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Abelian varieties attached to Hilbert modular surfaces

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§ Introductory speculations.

In this note, I would like to consider the Hodge realization of the following speculations via motives (cf. Deligne [1]).

Let $\mathbb{F}$ be a real quadratic field over the rational number field $\mathbb{Q}$. Suppose that a Hilbert modular cusp form f is given, which is a common eigenfunction of all Hecke operators. Let us call such a form f ($\neq 0$) primitive. And let K_f be the subspace of the complex number field $\mathbb{C}$, generated by the eigenvalues of all Hecke opertors. Then we have the following conjecture.

Tensor Product Conjecture. (0.1).

(i) We expect that there exists a motif M(f) defined over $\mathbb{Q}$ attached to f, on which K_f acts as endomorphisms on $M(f)\times_{\mathrm{Spec}(\mathbb{Q})}\mathrm{Spec}(\mathbb{F})$. Moreover the rank of M(f) over K_f should be $2^2=4$.

(ii) For any f which is given above, we expect that there exist two motives $M_{f,1}$ and $M_{f,2}$ defined over $\mathbb{F}$ with K_f actions on them, such that $M(f) \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{F}) \cong M_{f,1}\otimes_{K_f} M_{f,2}$ (an isomorphism as K_f-motives). Moreover these two motives $M_{f,1}$ and $M_{f,2}$, which will be of rank two over K_f, should be conjugate with respect to the extension $\mathbb{F}/\mathbb{Q}$.

(iii) Let f^{σ} be a primitive Hilbert cusp form obtained from f, applying an automorphism σ of $\mathbb{C}$ over $\mathbb{Q}$ to the Fourier coefficients of f. Then $M_{f^{\sigma},1} \cong M_{f,1}$ and $M_{f^{\sigma},2} \cong M_{f,2}$, or $M_{f^{\sigma},1} \cong M_{f,2}$ and $M_{f^{\sigma},2} \cong M_{f,1}$.

Let us apply the above conjecture to the problem of counting the number of algebraic 2-cycles on Hilbert modular surfaces. Let $M^2(S)$ be the 2-motif defined over $\mathbb{Q}$ attached to a Hilbert modular surface S, and let $W_2M^2(S)$ be the pure part of $M^2(S)$. Let Φ be a representative system of equivalence classes of primitive Hilbert modular cusp forms of <u>weight 2</u> with respect to the following equivalence relation:

<u>Two primitive forms</u> f <u>and</u> g <u>are equivalent</u>, <u>if and only if</u> $f^\sigma = cg$ <u>for some</u> σ Aut$(\mathbb{C}/\mathbb{Q})$ <u>and</u> $c \in \mathbb{C}$.

Then we should have a decompostion of the motif $W_2M^2(S)$

$$W_2M^2(S)\times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{F}) \cong \mathbb{F}[-1]\oplus \mathbb{F}[-1] \oplus\Big(\bigoplus_{f\in\Phi} M(f)\Big),$$

where $F[-1]$ the Tate motif of "poids" 2 over $\mathbb{F}$.

By Tensor Product Conjecture (0.1), there exist two abelian varieties $A_{f,1}$ and $A_{f,2}$ defined over $\mathbb{F}$ of dimension $d=[K_f:\mathbb{Q}]$ with endomorphism ring K_f, such that

$$M(f) \cong A_{f,1}\otimes_{K_f} A_{f,2},$$

where we regard these tw two abelian varieties as 1-motives, i.e. abelian varieties up to K_f-isogenie.

Therefore,

$$W_2M^2(S)\times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{F}) \cong \mathbb{F}[-1]^{\oplus 2} \oplus\Big\{\bigoplus_{f\in\Phi}(A_{f,1}\otimes_{K_f} A_{f,2})\Big\}.$$

Now let us consider the Tate twist of $W_2M^2(S)\times_{\mathrm{Spec}(Q)} \mathrm{Spec}(F)$:

$$(W_2M^2(S)\times_{\mathrm{Spec}(Q)} \mathrm{Spec}(\mathbb{F}))[1] \cong \mathbb{F}[0]\oplus \mathbb{F}[0]\oplus\Big(\bigoplus_{f\in\Phi} A_{f,1}[1]\otimes_{K_f} A_{f,2}\Big).$$

Since $A_{f,1}$ is an abelian variety, there exists a polarization

$$A_{f,1}\otimes_{K_f} A_{f,1} \longrightarrow K_f[-1].$$

Hence $A_{f,1}[1]$ is isomorphic to the dual motif $A^\vee_{f,1}$ of $A_{f,1}$.

