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On the algebraic varieties containing a curve in projective space

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Introduction.

Let Z be a projective curve in P^n over an algebraically closed field k; we want to investigate the linear system of all hypersurfaces of P^n through Z with degree $t \gg 0$, in order to see whether some member of such a system is smooth everywhere or has at least "good" tangent cones at the singularities of Z. Moreover we want to investigate the finite intersections of general hypersurfaces through Z, to see whether there are r-dimensional algebraic varieties containing Z and smooth (as far as it is possible). When Z is smooth or has just plane singularities (embedding dimension 2), then there is a surface, global complete intersection in P^n of hypersurfaces through Z, which contains Z and is smooth everywhere; more generally, when Z has at most points with embedding dimension $r \leq n-1$, then there is a smooth r-dimensional variety through Z, global complete intersection in P^n.
When Z has singular points of embedding dimension n, we look for hypersurfaces through Z smooth everywhere else and having at these points ordinary singularities (i.e. with tangent cone projectively smooth). There are examples of "bad" singular points of Z for which such a hypersurface cannot exist. So we introduce the concept of "quasi ordinary" singular point : when $P \in Z \subset P^n$ has embedding dimension n it is quasi ordinary if the projectivized tangent cone to Z at P, as a zero dimensional subvariety of P^{n-1}, is contained as a subscheme in some smooth variety. If all points of Z with em

bedding dimension n have such a property, then we are able to produce a hypersurface F through Z smooth everywhere except at these bad points, where the tangent cones to F are projectively smooth. In particular a curve Z in P^3 just with quasi ordinary singularities lies on a surface which is smooth elsewhere and has ordinary tangent cones at these points.

For embedding dimension $r \leq n-1$, the existence of a smooth r-fold through Z is also proved in [A-K] , with different techniques (which seem less elementary than ours).

One of the main tools of our work consists of a Bertini theorem (on the variable singular point), valid for any algebraically closed field useful, to deal with the behaviour of the hypersurfaces at the points of P^n-Z and at the points of embedding dimension n of Z. It includes the analogous Bertini theorem of [A-K], but our proof seems quicker and more elementary.

A Bertini type method works also to deal with the behaviour of the generic hypersurface through Z at the simple points of Z.

0. Conventions and terminology.

0.1 We work over an algebraically closed field k and "algebraic variety" means "scheme of finite type" over k.
By point we mean "closed point" and we often identify an algebraic variety with the ringed space of its closed points.
If Z,Y are closed subvarieties of X we write $Z \subset Y$ to mean that Z is a subvariety of Y.
We write P^n for projective space over k .

0.2 We consider only Cartier divisors. If a divisor is effective, we identify it with the closed subvariety corresponding to it.

0.3 The intersection of two closed subvarieties is always in the algebraic sense; that is if Y_1, Y_2 are the subvarieties of X given by the sheaves of ideals I_1, I_2, then $Y_1 \cap Y_2$ is the closed subvariety given by $I_1 + I_2$.

0.4 If $Y \subset X$ is a closed subvariety and D is a divisor of X not containing any component of Y, then $D \cap Y$ is a divisor on Y and is called "the divisor cut out by D on Y".
If S is a linear system on X, the restriction of S to Y is the set S' of divisors on Y which can be expressed in the form $D \cap Y$ where $D \in S$. It is clear that S' is a linear system. More precisely if $j : Y \to X$ is the embedding and if S corresponds to the vector space $V \subset H^0(X,L)$, L being an invertible sheaf, then S' corresponds to the vector space $V' = \varphi(V)$, where $\varphi : H^0(X,L) \to H^0(Y,j^*L)$ is the canonical map.

0.5 If S is a linear system on X and $D_0, \dots, D_r \in S$, $\lambda = (\lambda_0, \dots, \lambda_r) \in P^r$, we write D_λ for the linear combination $\lambda_0 D_0 + \dots + \lambda_r D_r$.

The "+" sign is always used for linear combinations (and not for sum of divisors) unless explicitly stated.

0.6 Let S be a finite dimensional linear system over X and let $D_0,\dots,D_r$ be a fixed base of S. We say that the generic element of S verifies a given property P if the set $\{\lambda \varepsilon P^r / D_\lambda$ has P$\}$ contains a non empty open set. This is clearly independent on the choice of the base and of the coordinates in P^r.

0.7 When considering local equations of the divisors of a linear system S (at a point or on an open set) we always take the equations in a coherent way. That is to say : if S corresponds to $V \subset H^0(X,L)$, the local equations in $U \subset X$, where $L_{/U} \cong O_{X/U}$ (or in $x \varepsilon X$) are given by one, and the same for all $D \varepsilon S$, fixed isomorphism $L_{/U} \cong O_{X/U}$ (resp. $L_x \cong O_x$).

0.8 The base locus of a linear system S on X is the set of points of X which belong to all $D \varepsilon S$; it is a closed subset of X.

1. A Bertini theorem.

In this section we give sufficient conditions for the generic member of a linear system to be smooth outside its base locus.

1.0 Notations. Throughout this section X is a proper integral variety of dimension $d > 0$ over k, $Z \subset X$ is a closed reduced subvariety and $U = X-Z$.

