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PERFECTNESS AND DISTRIBUTIVITY OF CONGRUENCES ON FINITE INVERSE SEMIGROUPS

Howard Hamilton and Takayuki Tamura

1. Introduction. Let S be a semigroup and ρ a congruence on S. For $a \in S$, $a\rho$ denotes the ρ-class containing a. A congruence ρ on S is called perfect if

for all $a, b \in S$, $(a\rho)(b\rho) = (ab)\rho$ as sets.

A semigroup S is called perfect if all congruences on S are perfect. This concept was introduced by Vagner [10], and perfect semigroups were studied by Fortunatov [3],[4],[5],[6]. Groups are typical examples of perfect semigroups; also it is well known that the lattice of congruences on a group is a modular lattice [1]. A semigroup is called modular (distributive) if its lattice of congruences forms a modular (distributive) lattice. In this paper we study the structure of finite perfect modular inverse semigroups and their lattices of congruences except distributive groups. It will be seen that the two concepts, perfectness and distributivity (modularity), are independent for finite inverse semigroups. The complete proof of a part of this paper will be published elsewhere [8].

2. Preliminaries. Perfectness is preserved under homomorphisms; a semilattice is perfect if and only if it is a chain; a proper ideal of a perfect semigroup is a completely prime ideal. Let S be a finite inverse semigroup and $S = \bigcup_{i \in \Gamma} S_i$ be the greatest semilattice decomposition of S where Γ is a finite chain, $\Gamma = \{0,1,\ldots,n\}$ with $i.j = \min\{i,j\}$. If S is perfect, then S_i is a group for all $i \in \Gamma$, $i \neq 0$, and S_0 is either a group or a Brandt semigroup. S is determined by the set of homomorphisms φ_{ij} of S_i into the right translation semigroup $\Omega_r(S_j)$ of S_j, $j < i$:

$$\varphi_{ij} : S_i \to \Omega_r(S_j) , \qquad p \mapsto p\varphi_{ij}, \qquad p \in S_i,$$
$$p\varphi_{ij} : S_j \to S_j \qquad x \mapsto x(p\varphi_{ij}), \quad x \in S_j.$$

In case $i, j > 0$, S_i and S_j are groups and $p\varphi_{ij}$ is regarded as an element of S_j. For $\bigcup_{i=1}^{n} S_i$, the set $\{\varphi_{ij} : i>j\}$ is the so-called transitive system [2]. Throughout this paper $\omega(X)$ denotes $X \times X$.

THEOREM 1. Let S be a finite inverse semigroup and $S = \bigcup_{i \in \Gamma} S_i$ be the greatest semilattice decomposition of S where Γ denotes a lower semilattice. Then S is perfect if and only if

(1) Γ is a finite chain : $\Gamma = \{0,1\ldots,n\}$ with $ij = \min\{i,j\}$.

(2) S_0 is either a group or a Brandt semigroup, and S_i is a group for all $i > 0$.

(3) S is composed by permutations, that is, $p\varphi_{ij}$, is a permutation of S_j for all $p \in S_i$, all $i, j \in \Gamma$, $j < i$.

(4) For all $i > j > 0$ (for all $i > j \geq 0$ if S_0 is a group) $\varphi_{ij} : S_i \to S_j$ is surjective.

If S_0 is a group of order ≥ 1, S is called type 1; if S_0 is a Brandt semigroup, namely, a completely 0-simple semigroup $\mathfrak{M}^0(m,G)$ whose basic group is G and sandwich matrix is an $m \times m$ identity matrix with $m>1$, then S is called type 2.

Let L be a lattice and L^0 the lattice obtained by adjoining a new element 0 to L such that 0 is the greatest element of L^0 preserving join and meet in L. The following is easily obtained.

LEMMA 2. L^0 is modular (distributive) if and only if L is modular (distributive).

Modularity (distributivity) of lattices is preserved under homomorphic images, sublattices and direct products.

LEMMA 3. A homomorphic image of a modular (distributive) semigroup *is modular (distributive).*

Let $L(2)$ denote the lattice of two elements 0,1 with $i \vee j = \max\{i,j\}$, $i \wedge j = \min\{i,j\}$. Let D be a semigroup and $D^0 \cup \{0\}$ be D with zero 0 adjoined. Let $L(D)$ denote the lattice of congruences on D.

