数理解析研究所講究録
第 400 巻 1980 年 42-50

Blowing-up characterization for local rings

by

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Let A be a Noetherian local ring with maximal ideal m . Then

Definition. A is called Buchsbaum if the difference

$$I(A) = l_A(A/q) - e_A(q)$$

is an invariant of A not depending on the particular choice of a parameter ideal q of A , where $l_A(A/q)$ and $e_A(q)$ denote the length of A/q and the multiplicity of A relative to q respectively.

Clearly every Cohen-Macaulay local ring A is Buchsbaum with I(A) = 0 . The converse is also true. In this sense the concept of Buchsbaum local rings is a generalization of that of Cohen-Macaulay local rings and the theory of Buchsbaum singularities has its root in an answer of W. Vogel to the following problem of D. A. Buchsbaum.

Problem (1965). Does the equality

$$l_A(A/q) - e_A(q) = \dim A - \operatorname{depth} A$$

hold for every parameter ideal q of A ?

Of course this is not true in general. W. Vogel first mentioned it in 1973 and started his research on local rings which satisfy the above condition. Buchsbaum rings were named by W. Vogel after this problem.

Example. (1) Let $A = k[[s^4, s^3t, st^3, t^4]] \subset B = k[[s, t]]$, where B is a formal power series ring over a field k . Then, as F. S. Macaulay has already mentioned in his famous book, A is not a Cohen-Macaulay ring. But A is Buchsbaum and, for every parameter ideal q of A , the following equality

$$l_A(A/q) - e_A(q) = \dim A - \operatorname{depth} A = 1$$

holds. The proof is very easy, so I will give it. Let q = (a, b) be a parameter ideal. Then, applying the functor $A/q \otimes_A \cdot$ to the exact sequence

$$0 \to A \to B \to B/A \to 0 ,$$

we have an exact sequence

$$0 \to \mathrm{Tor}_1^A(k, A/q) \to A/q \to B/qB \to k \to 0$$

because $B/A \cong k$. Thus

$$l_A(A/q) = l_A(B/qB) + 1 ,$$

as $\mathrm{Tor}_1^A(k, A/q)$ is of dimension 2 as a vector space over k . On the other hand,

$$e_B(q) = l_A(B/qB)$$

and

$$e_B(q) = e_A(q)$$

since B is a Cohen-Macaulay A-module and since $\dim_A B/A = 0 < 2$. Thus we have the assertion

$$l_A(A/q) = e_A(q) + 1 .$$

Of course $\dim A = 2$ and $\operatorname{depth} A = 1$.

(2) Similarly we can prove that

$$A = k[[x, y, z, w]]/(x, y) \cap (z, w)$$

is a Buchsbaum local ring with $I(A) = 1$, where $k[[x, y, z, w]]$ is a formal power series ring over a field k .

The following well-known diagram shows the classification of local rings from the view-point of homological algebra:

regular ==> hypersurfaces ==> C-I ==> Gorenstein ==> C-M ==> [ ].

I believe that Buchsbaum local rings will fill this blank. The main purpose of my talk is to give a characterization of Buchsbaum rings in terms of blowing-up.

Let $q = (a_1, \ldots, a_d)$ be a parameter ideal of A and put

$$R(q) = \bigoplus_{n \geqq 0} q^n .$$

The ring $R(q)$ is called the Rees algebra of q, The canonical morphism $\text{Proj } R(q) \to \text{Spec } A$ is the blowing-up of $\text{Spec } A$ with center $\text{Spec } A/q$. It is well-known that

$$\text{Proj } R(q) = \bigcup_{i=1}^{d} \text{Spec } A[\, x/a_i \;/\; x \in q \,] .$$

Let $H_m^i(A)$ denote the i-th local cohomology module of A relative to m, i.e.,

$$H_m^i(A) = \varinjlim_r \text{Ext}_A^i(A/m^r, A) .$$

With this notation the main result of my talk is stated as follows.

Theorem. $A/H_m^0(A)$ *is a Buchsbaum local ring if and only if* $\text{Proj } R(q)$ *is a Cohen-Macaulay scheme for every parameter ideal* q *of* A.

As a consequence we have

Corollary. $A/H_m^0(A)$ *is a Gorenstein local ring if and only if* $\text{Proj } R(q)$ *is Gorenstein for every parameter ideal* q *of* A.

Moreover

Corollary. *Suppose that* A *is a homomorphic image of a regular local ring. Then* $A/H_m^0(A)$ *is a C-I if and only if* $\text{Proj } R(q)$ *is locally complete intersection for every parameter ideal* q *of* A.

About the regularity we may add

Proposition. $A/H_m^0(A)$ *is a regular local ring if and only if* $\text{Proj } R(q)$ *is smooth for some parameter ideal* q *of* A. *In this case, if* $\dim A \geqq 2$, *the equality*

$$m = q + H_m^0(A)$$

must hold.