S is a linear system on X, of dimension r, and $D_0,\dots,D_r$ form a fixed base for S. For each $\lambda = (\lambda_0,\dots,\lambda_r) \varepsilon P^r$ we put : $D_\lambda =$
$= \Sigma\lambda_i D_i$ (0.5).

1.1 Definition : We say that D separates $x,y \in X$ if $x \in D, y \notin D$. We say that S separates the points of $U \subset X$ if, whenever $x,y \in U$ and $x \neq y$, there is $D \in S$ which separates x and y. Note that in this case the base locus of S is contained in Z = X-U (but not conversely). We say that $E_0,\ldots,E_n \in S$ separate the tangent vectors at $x \in X$ if $x \in E_0 \cap \ldots \cap E_n$ and the local equations of the E_i's at x generate the maximal ideal m_x of $O_{X,x}$, i.e. they are linearly independent modulo m_x^2 (see 0.7 for conventions about local equations).

We say that S separates the tangent vectors at $x \in X$ if there are $E_0,\ldots,E_n \in S$ as above (see [H] , 7.3, p.152). If this happens for all $x \in U$ we say that S separates the tangent vectors on U.

We begin with a well known lemma :

1.2 Lemma : *Let $f : V \to W$ be a dominant morphism of algebraic varieties. Then $\dim V \geq \dim W$.*

Proof : Let W_0 be an irreducible component of W having maximal dimension and let $V_0,\ldots,V_n$ be the irreducible components of V. Then we have : $W = \overline{f(V)} = \bigcup_i \overline{f(V_i)}$, hence $W_0 = \bigcup_i (\overline{f(V_i)} \cap W_0) = \bigcup_i (\overline{f(V_i)} \cap W_0)$; therefore we may assume that $W_0 = \overline{f(V_0)}$. Therefore we may assume that both V and W are irreducible; moreover it is easy to see that they may be assumed also reduced. Now use the transcendence degrees over the base field.

1.3 Theorem : *With the notation of 1.0 assume that*

(i) $U \subset$ *Reg*(X);

(ii) *The base locus of* S *is contained in* Z *(e.g.* S *separates the points of* U);

(iii) S *separates the tangent vectors on* U.

Then the generic element of S is nonsingular at the points of U (i.e. for $D \in S$ generic $Sing(D) \subset Z$).

Proof (compare with [H] , proof of Bertini theorem, II,8.18) : Put :
$B = \{(x,\lambda) \in U \times P^r / x \in Sing(D_\lambda)\}$.

Claim 1 : B is closed in $U \times P^r$.

Indeed let $U = \bigcup_j U_j$ be an open affine covering such that D_i is principal on U_j, for all i,j. Clearly it is sufficient to show that $B \cap (U_j \times P^r)$ is closed for all j; so we may assume that U be affine and D_i be principal on U for all i.

Fix a closed embedding $U \subset k^n$ and let $I = (G_1,\ldots,G_s) \subset k[T] = k[T_1,\ldots,T_n]$ be the corresponding ideal. For each $i = 0,\ldots,r$ there is a hypersurface F_i such that $D_i \cap U$, in the above embedding, corresponds to the divisor cut out by F_i on U. Then for each λ the variety $D_\lambda \cap U$ is isomorphic to the subvariety of k^n defined by the ideal $(G_1,\ldots,G_s, \Sigma\lambda_i F_i)$, so that claim 1 follows easily by the jacobian criterion.

Claim 2 : dim B < r. Indeed consider on B the reduced structure and let : $p : B \to U$ be the canonical morphism. For each $x \in U$ let B_x be the fiber of p at x. Since $\dim O_{X,x} = d$ for all x, it is sufficient to show that $\dim B_x = r-d-1$ for all x (see [M] ,(13.B)). Fix $x \in U$ and put : $A = O_{X,x}$. Let m be the maximal ideal of A. Denote by $q : U \times P^r \to P^r$ the canonical morphism and for each i let f_i be a local equation of D_i at x. Then we have :

$B_x \cong q(B_x) = \{\lambda \in P^r / \Sigma\lambda_i f_i \in m^2\}$.

Thus $B_x = \mathbb{P}(\mathrm{Ker}\,\varphi)$, where φ is the linear map from k^{r+1} to A/m^2 defined by $\varphi(\lambda_0,\ldots,\lambda_r) = \Sigma\lambda_i f_i \mod m^2$.

Now by (ii) and (iii) it follows that φ is surjective; by (i) we have $\dim A/m^2 = d+1$, so claim 2 follows.

Now we can conclude the proof of the theorem. Let $\bar{B}$ be the closure of B in $X \times P^r$ and let $\pi : X \times P^r \to P^r$ be the projection. Since X is proper over k we have that $\pi(\bar{B})$ is closed; and then by 1.2 we have $\dim \pi(\bar{B}) \leq \dim \bar{B} = \dim B < r$. Thus $P^r - \pi(\bar{B})$ is open and dense in P^r. On the other hand if $\lambda \in P^r - \pi(\bar{D})$ it is clear that $\mathrm{Sing}(D_\lambda) \cap U = \emptyset$ and the conclusion follows.