LEMMA 4. *Let D be a semigroup. Then $L(D^0)$ is a subdirect product of $L(D)$ and $L(2)$. Therefore D^0 is modular (distributive) if and only if D is modular (distributive).*

Proof. For each $\xi \in L(D)$, define a congruence $((\xi,1))$ on D^0 by

$$((\xi,1)) = \{(0,0)\} \cup \xi.$$

Then $L(D)$ is isomorphic to a sublattice of $L(D^0)$ under $\xi \mapsto ((\xi,1))$. If $\xi \in L(D)$ such that D/ξ has zero 0, let $I(\xi)$ denote the pre-image of 0 under $D \to D/\xi$, and define a congruence $((\xi,0))$ on D^0 by

$$((\xi,0)) = \omega(I(\xi) \cup \{0\}) \cup (\xi | D_\xi^*)$$

where $D_\xi^* = D \setminus I(\xi)$ and $\xi | D_\xi^*$ denotes the restriction of ξ to D_ξ^*. (If $\xi = \omega(D)$, $\omega(D^0) = ((\xi,0))$.) Then we have

$$L(D^0) = \{((\xi,1)) : \xi \in L(D)\} \cup \{((\xi,0)) : \xi \in L(D),\ D/\xi \text{ has zero}\}.$$

We can show that if $\xi, \eta \in L(D)$ and $i,j \in L(2)$ then

$$((\xi,i)) \vee ((\eta,j)) = ((\xi \vee \eta, \min\{i,j\})),$$

$$((\xi,i)) \wedge ((\eta,j)) = ((\xi \wedge \eta, \max\{i,j\}))\,..$$

Consequently $L(D^0)$ is a subdirect product of $L(D)$ and $L(2)$. The last statement is obvious.

LEMMA 5. *If D is a semilattice of two non-trivial groups, and if condition (4) of Theorem 1 is satisfied, then D is not modular.*

Proof. Let $D = D_1 \cup D_2$, $|D_1|>1$, $|D_2|>1$, D_1 and D_2 are groups. Let ξ be the smallest semilattice congruence on D, and η the smallest group congruence on D. Define a congruence ζ by

$$\zeta = \omega(D_1) \cup (\eta | D_2).$$

Then $\zeta \subset \xi$ but

$$\xi \vee \eta = \zeta \vee \eta = \omega(D). \quad \xi \wedge \eta = \zeta \wedge \eta = \iota D_1 \cup (\eta | D_2).$$

Hence D is not modular [1].

3. Structure of the Lattice, Type 1. From the results above we have

(6) If a finite perfect inverse semigroup of type 1 is modular (distributive), then S must be one of the following:

M 1. a chain; M 2. a group; M 3. a chain-group.

By a chain-group S, we mean an ideal extension S of a finite chain $T_0 = \{0,1,\dots,n-1\}$ by a finite group G with zero adjoined by means of identity mappings, namely, the identity left and right translations of T_0.

Chain. Let C(n) denote the chain of n elements, $n > 1$.

PROPOSITION 7. L(C(n)) is isomorphic to the direct product of n-1 copies of L(2). Hence C(n) is distributive.

Proof. In Lemma 4 let D be C(n-1). Since D is finite, D/ξ has zero for every $\xi \in L(D)$. Hence $L(C(n)) \cong L(C(n-1)) \times L(2)$. By induction on n we have the conclusion.

REMARK 8. Any chain is distributive without assuming finiteness. The congruences on a chain as a semilattice are its partitions into non-overlapping segments and this partition gives also a congruence on the chain as a lattice (Exercise 1, P.138 [1]). By Funayama and Nakayama's theorem ([7] or Theorem 9, P. 138, [1]), the lattice of congruences on any lattice is complete Browerian, but a Browerian lattice is distributive by Theorem 18, P. 45, [1]. Also we can show: The lattice of congruences on a lattice $S(\vee,\wedge)$ equals the lattice of congruences on the join (meet) semilattice $S(\vee)$ $(S(\wedge))$ if and only if $S(\vee,\wedge)$ is a chain.

