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QUASI-ORTHODOX SEMIGROUPS

Miyuki Yamada
Shimane University

As generalizations of groups, there are two important basic classes of regular semigroups. One is the class of inverse semigroups, and the other is the class of completely simple semigroups. The structure of inverse semigroups has been firstly investigated by Vagner [1952] and Preston [1954], and successively many papers concerning this class have appeared. On the other hand, a structure theorem for completely simple semigroups has been established by Rees [1940]. He has shown that every completely simple semigroup can be obtained, up to isomorphism, as a matrix semigroup called a Rees matrix semigroup over a group. These two classes are generalized to the class of orthodox semigroups and the class of completely regular semigroups respectively, and quite a lot of papers concerning these two classes have appeared during the last two decades (for example, see Hall [1969, 1970a, 1971] and the author [1967c, 1970b] etc. for orthodox semigroups; and Clifford [1941], Petrich [1967b] and Lallement [1967a] etc. for completely regular semigroups). As a class containing both the class of orthodox semigroups and the class of completely regular semigroups, we introduce the class of quasi-orthodox semigroups in this paper and discuss the structure of these semigroups. Throughout this paper, we shall use the following notations and terminology: For a completely regular semigroup M, the notation $M \sim \Sigma\{M_\lambda : \lambda \in \Lambda\}$ means that M is a semilattice Λ of completely simple semigroups $\{M_\lambda :$

$\lambda \in \Lambda$ } (that is, $M \sim \Sigma\{M_\lambda : \lambda \in \Lambda\}$ means the structure decomposition of M). Hereafter, the term " a completely regular semigroup $M \sim \Sigma\{M_\lambda : \lambda \in \Lambda\}$ " means that M is a completely regular semigroup and has $M \sim \Sigma\{M_\lambda : \lambda \in \Lambda\}$ as the structure decomposition. If an inverse semigroup Y has Λ as the semilattice of idempotents of Y, we shall denote it by $Y(\Lambda)$. For a regular semigroup S, the notation E(S) denotes the set of idempotents of S. In particular, if S is an orthodox semigroup then E(S) denotes the band of idempotents of S.

1. Basic properties

First, we shall give the definition of a quasi-orthodox semigroup.

Definition 1. Let S be a regular semigroup. If there exist an inverse semigroup $Y(\Lambda)$ and a surjective homomorphism $\phi: S \to Y(\Lambda)$ such that $\lambda\phi^{-1} = S_\lambda$ is a completely simple subsemigroup of S for each $\lambda \in \Lambda$, then S is said to be *quasi-orthodox*. In this case, it is obvious that $M = \bigcup\{S_\lambda : \lambda \in \Lambda\}$ is a completely regular subsemigroup of S, and the structure decomposition of M is $M \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$.

T. E. Hall has shown the following result (see the author [1979]): *A regular semigroup S is quasi-orthodox if and only if the subsemigroup $\langle E(S) \rangle$ of S generated by E(S) is completely regular*.

Now, we have the following results concerning quasi-orthodox semigroups:

Proposition 1.1. *Any homomorphic image of a quasi-orthodox semigroup is a quasi-orthodox semigroup.*

Proposition 1.2. *If ρ is a congruence on a quasi-orthodox semigroup, then S/ρ is a quasi-orthodox semigroup.*

Hereafter, for any element a of a regular semigroup S, V(a) denotes the set of inverses of a.

Proposition 1.3. *Let A and B be regular subsemigroups of a quasi-orthodox semigroup S. If*

(1.1) $A \cap B \neq \Box$ *and* $A \cap B \ni a$ *implies* $a^* \in A \cap B$ *for some* $a^* \in V(a)$,

then $A \cap B$ *is a quasi-orthodox subsemigroup of* S.

Definition 2. Let R and T be regular semigroup, and $f: R \to T$ a homomorphism. If the condition

(1.2) for any $a \in R$ and for any $(af)^* \in V(af)$, there exists $a^* \in V(a)$ such that $a^*f = (af)^*$

is satisfied, then f is called **-homomorphism*.

Proposition 1.4. *Let S and T be quasi-orthodox semigroups, and* $f: S \to T$ *a surjective *-homomorphism. For any regular subsemigroup K of T,* Kf^{-1} *is a regular subsemigroup of S. Hence,* Kf^{-1} *is a quasi-orthodox subsemigroup of S.*

If every H-class of a semigroup S consists of a single element, then S is said to be *H-degenerated*.

Proposition 1.5. *An H-degenerated quasi-orthodox semi-*

group is an orthodox semigroup.

