

On unstable manifolds of analytic  
diffeomorphisms of the plane.

\*Shigehiro USHIKI

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be an analytic diffeomorphism of the plane. We call a point  $x$  in  $\mathbb{R}^2$  a saddle point of  $f$  if  $x$  is a fixed point of  $f$  and that one of the absolute values of the eigenvalues of the Jacobian matrix  $df$  at the point is greater than one and the other is smaller than one.

For a saddle point  $x$  in  $\mathbb{R}^2$ , we denote by  $W_x^s$  ( resp. by  $W_x^u$  ) the stable manifold ( resp. the unstable manifold ) associated to  $x$ .

Let  $E_x^s$  ( resp.  $E_x^u$  ) denote the eigenspace of linear map  $df_x : T_x \mathbb{R}^2 \rightarrow T_x \mathbb{R}^2$  corresponding to the eigenvalue whose absolute value is smaller than one ( resp. greater than one ) . Stable manifold  $W_x^s$  ( resp. unstable manifold  $W_x^u$  ) is an injectively immersed one dimensional manifold and is tangent to  $E_x^s$  ( resp.  $E_x^u$  ) at the saddle point.

Let  $h : I \rightarrow \mathbb{R}^2$  be an embedding of the unit interval into  $\mathbb{R}^2$ . We call  $h$  a saddle connection if the following conditions i), ii) and iii) are satisfied.

- i) the image of two boundary points  $p=h(0)$  and  $q=h(1)$  are saddle points of  $f$ .
- ii) the image  $h(I-\{0,1\})$  contains no fixed point of  $f$ .
- iii) the image  $h(I)$  is invariant under  $f$ .

We have the following theorem.

Theorem 1 If an analytic diffeomorphism  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  can be extended to an automorphism  $F : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  of the two dimensional complex vector space considered as complex manifold, then there does not exist any saddle connection.

Proof We shall prove the theorem by contradiction. Suppose there was a saddle connection  $h : I \rightarrow \mathbb{R}^2$ . In the first place, we assume that saddle points  $p = h(0)$  and  $q = h(1)$  are distinct. As  $h$  is an embedding and the image  $h(I)$  is invariant under  $f$ , there is a neighborhood of  $0$  in  $I$  such that its image by  $h$  is included in  $W_p^s$  or  $W_p^u$ . We can assume, without loss of generality, it is included in  $W_p^u$ , the unstable manifold of  $p$ . In the contrary case, we can replace  $f$  by its inverse diffeomorphism  $f^{-1}$ . Embedding  $h$  is a diffeomorphism of the unit interval onto its image  $h(I)$ . Let  $g : I \rightarrow I$  be the continuous mapping of the unit interval into itself defined by  $g = h^{-1} \circ f \circ h$ . Since  $h$  and  $f$  are diffeomorphisms,  $g$  is also a diffeomorphism. The image  $h(I - \{0,1\})$  contains no fixed point of  $f$  by the assumption. Hence  $g$  has no fixed point in the interior of  $I$ . The boundary points  $0$  and  $1$  of  $I$  are fixed points of  $g$ . We see that

$$\frac{dg}{dt}(0) > 1$$

since some neighborhood of  $0$  in  $I$  is mapped diffeomorphically into the unstable manifold  $W_p^u$ . There is a neighborhood of  $1$  in  $I$  whose image by  $h$  is included in  $W_q^s$  or  $W_q^u$ . Hence

we have  $\frac{dg}{dt}(1) \neq 1$ .

We see that  $0 < \frac{dg}{dt}(1) < 1$  since  $g$  has no fixed points except the boundary points  $0$  and  $1$ . Therefore there is a neighborhood of  $1$  in  $I$  which is mapped into the stable manifold  $W_q^S$  of  $q$ . As manifolds  $W_p^u$  and  $W_q^S$  are invariant by  $f$  and that  $g$  is a diffeomorphism having no fixed points except  $0$  and  $1$ , the image  $h(I - \{1\})$  is included totally in  $W_p^u$  and  $h(I - \{0\})$  is included in  $W_q^S$ .

By a theorem obtained by the author [1], there exist analytic mappings  $\varphi_p : E_p^u \rightarrow W_p^u$  and  $\varphi_q : E_q^S \rightarrow W_q^S$  satisfying the following conditions.