Therefore, we have

$$(W_2M^2(S)\times_{\mathrm{Spec}(Q)} \mathrm{Spec}(F))[1] \cong F[0] \oplus F[0] \oplus \{\bigoplus_{f\in\Phi} \underline{\mathrm{Hom}}_{K_f}(A_{f,1},A_{f,2})\},$$

where $\underline{\mathrm{Hom}}_{K_f}(A_{f,1},A_{f,2})$ is the $\underline{\mathrm{Hom}}$-object in the category of K_f-motives.

Let $(W_2M^2(S)\times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(F))_{alg}$ be the submotif of $W_2M^2(S)\times_{\mathrm{Spec}(\mathbb{Q})}$ Spec(F), generated by algebraic 2-cycles.

Then Hodge-Tate conjecture for motives implies that

$$(\#) \qquad (W_2M^2(S)\times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(F))_{alg}[1] \cong F[0] \oplus F[0] \oplus \{\bigoplus_{f\in\Phi} \mathrm{Hom}_{K_f}(A_{f,1},A_{f,2})\}.$$

Here $\mathrm{Hom}_{K_f}(A_{f,1},A_{f,2})$ is the K_f-module of K_f-linear homomorphism of K_f-isogenie class of abelian varieties defined by $A_{f,1}$, to K_f-isogenie class of abelian varieties defined by $A_{f,2}$.

So, if (#) is true, we can reduce the problem of determination of algebraic cycles, to determination of endomorphisms of ableian varieties.

Here we consider the Hodge realization of the above argument. In this case, the validity of (#) is guaranteed by the Lefschetz criterion on algebraic cycles on surfaces:

<u>A 2-cycle on an algebraic surface is algebraic, if and only if it is rational and of (1.1)-type.</u>

§ 0. Main results.

Let F be a real quadratic field with discriminant D and with class number 1. And suppose that F has a unit with negative norm. Let f be a primitive form of weight 2 with respect to $\Gamma = SL_2(O_F)$. For any primitive form f of weight 2, we can attach, in general, two abelian varieties $A_{f,1}$ and $A_{f,2}$ of dimension $d=[K_f:Q]$, defined over $\mathbf{C}$, which have K_f as a subring of $End(A_{f,i}) \otimes_Z Q$ (i=1,2).

Definition. (0.1). A primitive form $f(z_1,z_2)$ is called <u>self-conjugate</u>, iff $f(z_1,z_2)=f(z_2,z_1)$.

<u>Remark</u>. We avoid the term "symmetric", because the associated 2-form $\omega_f=(2\pi i)^2 f(z_1,z_2)dz_1 \wedge dz_2$ is not symmetric, but anti symmetric with respect to the involution of Hilbert modular surfaces obtained from the mapping $(z_1,z_2) \longmapsto (z_2,z_1)$ on passing to the quotients. There is also a theoretical reason to call these forms self-conjugate.

For a self-conjugate form f, it is known that there exists a reellen Neben Typus elliptic cusp form φ of weight 2 with respect to $\Gamma_0(D)$ with multiplicator defined by Jacobi symbol $(\frac{D}{*})$, such that f is the lifting of φ (cf. Doi-Naganuma [11], Naganuma [12], Zagier [13], etc).

Let us recall the results of Shimura [20]. Let K_φ be the field generated by the eigenvalues of Hecke operator over Q. And let k_φ be the totally real subfield contained in K_φ with $[K_\varphi:k_\varphi]=2$. Then Shimura [20] naturally attached abelian variety

A_φ of dimension $[K_\varphi:Q]$, which decomposes into two conjugate abelian varieties B_φ and B'_φ defined over F of dimension $[k_\varphi:Q]$. Moreover B_φ and B'_φ are isogenous over F.

Theorem A. Let f be a self-conjugate form of weight 2 obtained from a reellen Neben typus elliptic modular cusp form φ of weight 2. Then $k_\varphi=K_f$, and the abelian varieties $A_{f,1}$ and $A_{f,2}$ are both isogenous to B_φ , as k_φ-abelian varieties defined over the complex number C.