Chain-Groups. By Lemma 4 we see that a chain-group is distributive if and only if the group is distributive. However we study more the structure of the lattice of congruences on a chain-group. Let $S = T \cup G$ where $T = \{s_i \ldots, s_{n-1}\}$ is a chain with $s_i s_j = s_{\min\{i,j\}}$, G is a finite group and S is the ideal extension of T by G^0 by means of identity mappings. If $\xi \in L(T)$ and $\eta \in L(G)$ then $\xi \cup \eta \in L(S)$. Let $\zeta \in L(S)$ and assume $a\zeta b$ for some $a \in T$, $b \in G$. Because of identity composition, $a\zeta xb$ for all $x \in G$, hence $a\zeta c$ for all $c \in G$; thus ζ can be obtained as follows: if $\xi \in L(T)$,

$$\zeta = \xi \mid (T\setminus(s_{n-1}\xi)) \cup \omega(s_{n-1}\xi \cup G).$$

Let $L(S) = L_1(S) \cup L_2(S)$ where $L_1(S)$ and $L_2(S)$ are defined by

$$L_1(S) = \{\zeta \in L(S) : a \not\zeta b \text{ for all } a \in T, \text{ all } b \in G\},$$

$$L_2(S) = \{\zeta \in L(S) : a\zeta b \text{ for some } a \in T, \text{ some } b \in G\}.$$

Then $L_1(S) \cong L(T) \times L(G)$ and $L_2(S) \cong L(T)$. We have

$$L_1(S) \cong \{((\xi,\eta)) : \xi \in L(T), \eta \in L(G)\}, \quad L_2(S) \cong \{((\xi,0)) : \xi \in L(T)\},$$

where

$$((\xi_1,\eta_1)) \vee ((\xi_2,0)) = ((\xi_1 \vee \xi_2, 0)),$$

$$((\xi_1,\eta_1)) \wedge ((\xi_2,0)) = ((\xi_1 \cap \xi_2, \eta_1)).$$

PROPOSITION 9. _If_ $S = T \cup G$ _is a chain-group, then_

$$L(S) \cong L(T) \times (L(G))^0.$$

S _is distributive if and only if_ G _is distributive_.

4. _Structure of the Lattice, Type_ 2. Let $S = S_0 \cup \bigcup_{i=1}^{n} S_i$ be of type 2. If S is modular (distributive), then $\bar{S} = \bigcup_{i=i}^{n} S_i$ must be one of M1, M2 and M3.

(10) _If a finite perfect inverse semigroup_ S _of type_ 2 _is modular (distributive), then_ S _must be one of the following_:

M4. _a Brandt semigroup_; M5. _a Brandt-group_;

M6. _a Brandt-chain_; M7. _a Brandt-chain-group_.

S is called a Brandt-group if $\bar{S}$ is M2; a Brandt-chain if $\bar{S}$ is M1; a Brandt-chain-group if $\bar{S}$ is M3. Each S is an ideal extension of S_0 by by $\bar{S}^0$ by means of permutations. In particular, S of M6 or M7 is an ideal extension of S_0 by $\bar{S}^0$ by means of identity mappings.

Brandt-Semigroups. Let $S = M^0(m,G)$, namely,

$$S = \{(i, x, j) : i, j = 1,\dots,m;\ x \in G\} \cup \{0\}$$

where

$$(i, x, j)(k, y, \ell) = \begin{cases} (i, xy, \ell) & \text{if } p_{jk} = e \\ 0 & \text{if } p_{jk} = 0. \end{cases}$$

Every non-universal congruence ρ on S is determined by a congruence τ on G as follows by [9]:

$(i, x, j)\ \rho\ (k, y, \ell)$ if and only if $i = k$, $j = \ell$, and $x\ \tau\ y$,

$\{0\}$ forms a singleton class.

PROPOSITION 11. If $S = M^0(m,G)$, then $L(S) \cong (L(G))^0$. S is distributive if and only if G is distributive.

Brandt-Groups. Let S be M5, $S = S_0 \cup H$ where $S_0 = M^0(m,G)$ and H is a group. Let $\zeta \in L(S)$ and assume $a\ \zeta\ b$ for some $a \in S_0$ and some $b \in H$. Since $b\varphi: S_0 \to S_0$ is a permutation for all $b \in H$, there is $c \in S_0$ such that $ca = 0$ but $cb \neq 0$. It follows that $x\ \zeta\ b$ for all $x \in S_0$, hence $z\ \zeta\ d$ for all $z \in S_0$, all $d \in H$, that is, $\zeta = \omega(S)$. Thus every non-universal congruence ζ on S has the form $\zeta = \xi \cup \eta$ where $\xi \in L(S_0)$, $\eta \in L(H)$. But ξ and η must satisfy:

(*) $y\ \eta\ u$ implies $xy\ \xi\ xu$ for all $x \in S_0$.