Later I will give a sketch of proof of the theorem and so, for a while, let me continue to express known results on Buchsbaum rings.

The theory of Buchsbaum rings is now developing rapidly and many important facts have been discovered. As well as their definition the properties of Buchsbaum rings are very similar to those of Cohen-Macaulay rings.

Proposition (1973, Stückrad-Vogel). *The following conditions are equivalent.*

(1) A *is a Buchsbaum local ring.*

(2) *Every system of parameters is a weak A-sequence, i.e., the equailty*

$$(a_1, \ldots, a_k) : a_{k+1} = (a_1, \ldots, a_k) : m$$

holds for every system $a_1, a_2, \ldots, a_d$ *of parameters for* A *and for every integer* $0 \leqq k < d = \dim A$.

This immediately yields the following

Corollary. Suppose that A *is a Buchsbaum ring. Then*

(1) A_p *is a Cohen-Macaulay local ring for every* $p \in \operatorname{Spec} A \setminus \{m\}$.

(2) $\dim A_p + \dim A/p = d$ *for every* $p \in \operatorname{Spec} A$.

Moreover the local cohomology modules $H_m^i(A)$ ($i \neq d$) are vector spaces over A/m . Namely

Proposition (1975, Renschuch-Stückrad-Vogel). *If* A *is a Buchsbaum local ring, then*

(1) $m \cdot H_m^i(A) = (0)$ *for every* $i \neq d$.

(2) $I(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \cdot l_A(H_m^i(A))$.

It is known that there are a lot of Buchsbaum rings.

Proposition (1979, Goto). *Let* $d > 0$ *and* $h_0, h_1, \ldots, h_{d-1} \geqq 0$ *be integers. Then there exists a Buchsbaum local ring* A *such that* $\dim A = d$ *and* $l_A(H_m^i(A)) = h_i$ ($0 \leqq i < d$) . *Moreover, if* $h_0 = 0$ (*resp.* $h_0 = h_1 = 0$) , *then the ring* A *may also be taken to be an integral domain* (*resp. a normal domain*) .

There is a cohomological criterion.

Definition. Let M *be a finitely generated A-module. Then* M *is called Buchsbaum if the difference*

$$I(M) = l_A(M/qM) - e_M(q)$$

is an invariant of M not depending on the particular choice of a parameter ideal q of M.

Thus A is a Buchsbaum ring if and only if A is a Buchsbaum module over itself. With this definition

Lemma (1979, Stückrad-Vogel). Let M be a finitely generated A-module.

(1) Suppose that the canonical homomorphisms

$$\mathrm{Ext}^i_A(A/m, M) \longrightarrow H^i_m(M)$$

are surjective for all $i \neq \dim_A M$. Then M is a Buchsbaum A-module.

(2) If A is a regular local ring, the converse is also true.

This is a very useful criterion and my existence theorem is one of its applications. The next is an easy consequence of this lemma.

Corollary. Suppose that $t = \operatorname{depth} A < d = \dim A$ and that $H^i_m(A) = (0)$ for every $i \neq t, d$. Then A is a Buchsbaum ring if and only if $m.H^t_m(A) = (0)$.

Now I hope that I have succeeded to persuade you about the importance of my theorem. From this point of view it seems to be natural to ask if the Rees algebra $R(q)$ is a Cohen-Macaulay ring in case A is a Buchsbaum ring. But this is not true in general. Namely

Proposition (1979, Goto-Shimoda). The following conditions are equivalent.

(1) A is a Buchsbaum local ring and $H^i_m(A) = (0)$ for $i \neq 1, d$.

(2) The Rees algebra $R(q) = \bigoplus_{n \geq 0} q^n$ is a Cohen-Macaulay ring for every parameter ideal q of A.

Thus we have

Corollary. Suppose that $\operatorname{depth} A \neq 1$. Then A is a Cohen-Macaulay local ring if and only if $R(q)$ is a Cohen-Macaulay ring for every parameter ideal q of A.

Corollary (1978, Shimoda). Suppose that A is an integral domain of $\dim A = 2$. Then A is a Buchsbaum local ring if and only if $R(q)$ is a Cohen-Macaulay ring for every parameter ideal q of A.

Shimoda's research was inspired by the following example due to Hochster and Roberts. Let $A = k[[s^4, s^3t, st^3, t^4]]$ and $q = (s^4, t^4)$. Then they showed that $R(q)$ is a Cohen-Macaulay ring and mentioned by this example that a ring retract of a Cohen-Macaulay ring is not necessarily again a Cohen-Macaulay ring. Our joint work gives a generalization of this remark and my main theorem about blowing-up characterizations is closely tied with this research. Notice that a lot of examples of such local rings that the Rees algebras of parameter ideals are always Cohen-Macaulay are given by my existence theorem or by gluing in the sense of Traverso.