1.4 Lemma : Let $X' \subset X$ be a closed irreducible subvariety and put : $U' = U \cap X'$. Let S' be the restriction of S to X' (see 0.4). If S separates the points (and the tangent vectors) on U, then S' separates the points (and the tangent vectors) on U'.

Proof : If $x, y \in U'$, $x \neq y$, then there is $D \in S$ which separates x,y; hence $D \not\supset X'$ and $D' = D \cap X'$ is an element of S' which separates x and y. Assume S separates the tangent vectors on U. Let $x \in U'$ and put : $S_x = \{D \in S / x \in D\}$. Then S_x is a linear system. If $U' = \{x\}$ there is nothing to prove; so we may assume that there is $y \in U'$, $y \neq x$. Then there is $D \in S_x$, with $y \notin D$, and hence the generic element of S_x does not contain X'. Hence the tangent vectors at x, viewed as a point of U, can be separated by $E_0, \ldots, E_n \in S$, with $E_i \cap X' \neq X'$. Therefore $E_0 \cap X', \ldots, E_n \cap X'$ are elements of S', which separate the tangent vectors at x, viewed as a point of U'.

1.5 Corollary : Let $Y \subset X$ be a closed subvariety and let $V = Y \cap U$. Assume that S separates points and tangent vectors on U and that $V \subset \mathrm{Reg}(Y)$. Then if $D \in S$ is generic we have : $\mathrm{Sing}(D \cap Y) \subset Y - V$.

Proof : Let X' be an irreducible component of Y and put : $U' = U \cap X'$. Then, if S' is the restriction of S to X', by 1.3 and 1.4 we have that the generic element of S' is smooth on U'. We may assume that

$D'_i = D_i \cap X'$ for $i = 1,\dots,s$ form a basis of S' and that $D_i \cap X'$ for $i > s$. Let $T' \subsetneq P^r$ be closed such that if $(\lambda_0,\dots,\lambda_r) \notin T'$ then $D'_\lambda = \Sigma\lambda_i D'_i$ is smooth on U'. Let $T = \{(\lambda_0,\dots,\lambda_r) \in P^r /$ $(\lambda_0,\dots,\lambda_s) \in T'\}$.

Then T is a closed subset of P^r, different from P^r, and it is easy to see that if $\lambda \notin T$ then $D_\lambda \cap X'$ is smooth at the points of U'. The conclusion follows by the irreducibility of P^r.

1.6 <u>Corollary</u> : <u>Let the notations and the assumptions be as in</u> 1.3 <u>and assume that</u> S <u>separates the points of</u> U. <u>Then for any</u> $n > 0$ <u>there are</u> $E_1,\dots,E_n \in S$ <u>such that</u>

(a) $E_1 \cap \dots \cap E_n \cap U$ <u>is smooth;</u>

(b) <u>dim</u> $E_1 \cap \dots \cap E_n = d-n$ ($E_1 \cap \dots \cap E_n = \emptyset$ <u>if</u> $d-n < 0$) ;

(c) E_1 <u>is generic in</u> S, E_2 <u>can be any</u> D_λ <u>where</u> λ <u>varies in a none-mpty open subset of</u> P^r <u>(depending on</u> E_1<u>) and so on</u>.

Proof : it follows from 1.5 by induction on n.

1.7 <u>Definition</u> : If $E_1,\dots,E_n$ verify (ii) and (iii) of 1.6 we say that they are generically independent (This clarifies the notion used in [A-K] , th. 1).

Now we want to apply the above to the case when X is projective.

1.8 <u>Lemma</u> : <u>Assume</u> $X \subset P^n$ <u>and for</u> $t > 0$ <u>let</u> $S^{(t)}$ <u>be the linear system cut out on</u> X <u>by the hypersurfaces of degree</u> t <u>which contain</u> Z. <u>Assume that for some</u> t <u>the base locus of</u> $S^{(t)}$ <u>is</u> Z. <u>Then for all</u> $t' > t$ $S^{(t')}$ <u>separates the points and the tangent vectors on</u> U.

Proof : Enough to prove the claim for $t' = t+1$. Let $x,y \in U$, $x \neq y$. Then there is $D \in S^{(t)}$ with $y \notin D$. Thus if H is a hyperplane section containing x but not y, D+H (sum of divisors) is an element of $S^{(t+1)}$ which separates x and y.

Let now $x \in U$ and let $H_1,\dots,H_\ell$ be hyperplane sections which separate the tangent vectors at x. Let $D \in S$ with $x \in D$. Then $D+H_1,\dots,D+H_\ell$ (sum of divisors) are element of $S^{(t+1)}$ which separate the tangent vectors at x.

1.9 Corollary : For all $t \gg 0$, $S^{(t)}$ separates points and tangent vectors on U (hence 1.3 is appliable).

Proof : There is t_0 such that Z is set theoretic intersection of hypersurfaces of degree t_0. Then any $t > t_0$ works by 1.8.