- i)  $\varphi_p(0) = p$ ,  $\varphi_q(0) = q$
- ii)  $d\varphi_p(0) : E_p^u \rightarrow T_p R^2$  and  $d\varphi_q(0) : E_q^S \rightarrow T_q R^2$  are inclusion mappings.
- iii)  $\varphi_p(E_p^u) = W_p^u$ ,  $\varphi_q(E_q^S) = W_q^S$ .
- iv)  $\varphi_p$  and  $\varphi_q$  are injective immersions.
- v)  $f(\varphi_p(\xi)) = \varphi_p(\alpha\xi)$ ,  $f^{-1}(\varphi_q(\xi)) = \varphi_q(\beta^{-1}\xi)$

where  $\alpha$  (resp.  $\beta$ ) denotes the eigenvalue of  $df_p$  (resp.  $df_q$ ) associated to the eigenspace  $E_p^u$  (resp.  $E_q^S$ ).

Moreover, in our case where  $f$  can be extended to an automorphism of  $C^2$ , these mappings  $\varphi_p$  and  $\varphi_q$  can be extended holomorphically to entire mappings

$$\Phi_p : C \rightarrow C^2 \quad \text{and} \quad \Phi_q : C \rightarrow C^2,$$

i.e.,  $\Phi_p$  and  $\Phi_q$  are holomorphic mappings defined globally on  $C$  and that  $\Phi_p|_R = \varphi_p$  and  $\Phi_q|_R = \varphi_q$ . Mappings  $\Phi_p$  and  $\Phi_q$  are injective immersions, too. Note that the image  $\Phi_p(C)$  does not contain  $q$  and  $\Phi_q(C)$  does not contain  $p$ . By the

assumption of the existence of a saddle connection  $h : I \rightarrow \mathbb{R}^2$ , we conclude that each of two complex curves  $\Phi_p(C)$  and  $\Phi_q(C)$  includes the image  $h(I - \{0,1\})$ . On the other hand, the intersection  $\Phi_p(C) \cap \Phi_q(C)$  of two complex curves must be of complex dimension one or zero. Now that it is at least of real dimension one, it follows that its complex dimension is one at least in the neighborhood of the image  $h(I - \{0,1\})$ .

Let  $U$  and  $V$  denote the inverse image of this intersection by  $\Phi_p$  and  $\Phi_q$  respectively. We have a holomorphic function  $\psi : U \rightarrow V$  defined by  $\psi = \Phi_q^{-1} \circ \Phi_p$ . Mappings  $\Phi_p$  and  $\Phi_q$  being injective immersions,  $\psi$  is a bijective mapping. Mapping  $\psi$  is also an isomorphism. Let  $M$  denote the one dimensional complex manifold defined by two holomorphic charts  $C_p$  and  $C_q$  with the coordinate transformation map  $\psi : U \rightarrow V$ . More precisely, manifold  $M$  is composed of two copies  $C_p$  and  $C_q$  of the complex plane  $C$  of which two points  $z \in C_p$  and  $w \in C_q$  are identified if  $w = \psi(z)$ . By an elementary argument using the uniqueness of analytic continuation, it is easily verified that this manifold  $M$  is Hausdorff. Manifold  $M$  is immersible in  $C^2$ . In fact, for a point  $z \in C_p$  or  $w \in C_q$ , define its image by  $\Phi_p(z)$  or  $\Phi_q(w)$ . We denote this mapping by  $\sigma : M \rightarrow C^2$ .

Define an entire mapping  $\gamma : C \rightarrow M$  by  $\gamma(z) = z \in C_p \subset M$ . Mapping  $\gamma$  is holomorphic on  $C$ . Mapping  $\gamma$  is not surjective since the origin of  $C_q$  is not contained in its image. Let  $\pi : \tilde{M} \rightarrow M$  be the universal covering of  $M$ . As the complex plane  $C$  is simply connected, mapping  $\gamma : C \rightarrow M$  can be lifted into