Proposition 1.6. A regular subsemigroup of a completely regular semigroup is completely regular.

Definition 3. Let S be a regular semigroup. A completely regular subsemigroup $G \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$ is called a kernel normal system of S if $G \supset E(S)$ and if there exists a congruence σ on S such that each S_λ $(\lambda \in \Lambda)$ is a complete σ-class. In this case, such a congruence σ is unique and is called the congruence determined by $G \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$. Of course, σ is an inverse semigroup congruence on S. If ρ is an inverse semigroup congruence on a quasi-orthodox semigroup S such that $e\rho$ is a completely simple subsemigroup of S for each $e \in E(S)$, then $G = \bigcup\{e\rho : e \in E(S)\}$ is a kernel normal system of S and has $G \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$ as its structure decomposition (where each S_λ is a ρ-class). In this case, the kernel normal system $G \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$ is called the kernel normal system determined by ρ, and conversely ρ is called an inverse semigroup congruence with kernel normal system (abbrev., k.n.s.). The completely regular subsemigroup $M \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$ in Definition 1 is a kernel normal system, which is called the kernel normal system determined by ϕ.

Let S be a quasi-orthodox semigroup. Let G_e be a subgroup (of S) containing e for each $e \in E(S)$ such that $G = \bigcup\{G_e : e \in E(S)\}$ is a subsemigroup of S. In this case, $G \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$ is not necessarily a kernel normal system of S. However, at least we have the following:

Proposition 1.7. Let S be a regular semigroup, and G_e a subgroup (of S) containing e for each $e \in E(S)$. If $G = \bigcup\{G_e : e \in E(S)\}$ is a subsemigroup of S, then there exists a kernel normal system $M \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$ of S such that $M \supset G$. Accordingly, S is a quasi-orthodox semigroup.

Definition 4. Let S be a regular semigroup, and M_e a maximal subgroup (of S) containing e for each $e \in E(S)$. If $M = \bigcup\{M_e : e \in E(S)\}$ is a subsemigroup of S, then S is said to be natural regular. By the result above, a natural regular semigroup is quasi-orthodox.

For the kernel normal systems of a quasi-orthodox semigroup, we have the following result:

Proposition 1.8. Let S be a quasi-orthodox semigroup, and $N \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$ a kernel normal system of S. Let A be a regular subsemigroup of S, and put $\Lambda' = \{\lambda \in \Lambda : S_\lambda \cap A \neq \square\}$. Then, $A \cap N \sim \Sigma\{A \cap S_{\lambda'} : \lambda' \in \Lambda'\}$ is a kernel normal system of A.

Definition 5. Let S be a regular semigroup. Let $M \sim \Sigma\{M_\lambda : \lambda \in \Lambda\}$ be a completely regular semigroup, and $Y(\Lambda)$ an inverse semigroup. If

(1) M is a subsemigroup of S, and

(2) there exists a surjective homomorphism $f: S \to Y(\Lambda)$ such that $\lambda f^{-1} = M_\lambda$ for each $\lambda \in \Lambda$,

then S is called a regular extension of $M \sim \Sigma\{M_\lambda : \lambda \in \Lambda\}$ by $Y(\Lambda)$.

Next, we shall show some characterizations of a quasi-orthodox semigroup:

Theorem 1.9. For a regular semigroup S, the following five conditions are equivalent:

(1) S is a quasi-orthodox semigroup.

(2) The subsemigroup $\langle E(S) \rangle$ is completely regular.

(3) There exists a completely regular subsemigroup C of S such that $S \supset C \supset E(S)$.

(4) S is a regular extension of a completely regular semigroup $M \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$ by an inverse semigroup $Y(\Lambda)$.

(5) S has a kernel normal system; accordingly, there exists an inverse semigroup congruence (on S) with kernel normal system.

Now, we can infer from the results in this section that an analogue to the process used in Hall-Yamada theory [1967c, 1969, 1970b, 1971] concerning the structure of orthodox semigroups will be applicable for the study of quasi-orthodox semigroups. In the following sections, we shall discuss the structure of quasi-orthodox semigroups under this direction.

2. Inverse semigroup congruences with k.n.s.

Proposition 2.1. The least inverse semigroup congruence η_S on S is an inverse semigroup congruence with k.n.s.

Let σ be an inverse semigroup congruence on S with k.n.s., and put $C_\sigma(S) = \{ \rho : \rho$ is an inverse semigroup congruence on

S with k.n.s. such that $\rho \supset \sigma$ }. On the other hand, let $I_\sigma(S)$ be the set of all idempotent separating congruences on S/σ .