1.10 Corollary : Let $Y \subset P^n$ and $W \subset Y$ be closed subvarieties. Assume that W is (set theoretically) the intersection of a family of hypersurfaces of degree t_0 and that Y-W is smooth. Then, if $G_1,\dots,G_n$ are generic independent hypersurfaces of degree $t > t_0$ (see 1.7), then $G_1 \cap \dots \cap G_n \cap (Y-W)$ is smooth (possibly empty).

Proof : Apply 1.8 and 1.9 with $X = P^n$, $U = P^n - W$.

1.11 Corollary : 1.10 remains true also if we delete the assumption "k algebraically closed", provided k is infinite.

Proof : It follows easily from 1.10, by extension of the ground field to the algebraic closure.

1.12 Corollary : Let $W \subset P^n_k$ (k infinite, not necessarily algebraically closed) and let dim $W = d$. Then for all e with $d < e \leq m$ there is a complete intersection Y_λ of dimension e containing W and such that Y-W is smooth.

1.13 Remarks : (i) In characteristic O theorem 1.3 holds without assumption (iii): this is in fact the classical "second theorem of Bertini" (on the variable singular point : see [Z] , section 5).
(ii) In positive characteristic theorem 1.3 is false without assumption (iii): see for instance the example given at the end of [Z] .
(iii) Other formulations of the second theorem of Bertini in characteristic $p > 0$ can be found in the literature : see e.g. [A] . For related results see also [F] , section 5.
(iv) It is not true that in 1.6 it is enough to take $E_0,\dots,E_n$ generic, linearly independent and such that $\dim E_1 \cap \dots \cap E_n = d-n$. For example let $U = X = P^2$ and let S be the linear system of all conics.
Assume there is an open non empty $W \subset P^5$ such that, whenever $\lambda,\mu \in W$, D_λ and D_μ are non singular and also $D_\lambda \cap D_\mu$ is nonsingular (i.e. D_λ and D_μ are not tangent). Fix $C = D_\lambda$, $\lambda \in W$, and fix $P \in C$. Let C' be a nonsingular conic, different from C, tangent to C at P. Then the pencil generated by C and C' corresponds to a line in P^5 which intersects W. Thus there is $\mu \neq \lambda$ such that $\mu \in W$ and D_μ is tangent to D_λ at P, a contradiction.
(v) The following conjecture regarding 1.7 (ii) seems reasonable : " there is a nonempty open subset $W \subset P^r \times \dots \times P^r$ (n times) such that $D_{\lambda_1} \cap \dots \cap D_{\lambda_n} \cap U$ is nonsingular for all $(\lambda_1,\dots,\lambda_n) \in W$".
(vi) The results of [A-K] and [F] and of the present paragraph seem to support the following conjecture : Let X be proper over k and irreducible of dimension $d > 0$. Let $Z \subset X$ be closed of codimension at least 2. Let S be a linear system on X which separates

points and tangent vectors on X-Z. Then the generic element of S is irreducible.

(vii) Note that in the above conjecture the assumption on the codimension of Z is necesary. Indeed let X be a nonsingular quadric in P^3 and let L_1, L_2 be two skew lines on X; then put : $Z = L_1 \cap L_2$. Then $S = S^{(2)}$ (notations of 1.8) separates points and tangent vectors on U = X-Z, but all elements of S contain the two skew lines, so being not irreducible.

(viii) Lemma 1.1 is false for general schemes. For example take $V = \mathrm{Spec}(A_P)$, $W = \mathrm{Spec}(A)$, A non local.

ix) The theorem of Bertini for S = linear system of all hypersurfaces of degree $t > 0$ in P^N is really stronger : the set of all nonsingular elements of S <u>is open</u> and $\neq \emptyset$, as it can be easily seen remarking that the set B of the proof of theorem 1.3 is closed in the projective variety $P^N \times P^r$, hence φ (B) is closed $\neq P^r$.

2. <u>Behaviour of divisors at the points of low embedding dimension of a base curve</u>.

In this section we keep the notations of 1.0 and we assume further that Z is a curve contained in every element of S.
Our aim is to study the behaviour of the generic element of S at the points of Z. In this section we begin with the points of low embedding dimension (see theorem 2.3).

If $x \in X$ and $D, D' \in S$ we say that D and D' are non tangent at x if $D \cap D'$ is non singular at x and $D \neq D'$. This is equivalent to say that the leading forms of their local equations at x (see 0.7) are linearly independent elements of $gr(O_{X,x})$ of degree 1.

2.1 Proposition : Let $V \subset Z$ be an open subset and let $I \subset V$ be finite. Assume that:

(*) for each $x \in V-I = V'$ there are $D, D' \in S$ non tangent at x;

and (**) for each $y \notin I$ there is $D \in S$ nonsingular at y.

Then the generic element of S is nonsingular at all points of V.

Proof : Clearly by (**) the generic element of S is nonsingular at each point of I and hence we may assume that $I = \emptyset$.

