

Notes on effective usage of double exponential formulas  
for numerical integration.

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## 1. Introduction

We shall describe the results of numerical evaluation  
of an integral

$$(1-1) \quad I = \int_a^b f(x) dx$$

using the double exponential formulas (DE formulas), and  
empirical evidences derived from these results.

The DE formulas have been developed as a class of  
numerical quadrature formulas by Takahashi and Mori [1] ,  
and they offer one of the most powerful methods to evaluate  
an improper integral on  $[a,b]$  , in which the integrand is  
singular at one endpoint or both. Moreover, we shall show  
a method to evaluate difficult integrals, where the integrand  
is highly oscillating.

## 2. Comparison of results by some numerical integration formulas.

As a numerical comparison of formulas among DE, IMT, CADRE  
and QUAD we show in Table 2-1 the number of integrand evaluations  
and the relative error estimated from the following convergence  
criterion (2-1) by choosing 15 test cases of Patterson [3] and  
Ichida and Kiyono [4].

A convergence criterion of our program is satisfied when

$$(2-1) \quad |S_{i+1} - S_i| \leq \varepsilon |S_{i+1}|$$

where  $S_i$  is a value of DE formula with mesh size  $h_i$  , and  $S_{i+1}$   
is a value of DE formula with mesh size  $h_{i+1} = h_i / 2$  , and  $\varepsilon$   
is some preassigned tolerance (e.g.,  $10^{-9}$ ).

## 2.1 DE formula and IMT formula

Table 2.1-1

DE	IMT
$I = \int_a^b f(x) dx$ $x = \phi(u)^*$	$I = \int_0^1 f(x) dx$ $x = \psi(u) = \frac{1}{Q} \int_0^u \exp\left(-\frac{1}{t} - \frac{1}{1-t}\right) dt$ $Q = \int_0^1 \exp\left(-\frac{1}{t} - \frac{1}{1-t}\right) dt$
$I = \int_{-\infty}^{\infty} f(\phi(u)) \phi'(u) du$ $\approx I_h = h \sum_{n=-\infty}^{\infty} f(\phi(nh)) \phi'(nh)$ <p>based on the fact that the trapezoidal rule gives the best result for <math>\int_{-\infty}^{\infty} g(u) du</math> with mesh size <math>h</math>.</p>	$I = \int_0^1 f(\psi(u)) \psi'(u) du$ $\approx S_n = \frac{1}{N} \sum_{n=1}^{N-1} f\left(\psi\left(\frac{n}{N}\right)\right) \psi'\left(\frac{n}{N}\right)$ <p>based on Euler-Maclaurin summation formula.</p>

\* where

$$(2.1-1) \quad x = \phi(u) = \tanh\left(\frac{\pi}{2} \sinh u\right) \quad \text{if} \quad I = \int_{-1}^1 f(x) dx$$

$$(2.1-2) \quad x = \phi(u) = \exp\left(\frac{\pi}{2} \sinh u\right) \quad \text{if} \quad I = \int_0^{\infty} f(x) dx$$

$$(2.1-3) \quad x = \phi(u) = \exp(u - \exp(-u)) \quad \text{if} \quad I = \int_0^{\infty} \exp(-x) f(x) dx$$

$$(2.1-4) \quad x = \phi(u) = \sinh\left(\frac{\pi}{2} \sinh u\right) \quad \text{if} \quad I = \int_{-\infty}^{\infty} f(x) dx$$

Table 2-1 Comparison of results by DE, IMT, CADRE and QUAD

	D E	IMT	CADRE	QUAD
P <sub>1</sub>	44 3.3 <sub>10</sub> -12	127 4.9 <sub>10</sub> -13	129 1.2 <sub>10</sub> -10*	361 2.2 <sub>10</sub> -10*
P <sub>2</sub>	96 1.7 <sub>10</sub> -12	127 2.4 <sub>10</sub> -12	33 1.3 <sub>10</sub> -13*	37 2.1 <sub>10</sub> -14*
P <sub>3</sub>	92 3.1 <sub>10</sub> -11	127 2.8 <sub>10</sub> -13	129 4.2 <sub>10</sub> -12*	73 1.5 <sub>10</sub> -12*
P <sub>4</sub>	40 1.0 <sub>10</sub> -11	127 8.1 <sub>10</sub> -13	529 2.7 <sub>10</sub> -10*	163 5.0 <sub>10</sub> -12*
P <sub>5</sub>	92 2.3 <sub>10</sub> -12	127 5.6 <sub>10</sub> -13	65 2.1 <sub>10</sub> -10*	73 0*
P <sub>6</sub>	724 1.2 <sub>10</sub> -10	509 2.2 <sub>10</sub> -12	785 8.4 <sub>10</sub> -12*	757 3.3 <sub>10</sub> -13*

$$P_1 \int_0^1 \sqrt{x} dx = \frac{2}{3}$$

$$P_2 \int_{-1}^1 (0.92 \cosh x - \cos x) dx = 1.84 \sinh 1 - 2 \sin 1$$

$$P_3 \int_{-1}^1 \frac{dx}{x^4 + x^2 + 0.9} \doteq 1.5822 \quad 32964$$

$$P_4 \int_0^1 x^{\frac{3}{4}} dx = \frac{2}{5}$$

$$P_5 \int_0^1 \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}} \left\{ \log(3 + 2\sqrt{2}) + \pi \right\}$$

$$P_6 \int_0^1 \frac{dx}{2 + \sin 10\pi x} = \frac{2}{\sqrt{3}}$$

P <sub>7</sub>	$\int_0^1 \frac{x}{e^x - 1} dx \doteq 0.77750 46341$	48	3.4 <sub>10</sub> -12	127	6.6 <sub>10</sub> -13	17	2.2 <sub>10</sub> -12*	37	2.1 <sub>10</sub> -14*
P <sub>8</sub>	$\int_0^1 \frac{\sin 100 \pi x}{\pi x} dx$	620	2.5 <sub>10</sub> -13	509	4.1 <sub>10</sub> -16	3505	3.2 <sub>10</sub> -12*	2773	5.6 <sub>10</sub> -13*
P <sub>9</sub>	$\int_0^{10} \frac{50}{\pi(2500x^2 + 1)} dx = \frac{1}{\pi} \tan^{-1} 500$	180	2.2 <sub>10</sub> -10	255	1.5 <sub>10</sub> -11	337	1.3 <sub>10</sub> -11*	343	3.3 <sub>10</sub> -12*
P <sub>10</sub>	$\int_0^\pi \frac{\cos(\cos x + 3 \sin x + 2 \cos 2x + 3 \sin 2x + 3 \cos 3x) dx}{}$	186	8.6 <sub>10</sub> -12	127	1.2 <sub>10</sub> -12	417	1.5 <sub>10</sub> -12*	343	1.0 <sub>10</sub> -12*
P <sub>11</sub>	$\int_0^1 \log x dx = -1$	44	3.9 <sub>10</sub> -13	127	2.0 <sub>10</sub> -11	369	4.5 <sub>10</sub> -9*	415	5.7 <sub>10</sub> -7*

The results with superscript \* are due to Ninomiya's paper.

