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About the conditions for the Rees ring to be C-M or Gorenstein.

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Let A be a Noetherian local ring of dimension d and $\underline{m}$ the maximal ideal of A. For an ideal $\underline{q}$ of A we put

$$R(\underline{q}) = \bigoplus_{n \geqq 0} \underline{q}^n$$

and call it the Rees algebra of A relative to $\underline{q}$. We identified with the subring $A[\{aT ; a \in \underline{q}\}]$ of the polynomial ring A[T].

Now we consider the following question:

Question. Is a given property of the ring A inherited by the Rees algebra of A?

For example, Noetherian, integral domain, integrally closed, regular local ring, C-M ring , Gorenstein ring and etc.

Here we will examine the property of being C-M or Gorenstein. Concening this property, the following results are well known.

J. Barshay ([1]) (1973).

① If A is C-M, $a_1,\ldots,a_s$ an A-sequence and $\underline{q} = (a_1,\ldots,a_s)$. Then $R(\underline{q})$ is C-M.

② If A is C-M, a_1,a_2 an A-sequence and $\underline{q} = (a_1,a_2)^n$, where n is a positive integer. Then $R(\underline{q})$ is C-M.

Moreover, J. Barshay presented in his paper the following.

Let A be a C-M ring, $a_1,\ldots,a_d$ a system of parameters, n a positive integer and $\underline{q} = (a_1,\ldots,a_d)^n$. Then is $R(\underline{q})$ a C-M ring?

G. Valla ([9]) (1976) has solved the above probrem and S. Goto ([2]) (1978) has given a shorter proof for this one.

Remark. We note that the converse of Barshay's probrem is not true in general. In fact, Hochster-Roberts ([5]) have given an example:

Let $A = k[x^2,x^3,y,xy]$ and $\underline{q} = (x^2,y)$. Then $R(\underline{q})$ is a Gorenstein ring, but A is not a C-M ring. (A is a Buchsbaum ring).

Thus the following probrem seems to be interesting.

Probrem. Find the conditions for $R(\underline{q})$ to be C-M or Gorenstein.

At first, as a partial answer

Proposition. If A ia a local integral domain of dim 2, a, b a system of parameters for A and $\underline{q} = (a,b)$. Then

(i) $R(\underline{q})$ is C-M if and only if $(a):b \cap (b):a \subseteq (a) \cap (b)$. In particular, $R(\underline{q})$ is C-M for every parameter ideal $\underline{q}$ if and only if A is Buchsbaum.

(ii) The following statements are equivalent.

1. A is Gorenstein.
2. $R(\underline{q})$ is Gorenstein for some parameter ideal $\underline{q}$.
3. $R(\underline{q})$ is Gorenstein for every parameter ideal $\underline{q}$.

For a general case, there is the following result.

S. Goto and Y. Shimoda ([3]).

Proposition. For a Noetherian local ring A, the following two conditions are equivarent.

1. The Rees algebra $R(\underline{q})$ is C-m for every parameter ideal $\underline{q}$
2. A is a Buchsbaum ring and the local cohomology module $H^i_{\underline{m}}(A) = (0)$ for $i \neq 1,d$.

Thus we have

Corollary. Let A be a Noetherian local ring and assume that depth $A \neq 1$. Then A is C-M if and only if so is $R(\underline{q})$ for every parameter ideal $\underline{q}$.

By the above result, we have done to find the conditions that $R(\underline{q})$ is a C-M ring for every parameter ideals $\underline{q}$. Now let us find the conditions that $R(\underline{q})$ is a C-M ring for a parameter ideal $\underline{q}$. A complete answer is

Theorem 1 (Goto-Shimoda). Let $0 = s = d$ be an integer, $a_{s+1},\ldots,a_d$ a subsystem of parameters for A and $\underline{q} = (a_{s+1},\ldots,a_d)$. Then the following conditions are equivalent.

(1) $R(\underline{q})$ is a C-M ring

(2) the following four properties are satisfied.

(i) $A/\underline{q}^n$ is a C-M ring for every integer $n > 0$.