Let $W = \{(x,\lambda) \in V \times P^r / x \in \mathrm{Sing}(D_\lambda)\}$. Then W is closed in $V \times P^r$ (same proof as in 1.2, claim 1). Moreover by the jacobian criterion the generic element of S has only finitely many singular points on V. Hence there is a non empty open set $U \subset P^r$ such that the fibers of $p : W \to P^r$ at the points of U are finite. By [EGA], IV_3, (9.7.8), we may also assume that the number of geometric connected components of the fibers over U be equal to the number of geometric connected components of the fiber over the generic point of P^r; but, if $x \in U$ is closed and if the base field is algebraically closed, such a number is simply the cardinality of the fiber over x ([EGA], IV_2, (4.5.2)). So we may assume that the fibers over the points of U are finite and of constant cardinality s. Our aim is to show that $s = 0$. Assume $s > 0$. Fix $D = D_\lambda$, $\lambda \in U$, and let $P_1, \ldots, P_s$ be its singular points on V. Let $V_1, \ldots, V_h$ be the irreducible components of V and let $Q_i \in V_i$ be simple for D ($i = 1, \ldots, h$). By (*) there are $E_0, \ldots, E_h \in S$ such that E_0 is nonsingular at P_1 and E_i is nontangent to D at Q_i for $i = 1, \ldots, h$. Then the generic element E of the linear system generated by $E_0, \ldots, E_h$ is nonsingular at P_1 and nontangent to D at $Q_1, \ldots, Q_h$; hence, by the jacobian criterion of simple points applied to $D \cap E$, it is tangent to D at only finitely many points (other than $P_1, \ldots, P_s$), say $R_1, \ldots, R_\ell$. Now the generic element of the pencil gene rated by D and E has at most $P_2,\dots,P_s$ as singular points. On the other hand this pencil corresponds to a line in P^r which intersects U, hence almost all the elements of the pencil are of the form D_μ, $\mu \varepsilon U$. But this is a contradiction.

2.2 *Corollary* : *Let the assumptions be as in* 2.1 *and assume further that* $U = X-Z$ *be smooth, that* Z *be the base locus of* S *and that* S *separates the tangent vectors on* U. *Then if* $D \varepsilon S$ *is generic we have* : *Sing*$(D) \subset Z-V$.

Proof : Apply 2.1 and 1.3.

The assumptions of 2.1 and 2.2 are verified by a large class of linear systems if X is projective. To see this we need first a lemma :

2.3 *Lemma* : *Assume* $X \subset P^n$ *and let* $x \varepsilon Z$. *Put* : $A = O_{X,x}$ *and* $B = O_{Z,x} = A/\mathfrak{a}$. *Let* $f_1,\dots,f_s$ *be non zero elements of* $\mathfrak{a}$. *Then for any* $t \gg 0$ *there are* $D_1,\dots,D_t \varepsilon S^{(t)}$ *whose local equations at* x (*see* 0.7) *are* $f_1,\dots,f_s$ *(as in the previous section* $S^{(t)}$ *is the linear system cut out on* X *by the hypersurfaces of degree* t *which contain* Z).

Proof : Let R be the graded ring of X in P^n and let $I,P \subset R$ be the homogeneous ideals of Z and X respectively. Then $\mathfrak{a}$ is the set of all fractions of the form a/b, where $a,b \varepsilon R_t$ (for some t depending on a and b), $a \varepsilon \mathfrak{a}, b \notin P$. Thus there are an integer t_o and $a_1,\dots,a_s \varepsilon \mathfrak{a} \cap R_{t_o}$, $s \varepsilon R_{t_o}-P$, such that $f_i = a_i/s$, and this is also true for all $t \geq t_o$. Then it is easy to see that $a_1,\dots,a_s$ define $D_1,\dots,D_s \varepsilon S^{(t)}$ whose local equations at x are $f_1,\dots,f_s$.

2.4 Corollary : The notations being as in 2.3, assume that A be regular of dimension d and that emdim $B < d$. Then for all $t \gg 0$ there is $D \in S^{(t)}$ which is smooth at x. Hence the generic element of $S^{(t)}$ is smooth at P.

Proof : Let m, n be the maximal ideals of A,B respectively. Then we have $\dim(m/m^2) > \dim(m/m^2 + a)$, hence $a \not\subset m^2$. The conclusion follows by 2.3.

2.5 Theorem : Let $X \subset P^n$ be an irreducible projective variety, let $Z \subset X$ be a closed reduced curve and let V be an open subset of Z. Let $x_1, \dots, x_s \in Z-V$ and let $p = (p_1, \dots, p_s)$ be an s-uple of positive integers. Let $S^{(t)}$ be as in 2.3 ; then put : $S_p^{(t)} = \{D \in S^{(t)} / e_{x_i}(D) \geq p_i\}$, where $e_{x_i}(.)$ denotes multiplicity at x_i. Assume that:

(i) $\dim X \geq 3$;

(ii) $V \subset \mathrm{Reg}(X)$;

(iii) emdim $O_{Z,x} < \dim X = d$ for all $x \in V$.

Then if $t \gg 0$ and $D \in S_p$ is generic, we have:

(a) D is nonsingular at each point of V;

(b) if moreover $X-Z \subset \mathrm{Reg}(X)$, then $\mathrm{Sing}(D) \subset Z-V$.