PROPOSITION 12. If $S = S_0 \cup H$ is M5 where $S_0 = M^0(m,G)$, then

$$L(S) = \{((\xi,\eta)) : \xi \in L(S_0),\ \eta \in L(H) \text{ with } (*)\}^0 \subseteq (L(S_0) \times L(H))^0.$$

S is distributive if and only if G and H are distributive.

Brandt-Chains, Brandt-Chain-Groups. If S is M6, $S = S_0 \cup S_1$ where $S_0 = M^0(m,G)$, $S_1 = \{1,2,\dots,n\}$. If S is M7, $S = S_0 \cup \bigcup_{i=1}^{n-1} S_i \cup S_n$, $n > 1$ where $S_i = \{s_i\}$ is a singleton class for $i = 1,\dots,n-1$; S_n is a group. Let V be S_1 if S is M6; let $V = \bigcup_{i=1}^{n} S_i$ if S is M7. Then $a\varphi: S_0 \to S_0$ is the identity mapping for all $a \in V$. As seen in M5, for any $\xi \in L(S_0)$ and $\eta \in L(V)$, $\xi \cup \eta \in L(S)$. In the same way as M5, we can show if $\zeta \in L(S)$ and $a \zeta b$ for some $a \in S_0$ and some $b \in V$ then $\omega(S_0) \subseteq \zeta$. Accordingly we have

$$L_1(S) = \{\zeta \in L(S) : a \not\zeta b \text{ for all } a \in S_0, \text{ all } b \in V\}$$
$$\cong \{((\xi,\eta)) : \xi \in L(S_0), \eta \in L(V)\} \cong L(S_0) \times L(V),$$
$$L_2(S) = \{\zeta \in L(S) : a \zeta b \text{ for some } a \in S_0, \text{ some } b \in V\}$$
$$\cong \{((0,\eta)) : \eta \in L(V)\} \cong L(V),$$

and

$$((\xi_1,\eta_1)) \vee ((0,\eta_2)) = ((0, \eta_1 \vee \eta_2)),$$
$$((\xi_1,\eta_1)) \wedge ((0,\eta_2)) = ((\xi_1,\eta_1 \cap \eta_2)).$$

Therefore $L(S) \cong (L(S_0))^0 \times L(V)$.

PROPOSITION 13. *If* $S = S_0 \cup S_1$ *is* M6 *where* $S_0 = \mathfrak{m}^0(m,G)$, *then* $L(S) \cong (L(S_0))^0 \times L(S_1)$.

S is distributive if and only if G is distributive.

PROPOSITION 14. *If* $S = S_0 \cup V$ *is* M7 *where* $S_0 = \mathfrak{m}^0(m,G)$, $V = \bigcup_{i=1}^{n} S_i$ *and* S_n *is a group, then*

$$L(S) \cong (L(S_0))^0 \times L(V).$$

S is distributive if and only if G and S_n are distributive.

5. Conclusion.

THEOREM 15. *Let S be a finite perfect inverse semigroup. Then S is modular if and only if S is one of the following:*

M1. *a chain*; M2. *a group*; M3. *a chain-group*;

M4. *a Brandt semigroup*; M5. *a Brandt-group*;

M6. *a Brandt-chain*; M7. *a Brandt-chain-group*;

equivalently, S has no more than one non-trivial group component in the *greatest semilattice decomposition of* S . *Futhermore* S *is distributive if and only if every subgroup of* S *is distributive*.

Abelian groups are distributive if and only if they are cyclic by Theorem 14, p. 173 [1]. Obviously if normal subgroups of a group G form a chain, G is distributive. As another example, if H and K are non-abelian simple groups and $H \ncong K$, then $H \times K$ is distributive.

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Department of Mathematics and Statistics
California State University, Sacramento
Sacramento, California 95819
U.S.A.

Department of Mathematics
University of California
Davis, California 95616
U.S.A.