(ii) $H^i_{\underline{m}}(A) = (0)$ for $i \neq s+1,d$

(iii) $\underline{q}H^{s+1}_{\underline{m}}(A) = (0)$ if $s+1 < d$

(iv) $\mathrm{Hom}_A(H^{s+1}_{\underline{m}}(A), E(A/\underline{m}))$ is an C-M A-module of dimension s if $s+1 < d$ and $H^{s+1}_{\underline{m}}(A) \neq (0)$.

By the above theorem,we immediately have the followings:

Corollary 1. ([8]) Under the same situation of Theorem 1, A is a C-M ring if and only if so is $R(\underline{q})$ for every ideal $\underline{q}$ generated by a subsystem of parameters of length s.

Corollary 2. We put $r = d - s$ and denote by $r(A)$ the type of a C-M ring A. Then we have

$$r(R(\underline{q}) = \begin{cases} (r-1)\cdot\mu_A(K_A) + \mu_A(\mathrm{Hom}\ (H^{s+1}_{\underline{m}}(A),\ E(A/\underline{m}))) & (r \geqq 2) \\ r(A) & (r = 1) \end{cases}$$

Therefore, $R(\underline{q})$ is a Gorenstein ring if and only if so is A and $r \leqq 2$.

Now we will consider the case of the Rees algebra relative to maximal ideal $\underline{m}$, that is, $R(\underline{m})$. At first we denote

$$G(\underline{m}) = \bigoplus_{n \geq 0} \underline{m}^n/\underline{m}^{n+1}$$

and call it the associated graded ring. The canonical map $f : \mathrm{Proj}\, R(\underline{m}) \longrightarrow \mathrm{Spec}(A)$ is the blowing up of Spec(A) with center $\mathrm{Spec}(A/\underline{m})$ and the fibre of f at the closed point is isomorphic to $\mathrm{Proj}(G(\underline{m}))$. Hence the properties of $\mathrm{Proj}(G(\underline{m}))$ determine some of properties of $\mathrm{Proj}(R(\underline{m}))$. For example, concerning the property of being C-M or Gorenstein we have

Hochster-Ratliff,Jr. (1973)

Suppose that A is C-M (resp. Gorenstein). If $G(\underline{m})$ is C-M (resp. Gorenstein), then so is $\mathrm{Proj}(R(\underline{m}))$.

Thus, to show that $\mathrm{Proj}(R(\underline{m}))$ is C-M (resp. Gorenstein) it suffices to show that $G(\underline{m})$ itself is C-M (resp. Gorenstein). Now about the question whether $G(\underline{m})$ is C-M or Gorenstein the following resuls are well known.

J. Sally ([6]) (1978)

(i) If A is a C-M ring of dimension d and embedding dimension $v(A) = e(A) + d - 1$, then $G(\underline{m})$ is a C-M ring.

(ii) If A is a Gorenstein ring of dimension d and $v(A) = e(A) + d - 2$, then $G(\underline{m})$ is a Gorenstein ring.

Thus we have partially done to solve the question whether Proj $(R(\underline{m}))$ is C-M or Gorenstein in the above sence. But the question whether $R(\underline{m})$ itself is C-M or Gorenstein still remains to be unsolved as far as I know. Here let us discuss about this probrem. A complete answer is

Theorem 2. Suppose that A is a C-M ring. Then the following conditions are equivalent.

(1) $R(\underline{m})$ is a C-M ring

(2) $G(\underline{m})$ is a C-M ring and $a(G(\underline{m})) < 0$.

Theorem 3. Suppose that A is a C-M ring with $\dim A \geq 2$. Then the following two conditions are equivalent.

(1) $R(\underline{m})$ is a Gorenstein ring

(2) $G(\underline{m})$ is a Gorenstein ring and $a(G(\underline{m})) = -2$.