Then,

Theorem 2.2.

(1) _For any_ $\rho \in C_\sigma(S)$, _the congruence_ $\bar{\rho}$ _defined by_

(2.1) $x\sigma \ \bar{\rho} \ y\sigma$ _if and only if_ $x \ \rho \ y$

is an element of $I_\sigma(S)$.

(2) _For any_ $\bar{\tau} \in I_\sigma(S)$, _the congruence_ τ _defined by_

(2.2) $x \ \tau \ y$ _if and only if_ $x\sigma \ \bar{\tau} \ y\sigma$

is an element of $C_\sigma(S)$.

(3) _The mapping_ $\psi: C_\sigma(S) \to I_\sigma(S)$ _defined by_ $\rho\psi = \bar{\rho}$ _is an order-preserving bijection (where ordering in each of_ $C_\sigma(S)$ _and_ $I_\sigma(S)$ _is given by the set-inclusion)_.

In particular, consider the case where $\sigma = \eta_S$. Then, ψ in Theorem 2.2 is an order-preserving bijection of the set of all inverse semigroup congruences on S with k.n.s. onto the set of all idempotent separating congruences on S/η_S. Hence, if $\bar{\zeta}_S$ is the greatest idempotent separating congruence on S/η_S then ζ_S given by (2.2) is the greatest inverse semigroup congruence on S with k.n.s.

Next for $\Delta \subset C_{\eta_S}(S)$, put $\rho_\Delta = \bigcap\{\rho: \rho \in \Delta\}$. Then $\rho_\Delta \in C_{\eta_S}(S)$. Accordingly, if $\sigma \cup \rho$ is defined by $\sigma \cup \rho = \bigcap\{\tau \in C_{\eta_S}(S): \tau \supset \sigma, \rho\}$ for $\sigma, \rho \in C_{\eta_S}(S)$ then $C_{\eta_S}(S)$ is a complete lattice with respect to $\cap, \cup$. Similarly, $I_{\eta_S}(S)$ is a complete lattice with respect to the ordering defined by the set-inclusion.

Theorem 2.3. <u>The complete lattices</u> $C_{\eta_S}(S)$ <u>and</u> $I_{\eta_S}(S)$ <u>are lattice isomorphic</u>.

Remark. It can be easily seen from Theorem 2.2 that a regular semigroup S is quasi-orthodox if and only if S is a regular extension of a completely regular semigroup $M \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$ by a fundamental inverse semigroup.

3. Construction

In this section, we shall consider the construction of quasi-orthodox semigroups. A construction theorem for general quasi-orthodox semigroups has been given by the author [1974], but it is given in a somewhat complicated form and we omit to show it in this paper. We shall only consider the construction of some special quasi-orthodox semigroups called

(A) an upwards [downwards] directed quasi-orthodox semigroup, and

(B) a splitting quasi-orthodox semigroup.

First, we introduce the concept of a partial chain as follows:

Definition 6. Let Λ be a semilattice, and T_λ a semigroup for each $\lambda \in \Lambda$. If a partial binary operation $\circ$ is defined in $T = \Sigma\{ T_\lambda : \lambda \in \Lambda \}$ (disjoint sum) such that

(1) $\lambda \geq \tau$, $a \in T_\lambda$ and $b \in T_\tau$ imply that $a \circ b$ $[b \circ a]$ is defined and $a \circ b$ $[b \circ a] \in T_\tau$,

(2) $a, b \in T_\lambda$ implies $a \circ b = ab$ (the product of a,b in T_λ), and

(3) $\lambda \geq \tau \geq \delta$, $a \in T_\lambda$, $b \in T_\tau$ and $c \in T_\delta$ imply $a \circ (b \circ c) =$

$(a \circ b) \circ c \quad [(c \circ b) \circ a = c \circ (b \circ a)]$,

then the resulting system $T(\circ)$ is called a lower [upper] partial chain Λ of $\{ T_\lambda : \lambda \in \Lambda \}$. We denote it by $T = LP\{ T_\lambda : \lambda \in \Lambda ; \circ \}$ $[T = UP\{ T_\lambda : \lambda \in \Lambda ; \circ \}]$.