$K_1 \int_{-1}^1 \frac{2^{-\alpha}}{4^{-\alpha} + x^2} dx$ $= 2 \tan^{-1} 2^\alpha$ $\alpha = 1, 2, \dots, 31$	$\alpha = 1$	202	$2.4_{10}^{-13}$	127	$2.3_{10}^{-13}$	49	$1.2_{10}^{-7}$ **	63	$1.6_{10}^{-8}$ **
	$K_1'$	103	$1.7_{10}^{-14}$	127	$7.0_{10}^{-13}$	1440	$3.4_{10}^{-8}$ **	567	$3.0_{10}^{-8}$ **
$\left( K_1' 2 \int_0^1 \frac{2^{-\alpha}}{4^{-\alpha} + x^2} dx \right)$	$\alpha = 8$	24450	$2.5_{10}^{-12}$	.....	.....	.....	.....	.....	.....
	$K_1'$	204	$2.0_{10}^{-12}$	255	$2.3_{10}^{-13}$	.....	.....	.....	.....
$K_2 \int_0^1 x^\alpha dx = \frac{1}{\alpha + 1}$ $\alpha = -.1, -.2,$ $\dots, -.9, -.99$	$\alpha = -.1$	53	$2.1_{10}^{-14}$	127	$4.5_{10}^{-12}$	2452	$1.0_{10}^{-7}$ **	357	$1.4_{10}^{-6}$ **
	$K_2'$	89	$7.3_{10}^{-10}$	509	$5.4_{10}^{-3}$	.....	.....	.....	.....
$\left( K_2' \int_0^\infty e^{-(1+\alpha)x} dx \right)$	$\alpha = -.8$	64	$7.3_{10}^{-16}$	509	$5.3_{10}^{-10}$	3043	$2.1_{10}^{-6}$ **	3297	$4.7_{10}^{-6}$ **
	$K_2'$	185	$1.4_{10}^{-13}$	.....	.....	.....	.....	.....	.....
$H_1 \int_0^1 \log  \log x  dx = -\gamma$	$\alpha = -.9$	2013	$6.2_{10}^{-10}$	509	$1.7_{10}^{-7}$	3532	$9.5_{10}^{-2}$ **	6699	$1.2_{10}^{-4}$ **
	$K_2'$	189	$4.8_{10}^{-12}$	.....	.....	.....	.....	.....	.....
	$\alpha = -.99$	.....	.....	.....	.....	.....	.....	.....	.....
	$K_2'$	394	$2.1_{10}^{-16}$	.....	.....	.....	.....	.....	.....
		48	$2.2_{10}^{-11}$	121	$3.1_{10}^{-11}$				

The results with superscript \*\* are due to Ichida and Kiyono [4],  $\epsilon = 10^{-7}$