Proof : To prove (a) it is enough to show that both (*) and (**) of 2.1 hold, with I = Sing(V). Assume first that $p = (1, \dots, 1)$, so that $S_p^{(t)} = S^{(t)}$ for all t. Let $x \in V$ and put : $A = O_{X,x}$, $B = O_{Z,x} = A/a$. Let m, n be the maximal ideals of A,B respectively. If $x \in \mathrm{Reg}(V)$, then $a = (f_1, \dots, f_e)$, where $e \geq 2$, and $f_1, \dots, f_e$ are contained in a regular system of parameters of A (use (i) and (ii)). Hence by 2.3 there are D, $E \in S^{(t)}$ (for some t) which are non tangent at x. By the jacobian criterion applied to $D \cap E$ we see that D and E are non tangent in a neighbourhood of x and, by the compactness of V, we see that $S^{(t)}$ verifies (*) for all $t \gg 0$.

As for (**) it follows immediately from 2.4.

Let now p be general. Fiw u such that $S^{(u)}$ verifies (*) and (**). Let e_i be the minimal multiplicity of the elements of $S^{(u)}$ at x_i (then e_i is the multiplicity of the generic element of $S^{(u)}$ at x_i). We may assume that $e_i \geq p_i$ if and only if $i \geq e$. Let $v = \sum_{i=1}^{e} (r_i - e_i)$. Then it is clear that, for all $t \geq u+v$, $S_p^{(t)}$ verifies (*) and (**) (add t-u suitable hyperplanes to suitable elements of $S^{(u)}$). This proves (a). To prove (b) apply 1.9 and the same argument used for (a).

2.6 Corollary : In 2.5 put : $e = \sup\{\text{emdim}(O_{Z,x})/x \in V\}$. Then for generic independent $E_1,\dots,E_n \in S^{(t)}$ (see 1.7), if $t \gg 0$ and $n \leq d-e$ we have: $E_1 \cap \dots \cap E_n$ is nonsingular at all points of V. If moreover $X-Z \subset \text{Reg}(X)$, then $E_1 \cap \dots \cap E_n$ is nonsingular also in X-Z.

Proof : It follows from 2.5 and 1.6, with an argument similar to the one used in 1.6.

2.7 Corollary : Let $Z \subset P^n$ be a curve and let s be the maximum embedding dimension of the points of Z. Then there is a complete intersection $Y \subset P^n$, of dimension r and smooth (hence irreducible by [H], III, ex. 5.5) containing Z. In particular:

(a) if $s < n$, Z is contained in a smooth hypersurface;

(b) if $s \leq 2$, Z is contained in a smooth irreducible surface.

Remark : When Z is contained in a smooth irreducible surface we say that Z has only plane singularities.

2.8 Proposition : Let $Z \subset P^3$ be a curve of degree d and let $S^{(t)}$ be the linear system of the surfaces of degree t which contain Z. Then:

(a) For $t \geq d$, $S^{(t)}$ is non empty and its generic element is smooth at all points of Reg(Z);

(b) For $t \geq d+1$ the generic element of $S^{(t)}$ is singular at most at the points of Sing(Z);

(c) If Z is smooth, then the generic element of $S^{(d+1)}$ is smooth. In particular Z is contained in a smooth surface of degree d+1.

Proof : (a) If F is a cone which projects Z from a point, then deg $F \leq d$ and hence $S^{(t)}$ is non empty for all $t \geq d$ (add hyperplanes to F). Moreover if $x \in$ Reg(Z) let $L_1 \neq L_2$ be two lines through x, not tangent to Z, not meeting Z outside x. Pick $y_i \in L_i$, $y_i \neq x$ and let F_i be the cone projecting Z from y_i. Then F_1 and F_2 are not tangent at x_j and by adding a suitable number of planes not containing x we see that, for $t \geq d$, $S^{(t)}$ verifies the assumptions of 2.1 and (a) follows.

(b) By 1.3 and 1.8 it is enough to show that $S^{(d)}$ separates the points of $U = P^3 - Z$; by the above remark it is enough to show that the cones which project Z from points of P^3 do the same. Let then $x, y \in U$, $x \neq y$, and let L be the line joining x and y. Let $z \in Z$, $z \notin L$, and let L' be the line joining x and z. Then if F is the cone which project Z from x, it is clear that $L' \not\subset F$. Let $w \notin L'$, $w \notin F$, and let G be the cone which projects Z from w. Then it is easy to show that $x \in G$, $y \notin G$. This proves (b), while (c) is an immediate consequence of (a) and (b).

2.9 Remarks : (i) The results of this section are in part contained in [A-K] . However our methods of proofs seem much simpler.

(ii) Corollary 2.5 and proposition 2.6 are false if dim $Z > 1$. Indeed if $Z \subset P^4$ is a smooth surface which is not a complete intersection, then every hypersurface which contain Z must be singular, for otherwise one could apply the Lefschetz - Grothendieck theorem to show that $Z = F \cap F_1$, where F_1 is another hypersurface. Note however that there are hypersurfaces containing Z and smooth outside Z (by 1.9).