First, we consider the construction of all $LP\{ T_\lambda : \lambda \in \Lambda ; \circ \}$ for a given semilattice Λ and for given right reductive semigroups $\{T_\lambda : \lambda \in \Lambda\}$. If G is a right reductive semigroup, the inner left translation semigroup $\Lambda_0(G)$ of G is a left ideal of the left translation semigroup $\Lambda(G)$ of G and the mapping ϕ: $G \rightarrow \Lambda(G)$ defined by $a\phi = \lambda_a$ (where λ_a is the inner left translation of G induced by a) is an injective homomorphism. Hereafter, $D(G)$ denotes an isomorphic copy of $\Lambda(G)$ such that $D(G)$ contains G as its left ideal and there exists an isomorphism ϕ_G: $D(G) \rightarrow \Lambda(G)$ satisfying $a\phi_G = \lambda_a$ for $a \in G$.

Theorem 3.1. Let Λ be a semilattice, and S_λ a right reductive semigroup for each $\lambda \in \Lambda$. For every pair (α, β) of $\alpha, \beta \in \Lambda$ with $\alpha \geq \beta$, let $\phi_{\alpha,\beta} : S_\alpha \rightarrow D(S_\beta)$ be a homomorphism such that the family $\{\phi_{\alpha,\beta} : \alpha, \beta \in \Lambda, \alpha \geq \beta\}$ satisfies the following (3.1)-(3.2):

(3.1) $\phi_{\lambda,\lambda}$ is the identity mapping on S_λ for each $\lambda \in \Lambda$,

(3.2) $(a\phi_{\alpha,\beta} * b)\phi_{\beta,\gamma} = (a\phi_{\alpha,\gamma}) * (b\phi_{\beta,\gamma})$ for $\alpha \geq \beta \geq \gamma$, $a \in S_\alpha$ and $b \in S_\beta$, where $*$ denotes the multiplication in $D(S_\tau)$ $(\tau \in \Lambda)$,

then $S = \Sigma\{ S_\lambda : \lambda \in \Lambda \}$ becomes a lower partial chain Λ of $\{ S_\lambda : \lambda \in \Lambda \}$ under the partial binary operation $\circ$ defined by

(3.3) $a \circ b = (a\phi_{\alpha,\beta}) * b$ for $\alpha \geq \beta$, $a \in S_\alpha$, $b \in S_\beta$.

Further, every $LP\{ S_\lambda : \lambda \in \Lambda ; \circ \}$ can be obtained in this way.

Dually, we can construct every UP{ S_λ: $\lambda \in \Lambda; \circ$} for a given semilattice Λ and left reductive semigroups {S_λ: $\lambda \in \Lambda$}.

(A) The construction of upwards [downwards] directed quasi-orthodox semigroups.

Definition 7. Let S be a quasi-orthodox semigroup, and $G \sim \Sigma\{S_\lambda : \lambda \in \Lambda\}$ a kernel normal system of S. If

(3.4) $E(S_\lambda)E(S_\mu) \subset E(S_\lambda)$ [$E(S_\mu)E(S_\lambda) \subset E(S_\lambda)$] for $\lambda < \mu$,

then S is said to be upwards [downwards] directed.

It is easily verified that this concept is independent from the selection of a kernel normal system of S.

It should be noted that orthodox semigroups and completely simple semigroups are of course upwards [downwards] directed quasi-orthodox semigroups. Now, let $Y(\Lambda)$ be a (fundamental) inverse semigroup, $L[\Lambda]$ a lower partial chain Λ of left groups { L_λ: $\lambda \in \Lambda$} , and $R[\Lambda]$ an upper partial chain Λ of right zero semigroups {R_λ: $\lambda \in \Lambda$}. Assume that $E(L_\lambda) \cap R_\lambda = \{u_\lambda\}$ for each $\lambda \in \Lambda$. For every pair (γ,δ) of $\gamma,\delta \in Y(\Lambda)$, let

$$f_{\langle\gamma,\delta\rangle}: R_{\gamma^{-1}\gamma} \times L_{\delta\delta^{-1}} \longrightarrow L_{\gamma\delta(\gamma\delta)^{-1}} \quad \text{and}$$

$$g_{\langle\gamma,\delta\rangle}: R_{\gamma^{-1}\gamma} \times L_{\delta\delta^{-1}} \longrightarrow R_{(\gamma\delta)^{-1}\gamma\delta}$$

be mappings such that the family $\Delta = \{f_{\langle\gamma,\delta\rangle}: \gamma,\delta \in Y(\Lambda)\} \cup \{g_{\langle\gamma,\delta\rangle}: \gamma,\delta \in Y(\Lambda)\}$ satisfies the following (3.5):