P6

$$f(x) = \frac{2}{2 + \sin 10\pi x}$$

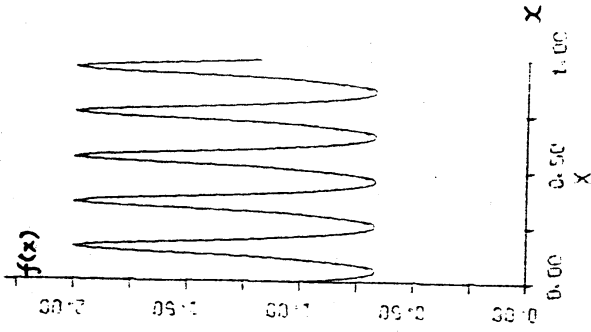


Figure 2-1

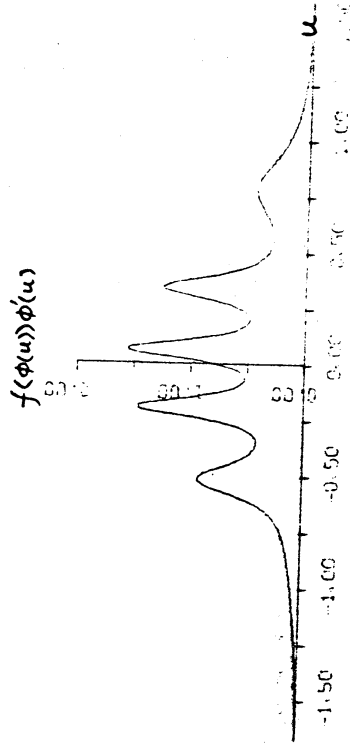


Figure 2-2

P7

$$f(x) = \frac{x}{e^x - 1}$$

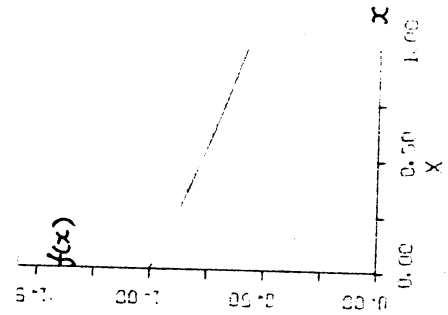


Figure 2-3

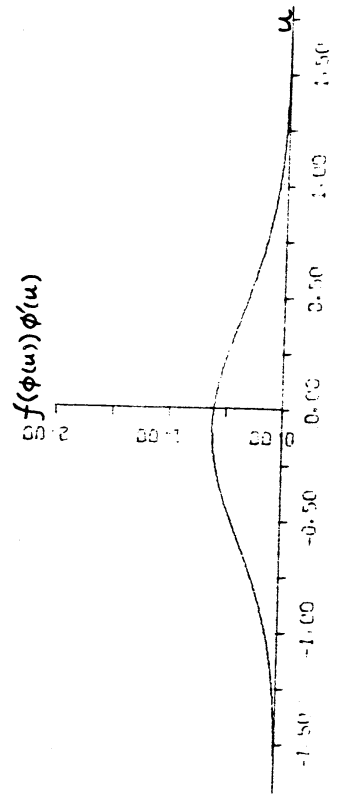


Figure 2-4

P 8

$$f(x) = \frac{\sin 100\pi x}{\pi x}$$

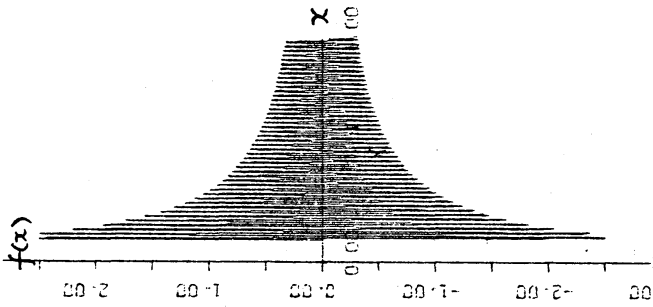


Figure 2-5

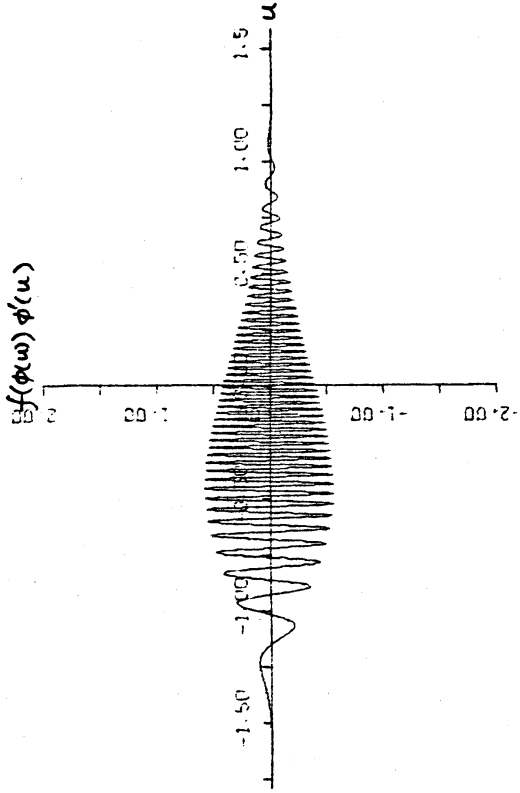


Figure 2-6

P 10

$$f(x) = \cos(\cos x + 3\sin x) + 2\cos 2x + 3\sin 2x + 3\cos 3x$$

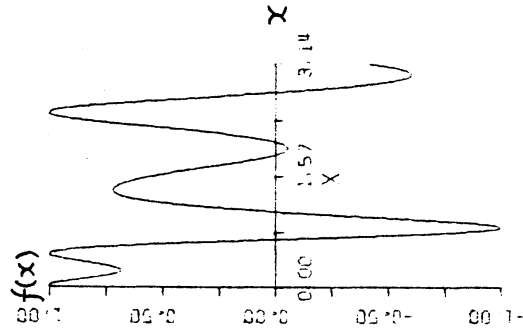


Figure 2-7

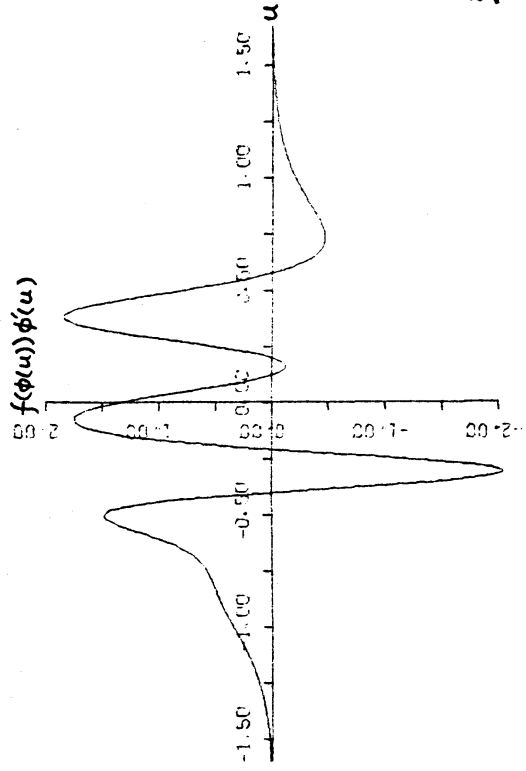


Figure 2-8

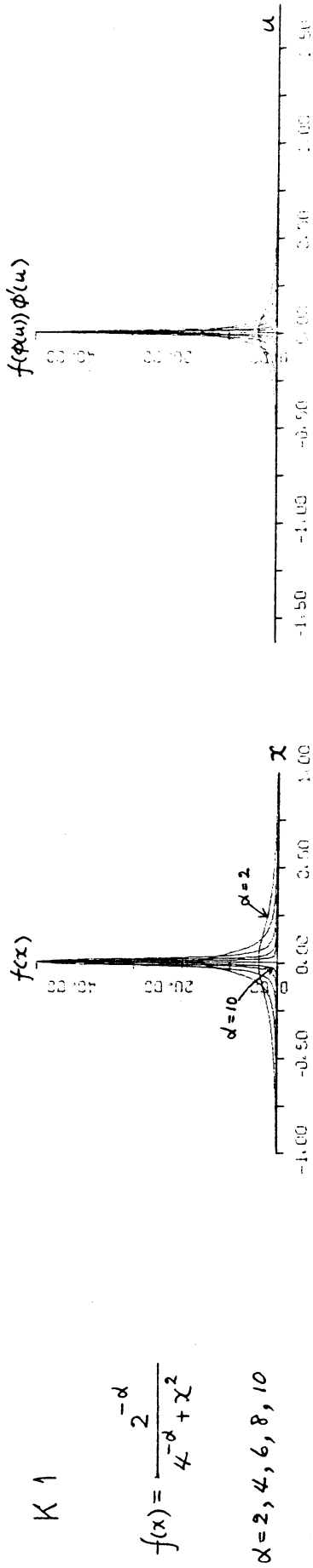


Figure 2-9

Figure 2-10-1

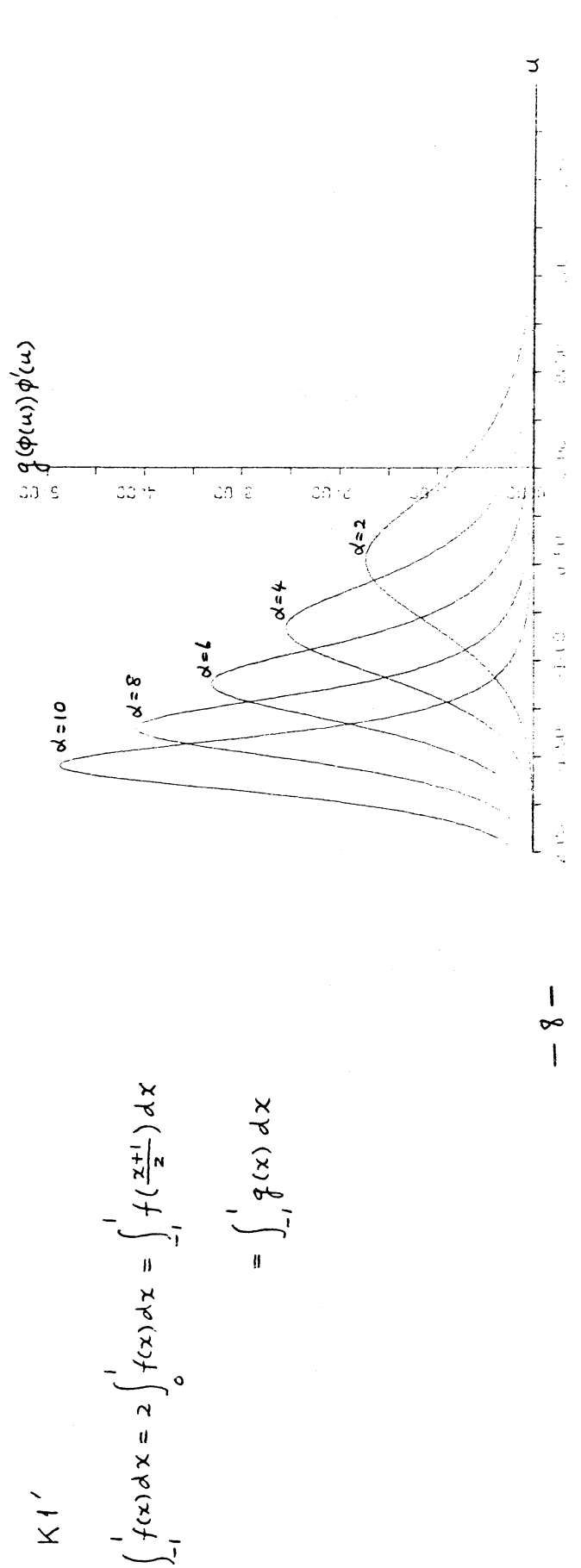