3. Points of high embedding dimension and quasi ordinary singularities.

In this section we study the behaviour of the generic element of a linear system of hypersurfaces containing a reduced curve $Z \subset P^n$, at the points of embedding dimension n, specially when these points are "quasi ordinary" singularities (in a sense we are going to introduce).

3.1 Definition : Let $F \subset P^n$ be a hypersurface. A point $x \in F$ is said to be ordinary if the projectivized tangent cone Proj(gr $(O_{F,x})$) of F at x is nonsingular.

If x is ordinary we have clearly:
(i) if x is singular, then the singularity of F at x can be resolved by the blowing up of F centered at x;
(ii) if $n \geq 3$, then the projectivized tangent cone at x is irreducible.

3.2 Definition : Let $X \subset P^n$ be a closed subvariety and let $x \in X$. We say that x is quasi ordinary (q.o) if there is a hypersurface F containing X such that x is ordinary for F.

If $x \in X$ is q.o. we put :

$\rho(x) = \min\{e_x(F)/F$ hypersurface with x ordinary, $Y \subset F\}$, where $e_x(V)$ denotes the multiplicity of the variety V at its point x.

Example : if X is a hypersurface, then x is q.o. if and only if x is ordinary and, if this is the case, $\rho(x) = e_x(X)$.

Observe that the notion of q.o. depends on the embedding $X \hookrightarrow P^n$. We begin with some more or less obvious remarks.

3.3 Lemma : Let $x \in X \subset P^n$ and put : $R = \underline{gr}(O_{P^n,x})$, $R' = \underline{gr}(O_{X,x}) = R/J$.

The the following are equivalent :

(i) x is q.o. and $\rho(x) = \rho$;

(ii) there is a form $\varphi \in J$, of degree ρ, such that $\mathrm{Proj}(R/\varphi R)$ is smooth.

Proof : Easy from 2.3.

3.4 Remark : Condition (ii) of 3.3 can be interpreted in the following way : the projectivized tangent cone Proj(R'), naturally embedded in $P^{n-1} = \mathrm{Proj}(R)$ (with its irrelevant component, if any) is a subscheme of a smooth hypersurface. Equivalently : the tangent cone Spec(R') is a subcone of a cone contained in $k^n = \mathrm{Spec}(R)$ and smooth outside the vertex.

3.5 Corollary : The point $x \in X$ is q.o. with $\rho(x) = 1$ if and only if emdim $O_{X,x} < n$.

Proof : Apply 3.3 and 2.4.

Now we want to give a sufficient condition for a curve singularity to be q.o.; for this we need a lemma.

3.6 Lemma : Let $n \geq 2$ and let $I \subset P^n$ be a finite set of cardinality r, not contained in any hyperplane. Let $S^{(t)}$ be the linear system consisting of the hypersurfaces of degree t which contain I. Then if $t \geq n-r+2$ the generic element of $S^{(t)}$ is smooth.

Proof : By 2.2 and 1.9 it is enough to show that if $u = r-n+1$ then

(i) the base locus of $S^{(u)}$ is I ;

(ii) for every $P \in I$ there is $D \in S^{(u)}$ which is smooth at P.

Let then $P \notin I$. Since I spans P^n there are $P_1,\ldots,P_n \in I$ which are independent and such that the hyperplane H_o they span does not contain P. Let $P_{n+1},\ldots,P_r$ be the remaining points of I and let H_i be a hyperplane containing P_{n+i} and not containing P. Then $H_o \cup \ldots \cup H_{n-r}$ is a hypersurface of degree u, which contains I but does not contain P. This proves (i).

The proof of (ii) is quite similar.

3.7 Proposition : Let $Z \subset P^n$ be a reduced curve and assume $n \geq 3$. Let $x \in Z$ be a point of embedding dimension n and assume that $gr(O_{Z,x})$ be reduced. Then x is q.o. and $\rho(x) \leq e_x(Z)-n+2$.

Proof : Let $I = Proj(gr(O_{Z,x})) \subset P^{n-1} = Proj(gr(O_{P^n,x}))$. Then I is finite and also a reduced subscheme of P^{n-1} containing $r = e_x(Z)$ points, which span P^{n-1}. Since $gr(O_{Z,x})$ is reduced it is easy to see that the conclusion follows from 3.6 and 3.3.

3.8 Corollary : Let $Z \subset P^n$, $n \geq 3$, be a reduced curve and let x be a seminormal point of Z of embedding dimension n. Then x is q.o. and $\rho(x)=2$.

Proof : Recall that x is seminormal if and only if $gr(O_{Z,x})$ is reduced and $e_x(Z) = emdim(O_{X,x})$ (see [B] or [D]). Then the conclusion follows from 3.7 and 3.5.

3.9 Lemma : Let S be a linear system in P^n and assume that S contains a smooth element. Then the generic element of S is smooth.

Proof : Let $S^{(t)}$ be the linear system of the hypersurfaces of degree t and identify $S^{(t)}$ with a projective space P^r (see section O). Then using remark 1.13, (ix), we see that the set $\{\lambda \varepsilon P^r / D_\lambda$ is smooth$\}$ is open and dense in P^r. The conclusion follows easily since S is a linear subsystem of some $S^{(t)}$ and corresponds to a linear subspace of P^r.