(3.5)
(1) For $a \in L_{\gamma\gamma^{-1}}$, $e \in R_{\gamma^{-1}\gamma}$, $b \in L_{\delta\delta^{-1}}$, $f \in R_{\delta^{-1}\delta}$, $c \in L_{\tau\tau^{-1}}$ and $h \in R_{\tau^{-1}\tau}$,

$$a(e,b((f,c)f_{\langle\delta,\tau\rangle}))f_{\langle\gamma,\delta\tau\rangle} = a((e,b)f_{\langle\gamma,\delta\rangle})((e,b)g_{\langle\gamma,\delta\rangle}f,c)f_{\langle\gamma\delta,\tau\rangle}$$

(2) For $a \in L_{\gamma\gamma^{-1}}$ and $e \in R_{\gamma^{-1}\gamma}$, there exist $b \in L_{\gamma^{-1}\gamma}$ and $f \in R_{\gamma\gamma^{-1}}$ such that

$$(e,b)f_{\langle\gamma,\gamma^{-1}\rangle}\ (f,a)f_{\langle\gamma\gamma^{-1},\gamma\rangle} \in E(L_{\gamma\gamma^{-1}}).$$

This Δ is called a factor set of $\{L[\Lambda], R[\Lambda]\}$ belonging to $Y(\Lambda)$. Now, $S = \{(x,\gamma,e)\colon x \in L_{\gamma\gamma^{-1}},\ e \in R_{\gamma^{-1}\gamma},\ \gamma \in Y(\Lambda)\}$ becomes a quasi-orthodox semigroup under the multiplication defined by

$$(x,\gamma,e)(y,\delta,f) = (x(e,y)f_{\langle\gamma,\delta\rangle}, \gamma\delta,\ (e,y)g_{\langle\gamma,\delta\rangle}f).$$

This S is called the regular product of $\{L[\Lambda], Y(\Lambda), R[\Lambda]\}$ determined by Δ , and denoted by

(3.6) $S = R(L[\Lambda] \times Y(\Lambda) \times R[\Lambda]\ ; \Delta)$.

Now, we have the following:

Theorem 3.2. If Δ above satisfies the following condition: (upwards directed condition)

$(u,z)f_{\langle\lambda,\lambda\rangle} \in E(L_\lambda)$, $(v,w)f_{\langle\mu,\mu\rangle} \in E(L_\mu)$ and $\lambda < \mu$ imply

$((u,w)g_{\langle\lambda,\mu\rangle}v,\ z(u,w)f_{\langle\lambda,\mu\rangle})f_{\langle\lambda,\lambda\rangle} \in E(L_\lambda)$,

then $S = R(L[\Lambda] \times Y(\Lambda) \times R[\Lambda]; \Delta)$ is an upwards directed quasi-orthodox semigroup. Further, every upwards directed quasi-orthodox semigroup can be constructed in this way.

(B) The construction of splitting quasi-orthodox semigroups.

Definition 8. Let T be a quasi-orthodox semigroup. If there exist an inverse semigroup $I(\Lambda)$, a homomorphism $\phi\colon I(\Lambda) \to T$ and a surjective homomorphism $\psi\colon T \to I(\Lambda)$ such that

(1) $\lambda\psi^{-1}$ is a completely simple subsemigroup of T for each $\lambda \in \Lambda$, and

(2) $\phi\psi = \iota_{I(\Lambda)}$ (the identity mapping on $I(\Lambda)$),

then T is called <u>split</u>.

Theorem 3.3. *In the regular product* $S = R(L[\Lambda] \times Y(\Lambda) \times R[\Lambda]; \Delta)$ *in* (3.6), *if* $u_\lambda u_\tau = u_\lambda$ *in* $R[\Lambda]$ *and* $u_\tau u_\lambda = u_\lambda$ *in* $L[\Lambda]$ *for* $\lambda \leq \tau$ *and if* Δ *satisfies the following condition*:

(*split condition*) *for* $\gamma, \delta \in Y(\Lambda)$,

$$(u_{\gamma^{-1}\gamma}, u_{\delta\delta^{-1}})f_{\langle\gamma,\delta\rangle} = u_{\gamma\delta(\gamma\delta)^{-1}} \quad \text{and}$$

$$(u_{\gamma^{-1}\gamma}, u_{\delta\delta^{-1}})g_{\langle\gamma,\delta\rangle} = u_{(\gamma\delta)^{-1}\gamma\delta},$$

then S *is a splitting quasi-orthodox semigroup*. *Further, every splitting quasi-orthodox semigroup can be constructed in this way*.

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