Figure 2-10-2



K 2

$$f(x) = x^\alpha, \quad \alpha = -0.1, -0.9$$

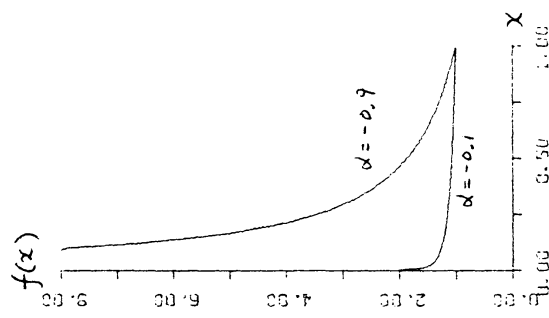


Figure 2-11

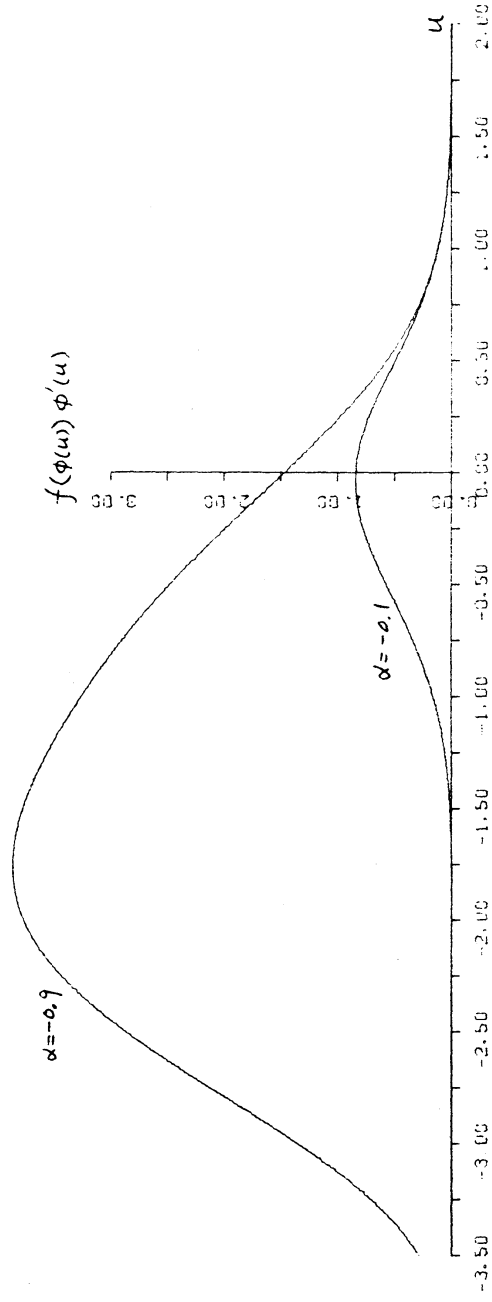


Figure 2-12

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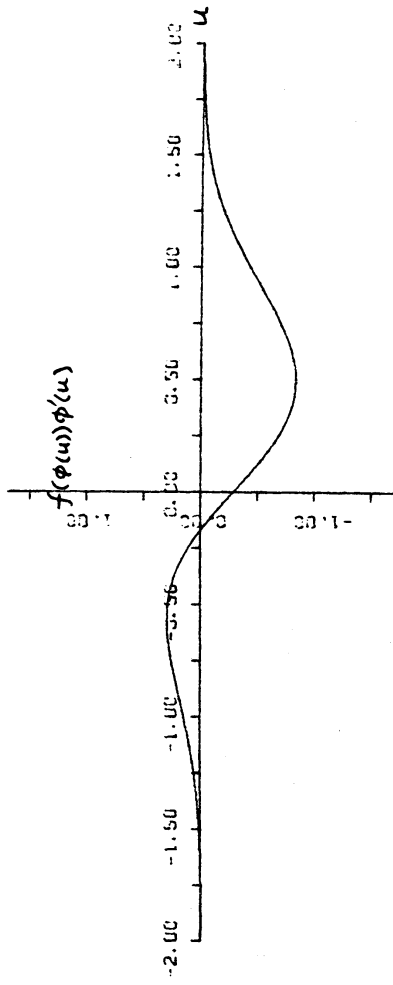


Figure 2-14-1

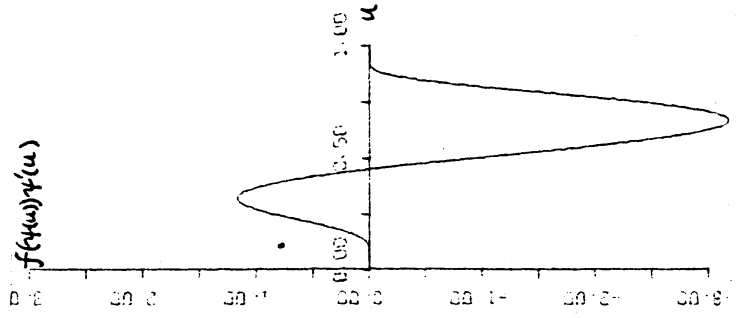


Figure 2-14-2

IMT quadrature

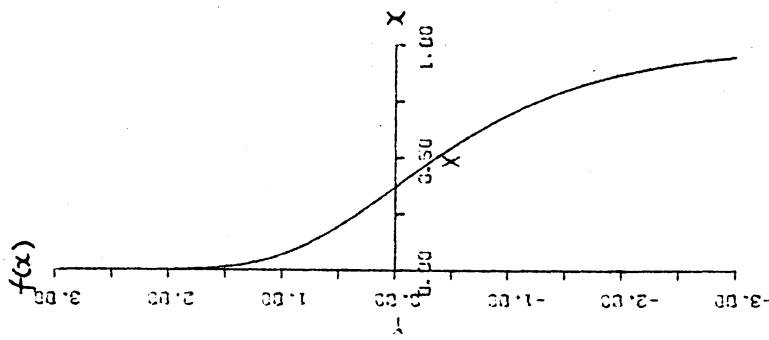


Figure 2-13

H 1

$$f(x) = \log |\log x|$$

H1'

$$\int_0^1 \log |\log x| dx = \int_0^{\infty} e^{-x} \log x dx = \int_0^{\infty} f(x) dx$$

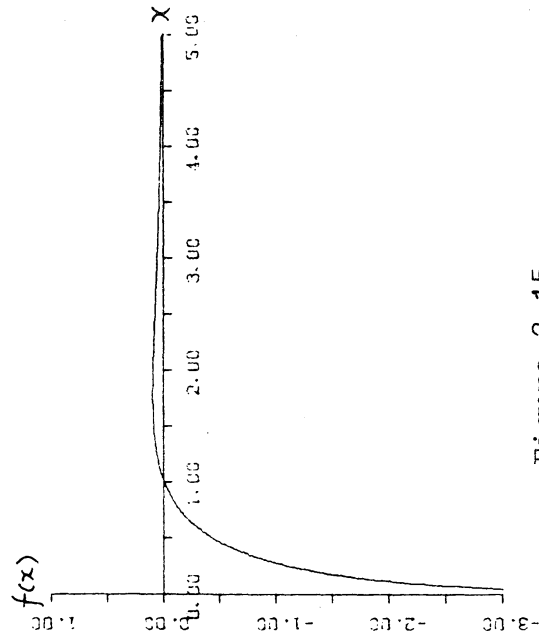


Figure 2-15

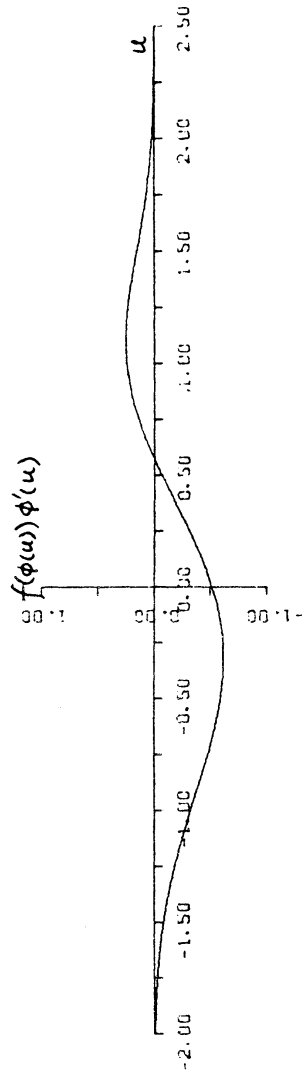


Figure 2-16

## 2.2 Choosing a quadrature formula

Table 2-1 supports the following statements:

- i) For integrals over finite intervals, if the integrand has no singularity through the interval, CADRE or QUAD gets an advantage over DE (P2, P5, P7).
- ii) For improper integrals over finite intervals with singularities at the endpoints of the interval, DE gets the best advantage and IMT gets the second best (P1, P4, P11).
- iii) For integrals over infinite and semi-infinite intervals, if the integrand is not regular at infinity, DE is better than all the other formulas (e.g.,  $\int_0^{\infty} \exp(-x) \cdot \log x \, dx$ ). However, if the integrand is regular at infinity, it can be reduced to the case i) by suitable change of variable (e.g.,  $\int_0^{\infty} \frac{1}{1+x^2} \, dx = \int_0^{\frac{\pi}{2}} \frac{\cos^2 \theta}{\cos^4 \theta + \sin^4 \theta} \, d\theta$ ). But even if DE may be used to evaluate the original integral, it will give slightly worse results than CADRE or QUAD.
- iv) For integrals with singularity close to the real axis in the interior of the interval (e.g., K1), it is hard to evaluate the integral with desired accuracy using any of all the methods presented here, and DE is the most unsuccessful among them. But by splitting the integral into two integrals at the real part of the singular point, it can be reduced to case ii) .

### 3. Notes on usage of DE formula on digital computer.

Some notes must be necessary to effective use of DE formula.

3.1 A numerical technique by Takahashi and Mori to avoid subtractive cancellation by the factor  $(1+x)^\alpha (1-x)^\beta$ ,  $\alpha, \beta < 0$ .

In the case of integral  $\int_{-1}^1 f(x) dx$ ,  $f(x)$  having a factor  $(1+x)^\alpha (1-x)^\beta$ ,  $\alpha, \beta < 0$  (e.g.,  $f(x) = 1/\sqrt{1-x}$ ,  $1/\sqrt{\tan((\pi/2)(1+x))}$ ), the number of its significant digits decreases as  $x \rightarrow \pm 1$ .

Thus the relative error in the  $f(x)$  will be greatly increased. In such a case, following Takahashi and Mori [1] p.735, we must make the change of variable to avoid the cancellation; splitting the integral into two integrals, we have

$$(3.1-1) \quad I = \int_{-1}^1 f(x) dx = \int_{-1}^0 f(x) dx + \int_0^1 f(x) dx = I_1 + I_2$$

and we make direct substitution of  $1+x$  as follows:

$$(3.1-2) \quad -t = 1+x = 1 + \tanh\left(\frac{\pi}{2} \sinh u\right) = \frac{\exp\left(\frac{\pi}{2} \sinh u\right)}{\cosh\left(\frac{\pi}{2} \sinh u\right)}$$

$$t \in [-1, 0]$$

$$(3.1-3) \quad t = 1-x = 1 - \tanh\left(\frac{\pi}{2} \sinh u\right) = \frac{\exp\left(-\frac{\pi}{2} \sinh u\right)}{\cosh\left(\frac{\pi}{2} \sinh u\right)}$$

$$t \in [0, 1]$$

Then we obtain

$$(3.1-4) \quad I_1 = \int_{-1}^0 f(-1-t) dt = \int_{-\infty}^0 f(-1-t(u)) \cdot \frac{\frac{\pi}{2} \cosh u}{\cosh^2\left(\frac{\pi}{2} \sinh u\right)} du$$

$$(3.1-5) \quad I_2 = \int_0^1 f(1-t) dt = \int_0^{\infty} f(1-t(u)) \cdot \frac{\frac{\pi}{2} \cosh u}{\cosh^2\left(\frac{\pi}{2} \sinh u\right)} du$$

## 3.2 Note on mechanizing DE formula on digital computer.

The essence of the technique described above is to substitute  $\mp \exp(\pm \frac{\pi}{2} \sinh u) / \cosh(\frac{\pi}{2} \sinh u)$  directly into  $t = \mp 1-x$ . Thus, it will be of no use to avoid cancellation to make the change of variable  $x = -1-t$  for  $x \in [-1, 0]$  and  $x = 1-t$  for  $x \in [0, 1]$  in library programs and to write  $f(x)$  in its original form. Since the term  $1+x$  will be evaluated not by  $-t$  but by  $1 + (-1-t)$ , the loss of significance will be very large. Then we must transform  $f(x)$  into the expression  $g(t)$  of the independent variable  $t = -1-x$  for  $x \in [-1, 0]$ , and  $t = 1-x$  for  $x \in [0, 1]$ . For example,

$$f(x) = (1+x)^{-\frac{1}{2}}(1-x)^{-\frac{1}{3}}$$

we must make transformation as follows:

$$f(x) = g(t) = \begin{cases} (-t)^{-\frac{1}{2}}(2+t)^{-\frac{1}{3}} & t \in [-1, 0] \\ (2-t)^{-\frac{1}{2}}t^{-\frac{1}{3}} & t \in [0, 1] \end{cases}$$

The same situation holds for the integral  $\int_a^b f(x) dx$ , if  $f(x)$  has a factor  $(x-a)^\alpha (b-x)^\beta$ ,  $\alpha, \beta < 0$ . In this case we must transform  $f(x)$  into the expression of the variable  $t = -\frac{2}{b-a}(x-a)$  for  $x \in [a, \frac{a+b}{2}]$  and  $t = \frac{2}{b-a}(b-x)$  for  $x \in [\frac{a+b}{2}, b]$ .

If this transformation is very tedious, we can transform the integral into an integral over  $[0, \infty)$  by suitable change of variable (e.g.,  $x = \exp(-t)$ ), and then apply (2.1-2) or (2.1-3). For H1, by the change of variable  $x = \exp(-t)$ , we get

$$H1' \quad I = \int_0^\infty \exp(-t) \log t \, dt$$

In this case, there is a problem how to choose the initial step size  $h_0$  for H1. We shall consider this problem in the next section.

### 3.3 The initial step size $h_0$ .

Our starting algorithm to find  $M, N$  such that

$S_0 = \sum_{n=-M}^N f(\phi(nh_0)) \cdot \phi'(nh_0)$ , where  $S_0$  is the first approximation of  $I$ , is as follows:

Input

$\varepsilon$  (some preassigned tolerance)

$h_0$  (initial step size)

Algorithm

$h \leftarrow h_0$ ;  $S_0 \leftarrow f(\phi(0)) \cdot \phi'(0)$ ;  $N \leftarrow 0$ ;  $M \leftarrow 0$

for  $n=1, 2, \dots$  do

if  $N=0$  then  $T_n \leftarrow f(\phi(nh)) \cdot \phi'(nh)$

if  $|T_n| \leq |S_0| * \max(\varepsilon * 10^{-3}, 10^{-16})$  then  $N \leftarrow n$

else  $S_0 \leftarrow S_0 + T_n$

endif

endif

if  $M=0$  then  $U_n \leftarrow f(\phi(-nh)) \cdot \phi'(-nh)$

if  $|U_n| \leq |S_0| * \max(\varepsilon * 10^{-3}, 10^{-16})$  then  $M \leftarrow n$

else  $S_0 \leftarrow S_0 + U_n$

endif

endif

endfor

next:

Thus,

$$(3.3-1) \quad |T_n| \leq |S_0| * \max(\varepsilon * 10^{-3}, 10^{-16})$$

$$|U_n| \leq |S_0| * \max(\varepsilon * 10^{-3}, 10^{-16})$$

must hold before  $nh_0$ ,  $n=1, 2, \dots$ , exceeds a certain threshold value which depends on representation of number by computer.