Sketch of Proof.

Let $q = (a_1, a_2, \ldots, a_d)$ be a parameter ideal of A and let M denote the unique graded maximal ideal of $R = R(q)$. Recall that

Proposition. The following conditions are equivalent.

(1) $\operatorname{Proj} R$ is Cohen-Macaulay.

(2) R_P is a Cohen-Macaulay local ring for every $P \in \operatorname{Spec} R \setminus \{M\}$.

(3) $l_R(H^i_M(R))$ is finite for every $i \neq d+1$.

In this case $l_A(H^i_m(A))$ is also finite for every $i \neq d$.

This suggests that the general theory of local rings with finite local cohomology modules may be very important for our purpose. In fact the following fact is essential.

Lemma. The length $l_A(H^i_m(A))$ is finite for every $i \neq d$ if and only if there exists an ideal I of A containing some power of m and such that the equality

$$(b_1, \dots, b_k) : b_{k+1} = (b_1, \dots, b_k) : I$$

holds for every system $b_1, b_2, \dots, b_d$ *of parameters for* A *contained in* I *and for every integer* $0 \leqq k < d$.

Moreover we have

Proposition. Suppose that $l_A(H^i_m(A))$ *is finite for every* $i \neq d$ *and let* I *be as above. If* $a_1, a_2, \dots, a_d \in I$, *then* $a_1, a_2/a_1, \dots, a_d/a_1$ *is a* B*-regular sequence where*

$$B = A[a_2/a_1, a_3/a_1, \dots, a_d/a_1] .$$

A half of the theorem follows from this lemma.

Corollary. Suppose that A *is a Buchsbaum local ring. Then* Proj R *is Cohen-Macaulay.*

Proof. We may choose $I = m$. Moreover we may assume that the field A/m is algebraically closed. Then every maximal ideal of the ring B contains a regular sequence of the form $a_1, a_2/a_1, \dots, a_d/a_1$. Thus B is a Cohen-Macaulay ring. Recall that

$$\text{Proj } R = \bigcup_{i=1}^{d} \text{Spec } A[x/a_i \;/\; x \in q] .$$

Now consider the opposite implication. Let $M = (m, a_2/a_1, \dots, a_d/a_1)B$ and $Q = (a_1, a_2/a_1, \dots, a_d/a_1)B$. The following is the key lemma.

Lemma. Suppose that $l_A(H^i_m(A)$ *is finite for every* $i \neq d$. *Then*

(1) $e_{B_M}(QB_M) = e_A(q)$.

(2) *Assume that* $d \geqq 2$. *Then* B_M *is a Cohen-Macaulay local ring if and only if* $U(a_2, \dots, a_d) \subset (a_2/a_1, \dots, a_d/a_1)B \cap A$, *where*
$U(a_2, \dots, a_d) = \bigcup_{r>0} (a_2, \dots, a_d) : m^r$.

The second assertion follows from the first and the proof is not very easy. But this allows us to reduce the problem to the case $\dim A = 2$. In this situation the problem is pretty easy.

Using the same technique we can prove the following result and it may have its own interest. In fact our main theorem is a consequence of it.

Theorem. The following conditions are equivalent.

(1) $l_A(H_m^i(A))$ is finite for every $i \neq d$.

(2) There is an ideal I of A containg some power of m and such that the equality

$$(a_1, \dots a_k) : a_{k+1} = (a_1, \dots , a_k) : m$$

holds for every system $a_1, a_2, \dots , a_d$ of parameters for A contained in I and for every integer $0 \leqq k < d$.

(3) Proj R(q) is Cohen-Macaulay for some parameter ideal q of A .

In this case, for every parameter ideal q of A contained in I , Proj R(q) is always Cohen-Macaulauy. Moreover, if depth $A > 0$, then the Rees algebra R(a) is a Cohen-Macaulay ring for some m-primary ideal a of A .

It is easy to show that the associated graded ring $G = G_q^{\cdot}(A)$ has also finite local cohomology modules if so does $R = R(q)$. Moreover I can prove that G_M is a Buchsbaum local ring if so is A and if $\dim A \leqq 3$. Thus I have the following

Conjecture. Suppose that A is a Buchsbaum local ring and let q be a parameter ideal of A . Let $G = G_q^{\cdot}(A)$ and M the unique graded maximal ideal of G . Then

$$[H_M^i(G)]_n = \begin{cases} H_m^i(A) & (n = -i) \\ (0) & (n \neq -i) \end{cases}$$

for every integer n and $0 \leqq i < d$. Moreover G_M is a Buchsbaum local ring with $I(G_M) = I(A)$.

I will finish my talk with the following

Question. *Is* R(q) *a Buchsbaum ring if so is* A ?

This is quite open.

Added in proof. The author succeeded in proving the above conjecture in June of 1980.