3.10 Corollary : Let S be a linear system in P^n and let x be a base point for S. Assume that x is ordinary with multiplicity e for some element of S and that $e_x(D) \geq e$ for all $D \varepsilon S$. Then x is ordinary with multiplicity e for the generic element of S.

Proof : The generic element of S has multiplicity e at x and hence there is a basis $D_0,\dots,D_r$ of S such that $e_x(D_i) = e$ for all i. Let φ_i be the leading form in $gr(O_{P^n,x})$ of the local equation of D_i at x (see 0.7).

Then (with the conventions of 0.5) if φ_λ is the leading form of the local equation of D_λ, where $\lambda = (\lambda_0,\dots,\lambda_m)$, we have that for λ generic the degree of φ_λ is e and, if this is the case, $\varphi_\lambda = \Sigma\lambda_i\varphi_i$. The conclusion follows from 3.9.

3.11 Proposition : Let $n \geq 3$ and let $Z \subset P^n$ be a reduced curve. Let V be the set of q.o. points of Z and let $I = \{x_1,\dots,x_s\}$ be the set of points of V with embedding dimension n. Let $e_i = \rho(x_i)$ and let $S_e^{(t)}$ be the linear system of the hypersurfaces of degree t which contain Z and which have multiplicity at least e_i at x_i. Then if $t \gg 0$ and $F \in S_e^{(t)}$ is generic we have:

(a) $\mathrm{Sing}(F) \subset (Z-V) \cup I$;

(b) x_i is ordinary for F_i with multiplicity e_i.

Proof : It follows from 3.9 and 2.5.

3.12 Corollary : Let $n \geq 3$ and let $Z \subset P^n$ be a reduced curve and let I be the set of points of Z which have embedding dimension n. Assume that $gr(O_{Z,x})$ be reduced for all $x \in I$. Then the generic hypersurface F of degree $t \gg 0$ containing Z is singular only at the points of I. Moreover each $x \in I$ is ordinary for F and $e_x(F) \leq e_x(Z)-$ $-n+2$.

Proof : Apply 3.10 and 3.7.

We do not know much about q.o. singularities in higher dimension. We can prove only the following.

3.13 Proposition : Let $x \in X \subset P^n$ and assume that $n \geq 2+\dim O_{X,x}$. Consider the following conditions :

(i) x is q.o. ;

(ii) if $Z = \mathrm{Proj}(O_{X,x})$ and $z \in Z$, then $\mathrm{emdim}(O_{Z,z}) \leq n-2$.

Then (i) $\rightarrow$ (ii). If moreover $gr(O_{X,x})$ is reduced and $\dim O_{X,x} = 2$, the converse is also true.

Proof : By 3.3 and 3.4 we have easily (i) → (ii). Conversely if $\dim O_{X,x} = 2$ we have that Z is a reduced curve in $P^{n-1} = \mathrm{Proj}(gr(O_{P^n,x}))$. Hence if (ii) holds Z is contained in a smooth hypersurface by 2.5. Since $gr(O_{X,x})$ is reduced, it is easy to deduce (i), using 3.3.

3.14 Remarks : (i) Consider the curve C in P^3 given by the ideal $I = (x_o^2x_3-x_1^3, x_1^2x_3-x_2^3)$ and let $x = (0,0,0,1)$. Then x is singular for C and moreover the leading ideal of I_x in $gr(O_{P^n,x}) = k[T_0, T_1, T_2]$ is $\mathfrak{a} = (T_0^2, T_1^2)$ (e.g. apply [V-V] , prop. 1.2). Now it is clear that if $\varphi \in \mathfrak{a}$ is any form, then $\mathrm{Proj}(k[T_0,T_1,T_2]/(\varphi))$ must be singular at (0,0,1). Hence x is not a q.o. singularity for C.

(ii) In 3.6 we have shown that any set I of r points in P^n which spans P^n is contained in a smooth hypersurface of degree $t=r-n+2$. Given n and r, the above number is clearly not the minimum one that has the property: we do not know which is the minimum one. Note its relation with the character $\rho(x)$ of a q.o. singularity (see 3.7).

(iii) The implication (ii) → (i) of 3.13 is false if $\dim O_{X,x} > 2$. Indeed let $Z \subset P^n$ be a smooth irreducible surface which is not a complete intersection and let $V \subset k^5$ be the corresponding affine cone. Embed k^5 in P^5 and let X be the closure of V in P^5. Let $x \in X$ be the vertex of the cone V. Then $\dim O_{X,x} = 3$, $gr(O_{X,x})$ = ring of coordinates of $V \subset k^5$ and hence $\mathrm{Proj}(gr(O_{X,x})) = Z$. So the given embedding of Z in P^4 coincides with the embedding $Z \subset \mathrm{Proj}(gr(O_{P^5,x})) = P^4$. Hence Z verifies (ii) of 3.13 and moreover $gr(O_{X,x})$ is reduced. However Z is not contained in any smooth hypersurface (see 2.8 (ii)); hence x is not q.o.

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