For example, in the case of H1, we have

$$\begin{aligned} I &= \int_0^1 \log |\log x| dx = \int_{-1}^1 \frac{1}{2} \log \left| \log \frac{1+x}{2} \right| dx \\ &= \int_{-1}^0 \frac{1}{2} \log \left| \log \left( -\frac{t}{2} \right) \right| dt + \int_0^1 \frac{1}{2} \log \left| \log \frac{2-t}{2} \right| dt \end{aligned}$$

where

$$\begin{aligned} -t &= \frac{\exp\left(\frac{\pi}{2} \sinh u\right)}{\cosh\left(\frac{\pi}{2} \sinh u\right)} & t \in [-1, 0] \\ t &= \frac{\exp\left(-\frac{\pi}{2} \sinh u\right)}{\cosh\left(\frac{\pi}{2} \sinh u\right)} & t \in [0, 1] \end{aligned}$$

Since  $2-t$  and  $2$  are considered as the same values (additive no contribution) for  $nh_0 > 3.45$  with double precision arithmetic in T-56, the second integrand  $\frac{1}{2} \log \left| \log \frac{2-t}{2} \right|$  is regarded as  $\log 0$  (error message "not allowed").

In the case of P7, we have

$$\begin{aligned} I &= \int_0^1 \frac{x}{\exp x - 1} dx = \int_{-1}^1 \frac{1}{2} \cdot \frac{\frac{1+x}{2}}{\exp\left(\frac{1+x}{2}\right) - 1} dx \\ &= \int_{-1}^0 \frac{1}{2} \cdot \frac{-\frac{t}{2}}{\exp\left(-\frac{t}{2}\right) - 1} dt + \int_0^1 \frac{1}{2} \cdot \frac{\frac{2-t}{2}}{\exp\left(\frac{2-t}{2}\right) - 1} dt \end{aligned}$$

In the denominator of the first integral  $\exp\left(-\frac{t}{2}\right)$  is considered to be 1 for  $nh_0 \geq 3.31$ .

In both cases for large step size  $h_0$ , our starting algorithm results in failure by evaluating  $f(\phi(nh_0)) \cdot \phi'(nh_0)$  or  $f(\phi(-nh_0)) \cdot \phi'(-nh_0)$  at a point  $nh_0$  beyond the limit 3.45 or 3.31 respectively. In contrast, it can be expected that by taking  $h_0$  small enough depending on the tolerance  $\varepsilon$  (3.3-1) can hold before  $nh_0$  exceeds that threshold value. This is illustrated in Fig. 3.3-1.



P7-1

ERROR REQUIREMENT 0.1000E-14 HO = 0.50000

FT957W ERROR AT(020531) DIVIDE CHECK

0.50000	7	0.7774990274260581	
0.25000	13	0.7775046341101479	0.7211E-05
0.12500	26	0.7775046341122483	0.2701E-11

S = 0.7775046341122483 ER = 0.1314E-16

P7-2

ERROR REQUIREMENT 0.1000E-14 HO = 0.25000

0.25000	13	0.7775046341101479	
0.12500	26	0.7775046341122483	0.2701E-11

S = 0.7775046341122483 ER = 0.1358E-16

H1-1

ERROR REQUIREMENT 0.1000E-11 HO = 0.50000

FT911W ERROR AT(020621)	DLOG	ARG= 0.0	( ARG,LE,0 )	RESULT=(MIN VALUE)
FT911W ERROR AT(020621)	DLOG	ARG= 0.0	( ARG,LE,0 )	RESULT=(MIN VALUE)
0.50000	8	7	-0.2026589570990376E+56	

H1-2

ERROR REQUIREMENT 0.1000E-11 HO = 0.25000

0.25000	13	-0.5772156649014750	
S = -0.5772156649015324	ER = 0.9944E-13	TRUE VALUE -0.5772156649015328608	ERROR -0.8648E-15

Figure 3.3-1

## 3.4 On the criterion of convergence.

Since the error term for DE formula with mesh size  $h$  is roughly estimated (Ref. [5] p.265)

$$(3.4-1) \quad |\Delta I_h| \simeq \exp\left(-\frac{A}{h}\right)$$

where  $A$  is a constant, we have

$$(3.4-2) \quad \left|\Delta I_{\frac{h}{2}}\right| \simeq \left|\Delta I_h\right|^2$$

So that if we halve the mesh size, then the number of significant digits are doubled. Thus we can usually get the desired accuracy when

$$(3.4-3) \quad |S_{i+1} - S_i| \leq \sqrt{\varepsilon} |S_{i+1}|$$

holds, except in the case where the integrand oscillates (e.g., P8 and P10).

In the above-mentioned case, however, on the safe side, we should like to test whether both

$$|S_{i+1} - S_i| \leq \varepsilon |S_{i+1}|$$

and

$$|S_{i+2} - S_{i+1}| \leq \varepsilon |S_{i+2}|$$

hold as the criteria of convergence. From Fig. 3.4-1, we see that (3.4-2) does not hold until mesh size  $h$  becomes small enough.

P6

ERROR REQUIREMENT 0.1000E-14 HO = 0.50000

0.50000	7	1.145065201836647	0.3215E-01
0.25000	13	1.109398079973005	0.6951E-02
0.12500	26	1.101740303515750	0.4805E-01
0.62500E-01	52	1.157349192484848	0.2285E-02
0.31250E-01	103	1.154711000023246	0.9060E-05
0.15625E-01	205	1.154700538516411	0.1188E-09
0.78125E-02	409	1.154700538379251	

S = 1.154700538379251 ER = 0.7079E-17 TRUE VALUE 1.154700538379251529 ERROR -0.5033E-16

P8

ERROR REQUIREMENT 0.1000E-14 HO = 0.50000

0.50000	6	0.2455391953890915	0.6189E-01
0.25000	11	0.2312281657296014	0.1053E+01
0.12500	22	0.1126117247672289	0.8368E+00
0.62500E-01	44	0.6130785022273157E-01	0.5753E+00
0.31250E-01	87	0.3891818870620057E-01	0.3277E+01
0.15625E-01	173	0.9098637539166843E-02	

S = 0.9098637539166843E-02 ER = 0.3978E-16

P10

ERROR REQUIREMENT 0.1000E-14 HO = 0.50000

0.50000	7	1.873635947230349	0.1294E+01
0.25000	14	0.8168465615955236	0.2712E-01
0.12500	27	0.8396163176771946	0.1121E-02
0.62500E-01	53	0.8386763427025238	0.9651E-11
0.31250E-01	105	0.8386763426944296	

S = 0.8386763426944296 ER = 0.3249E-17

Figure 3.4-1 - 19 -

#### 4. Difficult problems of integration

##### 4.1 The restriction imposed by the number of bits for exponent.

The number of bits for exponent is usually taken to be 7 or 8 . Then, for example, the integral to get the abscissas of IMT formula

$$I = \int_0^t \exp\left(-\frac{1}{x} - \frac{1}{1-x}\right) dx$$

$$= t \int_0^{\infty} \exp\left(-\frac{1}{t \exp(-x)}\right) \exp\left(-\frac{1}{1-t \exp(-x)}\right) \exp(-x) dx$$

with  $t = 3/512, 2/512, 1/512$ , has the value far smaller than  $10^{-8}$ . Thus, if we wish to evaluate this integral using DE formula (2.1-3), our algorithm will fail, since (3.3-1) is not satisfied before  $nh=5.5$ , with this  $nh$   $x = \exp(nh - \exp(-nh))$  becomes 243.7 and  $\exp(-x)$  vanishes.