mapping  $\tilde{\gamma} : C \rightarrow \tilde{M}$ . By the theorem of Koebe on the classification of simply connected one dimensional complex manifolds,  $\tilde{M}$  is isomorphic to the complex plane  $C$ , the interior of the unit disc  $D = \{z \in C \mid |z| < 1\}$  or the complex projective space  $CP$ . As there exists an entire function  $\tilde{\gamma} : C \rightarrow \tilde{M}$  which is injective and not surjective,  $\tilde{M}$  cannot be isomorphic to  $C$  nor to  $D$ . Finally if  $\tilde{M}$  is isomorphic to  $CP$ , the composed map  $\sigma \circ \pi \circ \tilde{\gamma} = \sigma \circ \gamma : C \rightarrow C^2$  will be entire. But the projective space  $CP$  is compact hence its image by  $\sigma$  is bounded. Therefore  $\sigma \circ \gamma$  is a constant map, which contradicts to the injectivity of the composed map  $\sigma \circ \gamma = \Phi_p$ .

Now consider the case where  $p$  and  $q$  are identical. The point  $p = q$  is a saddle point so that the unstable manifold  $W_p^u$  and  $W_q^s$  intersect at the point transversally. We blow up the space  $C^2$  at  $p$ . Then we have a complex manifold of dimension two. We denote this manifold by  $H$ . Let  $\rho : H \rightarrow C^2$  be the canonical projection map. Mapping  $\rho$  is holomorphic and bijective on  $\rho^{-1}(C^2 - \{p\})$ . The inverse image  $\rho^{-1}(p)$  is isomorphic to the complex projective space  $CP$ . Eigenspaces  $E_p^u$  and  $E_q^s$ , which are tangent to  $W_p^u$  and  $W_q^s$  respectively at  $p$ , define points  $p'$  and  $q'$  respectively in  $\rho^{-1}(p)$ . The transversality of  $W_p^u$  and  $W_q^s$  implies that  $p'$  and  $q'$  are distinct in  $H$ . In place of entire mappings  $\Phi_p$  and  $\Phi_q$ , take modified mappings  $\Psi_p : C \rightarrow H$  and  $\Psi_q : C \rightarrow H$  defined as follows :

$$\begin{aligned} \Psi_p(0) &= p' , & \Psi_q(0) &= q' \\ \Psi_p(\xi) &= \rho^{-1} \circ \Phi_p(\xi) , & \Psi_q(\xi) &= \rho^{-1} \circ \Phi_q(\xi) \quad \text{for } \xi \neq 0. \end{aligned}$$

By the same argument, we will have a contradiction. We proved our theorem also for saddle connection curve whose starting point and ending point are same.

Remark We used the supposition of the existence of saddle connection only to derive that the complex manifolds  $\Phi_p(\mathbb{C})$  and  $\Phi_q(\mathbb{C})$  intersect in a certain portion of complex dimension one. We have the following generalization of our theorem.

Theorem 2 Let  $V$  be a finite-dimensional complex manifold. Assume  $V$  can be immersed holomorphically in finite-dimensional complex vector space  $\mathbb{C}^N$ . Let  $f : V \rightarrow V$  be an automorphism of  $V$  and let  $p$  and  $q$  be hyperbolic fixed points. We assume that the unstable manifold of  $p$  and the stable manifold of  $q$  of dynamical system  $(V, f)$  are one-dimensional. Then there exists no embedding  $h : I \rightarrow V$  of the unit interval into  $V$  whose image  $h(I)$  is included in the unstable manifold of  $p$  and the stable manifold of  $q$  simultaneously.

The proof is similar to that for theorem 1.

We employ a theorem of the author [1],[2] for the case of automorphisms of complex manifolds in order to obtain the entire mappings parametrizing unstable manifolds and stable manifolds. The author would like to express his gratitude to professors M.Yamaguti and K.Ueno for discussions and suggestions.

\*Department of Mathematics  
Kyoto University

REFERENCES

- [1] S.Ushiki : Analytic expressions of unstable manifolds.  
Proc.Jap.Acad. vol.56, ser.A,p239-244,1980.
- [2] S.Ushiki : Unstable manifolds of analytic dynamical systems.  
(to appear).
- [3] S.Ushiki : Sur liaison-cols de système dynamique analytique,  
(to appear).
- [4] M.Yamaguti et S.Ushiki : Discrétisation et chaos,C.R. t.290,  
N°14,p637-640,1980.
- [5] M.Yamaguti and S.Ushiki : Chaos in numerical analysis of  
ordinary differential equations (to appear).
- [6] S.Ushiki : Central difference scheme and chaos. (to appear).