Here $a(G(\underline{m}))$ denotes the invariant of $G(\underline{m})$ as follows: Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian graded ring with R_0 a field and M the irrevant maximal ideal of R. We put dim R = r. we define

$$a(R) = \max \{ k \in \mathbb{Z} \ : \ [H^r_{\underline{M}}(R)]_k \neq (0) \}.$$

Notice that if R is a C-M ring and if $h_1, \ldots, h_s$ is a homogeneous R-regular sequence, then

$$a(R) = a(R/(h_1, \ldots, h_s)) - \sum_{i=1}^{s} \deg h_i.$$

In the above theorems, as we can replace $A[U]_{\underline{m}A[U]}$ instead of A, we may assume that $A/\underline{m}$ is an infinite residue field. Then it is well known that there exist $a_1, \ldots, a_d$ in $\underline{m}$ such that $\underline{m}^{t+1} = (a_1, \ldots, a_d)\underline{m}^t$ for some $t \geq 0$. Therefore we may restate the condition (2) of Theorem 2 (resp. Theorem 3) as follows:

(2)' $G(\underline{m})$ is a C-M (resp. Gorenstein) ring and $\underline{m}^d = (a_1, \ldots, a_d)\underline{m}^{d-1}$ (resp. $\underline{m}^{d-1} = (a_1, \ldots, a_d)\underline{m}^{d-2}$ and $(a_1, \ldots, a_d) \not\supset \underline{m}^{d-2}$).

In case dim A = 2, we have

Proposition. Suppose that dim A = 2. Then $R(\underline{m})$ is C-M (resp. Gorenstein) if and only if A is C-M and $v(A) = e(A) + 1$ (resp. A is regular).

Now we will give some examples satisfying the above theorems.

1. If A is a rational surfase singularity of dim A = 2, then A satisfies (2) of the above proposition and hence $R(\underline{m})$ is C-M.

2. Suppose that A is C-M and dim A = 3. Then $R(\underline{m})$ is a Gorenstein ring if and only if A is abstract hypersuface (i.e., A is hypersurface) and e(A) = 2.

3. Let $d > 0$ be an integer. We put

$$A = k[[(X_1,\dots,X_d)^d]].$$

Then we have $G(\underline{m}) = k[(X_1,\dots,X_d)^d]$ and $\underline{m}^d = (X_1^d,\dots,X_d^d)\underline{m}^{d-1}$.

Thus $G(\underline{m})$ is a Gorenstein ring and $a(G(\underline{m})) = -1$. Hence $R(\underline{m})$ is a C-M ring.

4. Let n,r be integers with $r + 2 = n$ and $P = k[[X_1,\dots,X_n]]$. Let $f_1,\dots,f_r$ be a homogeneous P-regular sequence. We put $A = P/(f_1,\dots,f_r)$. Then $G(\underline{m}) = k[X_1,\dots,X_n]/(f_1,\dots,f_r)$ is a Gorenstein ring and

$$a(G(\underline{m})) = a(k[X_1,\dots,X_n]) + \sum_{i=1}^{r} \deg f_i$$

$$= -n + \sum_{i=1}^{r} \deg f_i$$

Thus $R(\underline{m})$ is C-M (resp. Gorenstein) if and only if

$\sum_{i=1}^{r} \deg f_i < n$ (resp. $\sum_{i=1}^{r} \deg f_i = n - 2$).

5. Let $e \geq 3$ and $d > 0$ be integers. Put $P = k[[X, Y_1,\dots,Y_{d-1}]]$ and $A = k[[X^e, X^{e+1},\dots,X^{2e-2}, Y_1,\dots,Y_{d-1}]]$. Then A is Gorenstein with dim A = d, e(A) = e and v(A) = e + d - 2. By virtue of the result in J. Sally's paper, we can see that $G(\underline{m})$ is Gorenstein and $\underline{m}^3 = \underline{q}\underline{m}^2$ ($\underline{q} \not\supset \underline{m}^2$). Hence if $d \geq 3$, $R(\underline{m})$ is C-M and $R(\underline{m})$ is Gorenstein if and only if d = 4.

Remark. Concerning the properties of being complete intersection or regular we have

(i) $R(\underline{m})$ is complete intersection if and only if A is regular and dim A $\leqq$ 2.

(ii) $R(\underline{m})$ is regular if and only if A is D.V.R.

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