Such a problem requires the use of multiple-precision arithmetic [6] .

#### 5. Conclusion

We conclude from the experiment described above that DE formulas offer a very powerful method for numerical integration, except for the special cases described in the previous section. It is recommended to use DE formulas especially for integrals over infinite or semi-infinite intervals.

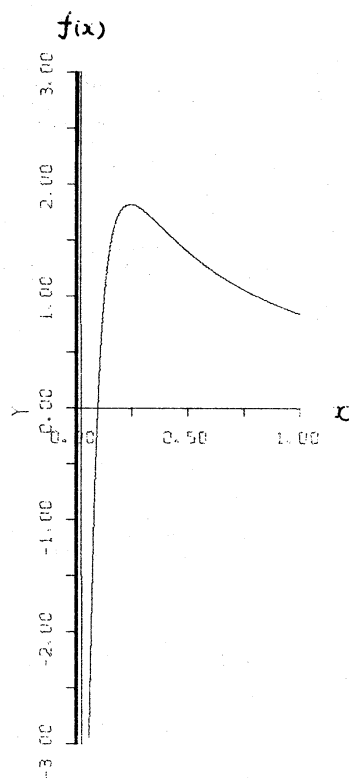
## Appendix.

Dr. Rabinowitz presented the following problem when he visited ETL at Tsukuba on June 4, 1980.

$$I = \int_0^1 f(x) dx = \int_0^1 \frac{\sin \frac{1}{\sqrt{x}}}{\sqrt{x}} dx \doteq 1.00813 \ 41238 \ 139$$

at  $x=0$  value=0

We will show the results obtained by DE formulas below.



LIST 10@0170-0270

```

0170 DOUBLE PRECISION FUNCTION FMPI1(X)
0180 DOUBLE PRECISION X, T, G
0190 IF(X) 1,2,2
0200 T = -0.500*X
0210 GO TO 3
0220 T = 1.000 - 0.5*X
0230 CONTINUE
0240 G = DSIN(1.000/DSQRT(T)) /DSQRT(T)
0250 FMPI1 = G / 2.000
0260 RETURN
0270 END

```

\*RUN

PLEASE TYPE IN H0, EPS, LIST

```

= 0.12500 23
0.62500E-01 46
0.31250E-01 91
0.15625E-01 181
0.78125E-02 361
0.39063E-02 721
0.19531E-02 1441
0.97656E-03 2881
0.48828E-03 5761
0.24414E-03 11521
0.12207E-03 23041
0.61035E-04 46081
0.30518E-04 92161
0.15259E-04 184321
0.76294E-05 368641
0.38147E-05 737281
0.19073E-05 1474561
0.95367E-06 2949121

```

```

28 1.00-9 1
56 0.9963033435784173
112 1.008041952064974
224 1.003388731021478
447 1.005907923892132
893 1.008449386893941
1785 1.007959610813495
3569 1.008295296649887
7137 1.008103976816533
14273 1.008165123910790
28545 1.008143849296904
57089 1.008135239122684
114177 1.008135329025766
228353 1.008134649610200
456705 1.008134257201770
913409 1.008134050475504
1826817 1.008134144078395
3653633 1.008134131626789

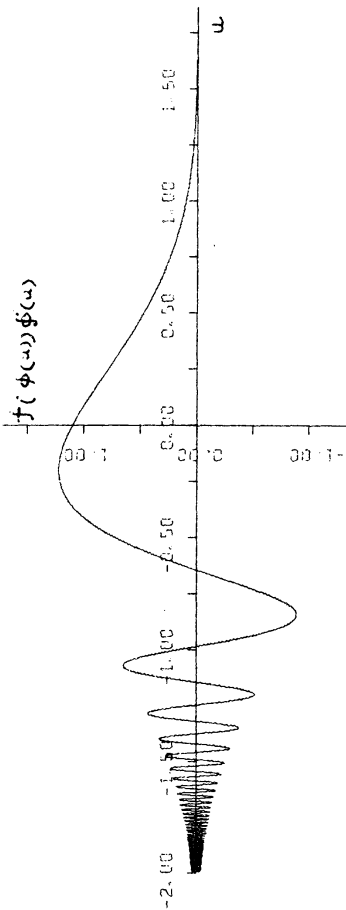
```

```

0.1534E-01
0.2657E-01
0.4638E-02
0.2504E-02
0.2520E-02
0.4859E-03
0.3329E-03
0.1898E-03
0.6065E-04
0.2110E-04
0.8541E-05
0.8918E-07
0.6739E-06
0.3892E-06
0.2051E-06
0.9285E-07
0.1235E-07

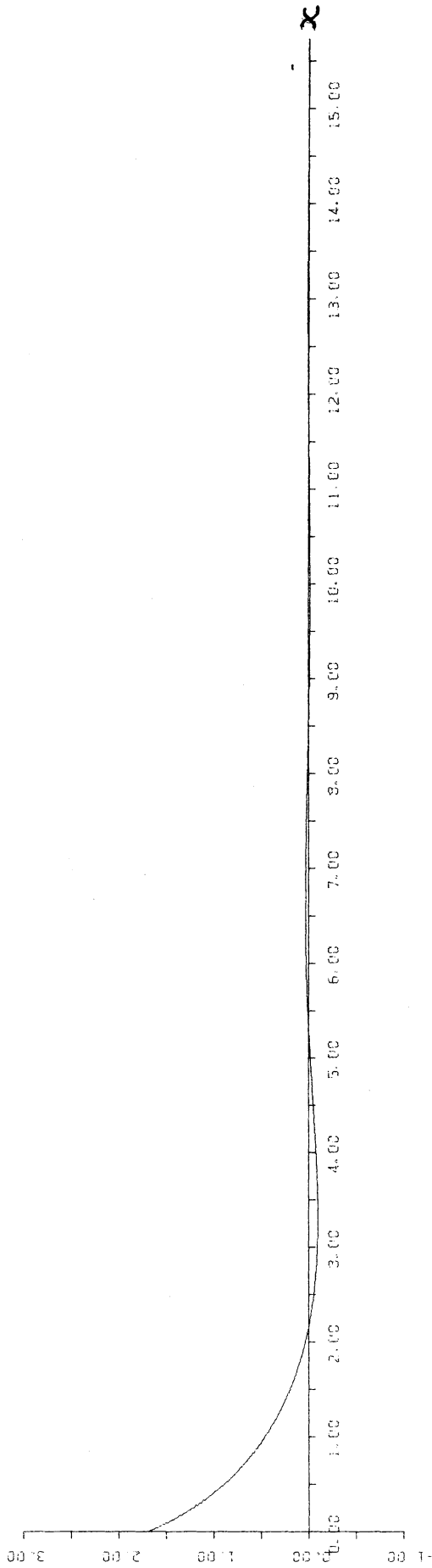
```

$$I = \int_{-1}^0 \frac{1}{2} \frac{\sin \sqrt{-t/2}}{\sqrt{-t/2}} dt + \int_0^1 \frac{1}{2} \frac{\sin(\frac{1}{\sqrt{2-t/2}})}{\sqrt{(2-t)/2}} dt$$

