ } X,$$

and assume that φ is a dominant morphism. Since $\dim(C \times \mathbb{A}^1_k) = \dim V = 2$, we have

$$-\infty = \bar{\kappa}(C \times \mathbb{A}^1_k) \geqq \bar{\kappa}(X) \geqq 0,$$

which is a contradiction. Hence φ is not a dominant morphism. Let D be the closure of $\varphi(C \times \mathbb{A}^1_k)$ in X. Then $C \times \mathbb{A}^1_k \subset \pi^{-1}(D)$. Suppose $C \cap W \neq \phi$. Then the general fibers of $\pi : \pi^{-1}(D) \longrightarrow D$ are isomorphic to $\mathbb{A}^1_k$. This implies that $\pi^{-1}(D)$ is irreducible and reduced. Since $\dim(C \times \mathbb{A}^1_k) = \dim \pi^{-1}(D) = 2$, we have $C \times \mathbb{A}^1_k = \pi^{-1}(D)$.

Let Q be a point on H, and let $C_1, \ldots, C_r$ be irreducible curves on H such that $C_1 \cap \dots \cap C_r = \{Q\}$ and that $C_i \cap W \neq \phi$ for $1 \leqq i \leqq r$. For any point Q on H, we can find such a set of irreducible curves. Indeed, H is the affine plane $\mathbb{A}_k^2$ and $H \cap (Y-W)$ has dimension $\leqq 1$; thus we have only to take a set of suitably chosen lines on H passing through Q. Let D_i be the irreducible curve which is the closure of $\pi(C_i \times \mathbb{A}_k^1)$ on X for $1 \leqq i \leqq r$. Then $C_i \times \mathbb{A}_k^1 = \pi^{-1}(D_i)$ for $1 \leqq i \leqq r$. Since we have

$$(Q) \times \mathbb{A}_k^1 = (C_1 \cap \dots \cap C_r) \times \mathbb{A}_k^1 = (C_1 \times \mathbb{A}_k^1) \cap \dots \cap (C_r \times \mathbb{A}_k^1)$$
$$= \pi^{-1}(D_1) \cap \dots \cap \pi^{-1}(D_r) = \pi^{-1}(D_1 \cap \dots \cap D_r),$$

we know that $D_1 \cap \dots \cap D_r = \{P\}$, P being a point on X. The correspondence $Q \longmapsto P$ defines a morphism $\psi : H \longrightarrow X$ such that $(Q) \times \mathbb{A}_k^1 = \pi^{-1}(P)$. If ψ is a dominant morphism, we have

$$-\infty = \bar{\kappa}(H) \geqq \bar{\kappa}(X) \geqq 0,$$

which is a contradiction. Hence ψ is not a dominant morphism. Let F be the closure of $\psi(H)$ in X. Then, for every point P of F, we have $\dim \psi^{-1}(P) \geqq 1$, and $\pi(\psi^{-1}(P) \times \mathbb{A}_k^1) = \psi(\psi^{-1}(P)) = P$. This contradicts the assertion proved in the step (ii). Therefore $\bar{\kappa}(X) = -\infty$. Q.E.D.

Natural as it is, the situation of G_a-actions on a polynomial ring $R = k[x_1,\dots,x_n]$ becomes complicated and worse as n increases. If $n \leqq 2$, R is a polynomial ring in one variable over the subring A of G_a-invariants (cf. [9; §1, Chap.I]). However, this does not hold in the case where $n = 3$. Still, the property that A be a polynomial ring seems to hold without the assumption that A is regular. When $n = 3$, another criterion for A to be a polynomial ring is that A contains one of coordinates x_1, x_2, x_3. Thus, if G_a acts linearly on $R = k[x_1,x_2,x_3]$, then A is a polynomial ring. Perhaps, A no longer is a polynomial ring for a general G_a-action on R if $n \geqq 4$.

2.3. Finally, we shall state the following result without proof:

THEOREM (cf. [11]). *Assume that* char(k) = o. *Let* X = Spec(A) *be a normal affine surface defined over* k, *possessing a non-trivial action of the additive group* G_a. *Then every singular point of* X *is a cyclic quotient singularity*.

The result no longer holds if X is not affine.

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