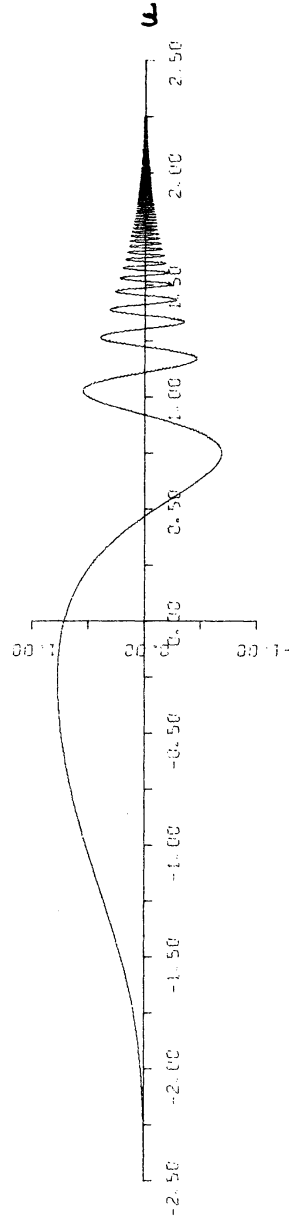


$$I = \int_0^1 \frac{\sin\left(\frac{1}{\sqrt{x}}\right)}{\sqrt{x}} dx = \int_0^\infty \frac{2 \sin(1+x)}{(1+x)^2} dx \quad (\text{by Dr. Radinowitz's comment})$$

$$f(x) = \frac{2 \sin(1+x)}{(1+x)^2}$$



$$f(\phi(\omega) \phi(\omega))$$



LIST 180-1000

```

180 DOUBLE PRECISION FUNCTION F0INF(X)
190 DOUBLE PRECISION X
200 F0INF = 2.000*DSIN(1.000*X)/(1.000+X)**2
210 RETURN
220 END

```

$$I = 2 \int_0^{\infty} \frac{\sin(x)}{(1+x)^2} dx$$

\*RUN

(C) INTEGRATE(F0INF, 0, INF)

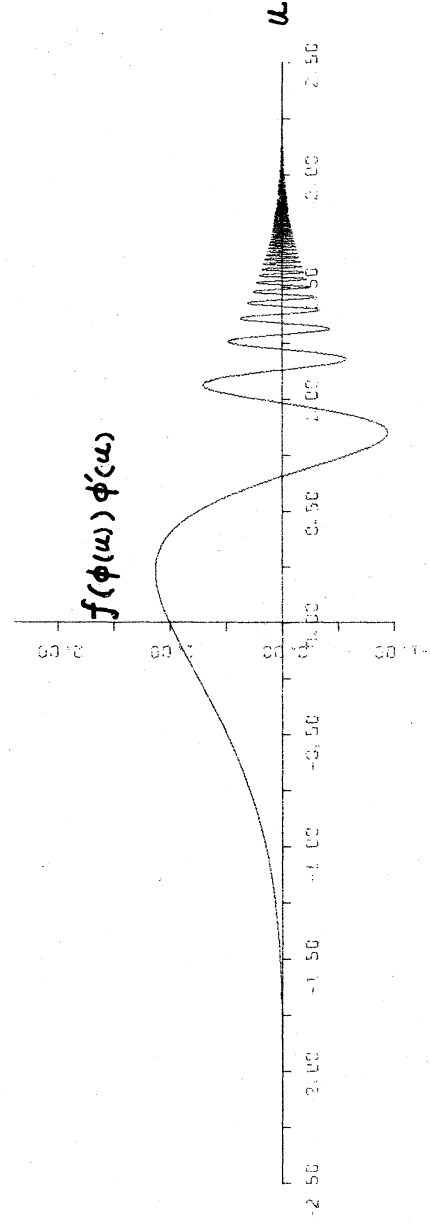
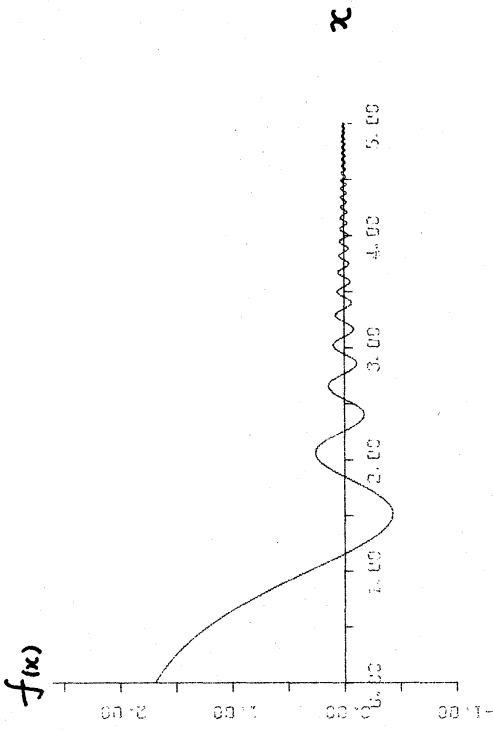
PLEASE TYPE IN H0, EPS, LIST

H0	EPS	LIST	Value
0.0312500	1.00-7	1	0.7562E-02
100	1.000068727537460	105	0.9063E-03
200	1.007688400674151	210	0.7320E-03
400	1.006775973599763	419	0.3812E-03
800	1.007513512037899	837	0.3029E-03
1600	1.007897695988269	1673	0.2271E-04
3199	1.008203125717950	3345	0.1012E-04
6397	1.008180234159500	6689	0.3180E-04
12793	1.008170029883467	13377	0.1042E-05
25585	1.008137971140978	26753	0.3974E-05
51169	1.008139021682263	53505	0.1115E-05
102337	1.008135015685097	107009	0.2620E-06
204673	1.008133892035357	214017	0.2691E-07
409345	1.008134156187340	428033	
818689	1.008134129054916	856065	

1.008134129054916, 0.2691E-07



$$I = \int_0^1 \frac{\sin(\frac{1}{\sqrt{x}})}{\sqrt{x}} dx = \int_0^1 2 e^{-x} \sin(e^x) dx \equiv \int_0^{\infty} f(x) dx$$



$$I = \int_0^{\infty} 2 e^{-x} \sin(e^x) dx$$

```

LIST 180-10000
180 DOUBLE PRECISION FUNCTION FEXP(X)
190 DOUBLE PRECISION X, DEXPF
200 FEXP= 2.0D0*DEXP(-X)*DSIN(DEXP(X))
210 RETURN
220 END

```

\*RUN

(D) INTEGRATE(FEXP, 0, INF)

PLEASE TYPE IN H0, EPS, LIST

H0	EPS	LIST	Value
0.12500	27	26	1.00-9 1
0.62500E-01	54	51	1.071704042186671
0.31250E-01	107	101	1.004933608340182
0.15625E-01	213	201	0.9969797797212950
0.78125E-02	425	401	1.015120956996488
0.39063E-02	849	801	1.009720373317079
0.19531E-02	1697	1601	1.007896100079553
0.97656E-03	3393	3201	1.008593032756302
0.48828E-03	6785	6401	1.008131668867897
0.24414E-03	13569	12801	1.00811677324365
0.12207E-03	27137	25601	1.008157314514286
0.61035E-04	54273	51201	1.008136230970505
0.30518E-04	108545	102401	1.008132923350206
0.15259E-04	217089	204801	1.008134416324451
0.76294E-05	434177	409601	1.008133909877286
0.38147E-05	868353	819201	1.008134085382540
0.19073E-05	1736705	1638401	1.008134103770320
0.95367E-06	3473409	3276801	1.008134109726709

\*↑

BREAK

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