In order to state our next theorem, we need more terminology. Let p_{sc} be the dimension of the subspace of cusp forms of weight 2, generated by self-conjugate forms, and let p_{nsc} be p_g-p_{sc} i.e. p_{nsc} is the dimension of the non-self-conjugate forms.
Let λ be $b_2(\tilde{S})-\rho(\tilde{S})$, where $\tilde{S}$ is a smooth proper model of S, ~~XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX~~ and $b_2(\tilde{S})$ and $\rho(\tilde{S})$ are the second Betti number and the Picard number of $\tilde{S}$, respectively. Since λ is a biratinal invariant, it does not depend on the choice of the smooth proper model $\tilde{S}$

Theorem B. The following two statements are equivalent.

(i) $\lambda=3p_{sc}+4p_{nsc}$.

(ii) For any non-self-conjugate primitive form f, the abelian varieties $A_{f,1}$ and $A_{f,2}$ are not isogenous as K_f-abelian varieties. For any self-conjugate primitive form f, $A_{f,1}$ (or equivalently $A_{f,2}$) is not of CM-type.

Corollary. If $p_{nsc}=0$, then $\lambda=3p_{sc}$. (More generally, $3p_{sc}+2p_{nsc}\leq\lambda$).

Remark. The equality $\lambda \leqq 3p_{sc}+4p_{nsc}$ is proved by Hirzebruch [6].

Let us explain the outline of our proofs. In the first four sections, we construct abelian variety A(f) of dimension 4d defined over $\mathbb{C}$, attached to a primitive form f, where $d=[K_f:Q]$. Here we use an idea of Satake [9], Kuga-Satake [8], and Deligne [2]. More precisely speaking, we attach a polarized Hodge structure H(f) of "poids" 2 with K_f action to each primitive form (§2). And next, following Kuga-Satake [8], we construct abelian varieties, by using the Cifford algebras attached to polarization forms on H(f).

In section 5, we consider two topological involutions on Hilbert modular surfaces. And by means of these involutions, we show that our Clifford algebras are isomorphic to a direct product of two 2×2 matrix algebra over K_f. By construction, our Clifford algebras act on A(f). Accordingly, we have a decomposition up to isogenie $A(f) \sim A_{f,1} \times A_{f,1} \times A_{f,2} \times A_{f,2}$.

In section 6, we calculate explicitly the period lattices of these two abelian varieties. In this calculation, the period relation of Riemann-Hodge plays a key role.

The rest of this note discusses the application of theses result to the problem of counting the number of transcendental cycles on Hilbert modular surfaces. The sources are following: Manin's idea [18] to represent the period integrals of modular forms by the twisted L-functions of these modular forms; liftings of Doi-Naganuma [11], and Naganuma [12]; Shimura's results on period integrals of elliptic modular forms [15],[16]; the determination of endomorphisms rings of abelian varieties attached to elliptic cusp forms by Ribet [18] and Momose [19].

§ 1. Polarized Hodge structures attached to Hilbert modular surfaces.

(1.0) Definitions.

We fix a real quadratic field $F=Q(\sqrt{d})$ with discriminant d (>0), and assume that F has a unit ε with negative norm $N\varepsilon=-1$. Let H be the complex upper half plane $\{z\in C \mid \text{Im } z>0\}$, and let $SL_2(\mathbb{R})$ be the special linear group of degree 2 with entries in the real number field $\mathbb{R}$. As usual $SL_2(\mathbb{R})$ acts on H via

$$g(z)=(az+b)/(cz+d) \quad \text{for } g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbb{R}), \text{ and } z\in H.$$

The product group $SL_2(\mathbb{R})\times SL_2(\mathbb{R})$ acts on the product $H\times H$ factorwise. Let α_1 and α_2 be two embeddings of an element α of F into $\mathbb{R}$. Then the mapping

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longrightarrow \left(\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}\right)$$

defines an injective homomorphism of groups $SL_2(F) \longrightarrow SL_2(\mathbb{R})\times SL_2(\mathbb{R})$. By means of this injection, the group $SL_2(F)$ acts on $H\times H$, and accordingly its subgroup $\Gamma=SL_2(O_F)$ also acts on $H\times H$. Here O_F is the ring of integers of F. Γ acts properly discontinuously on $H\times H$, and the quotient analytic space $S=\Gamma\backslash H\times H$ has a natural structure of quasi-projective algebraic surface (cf. Baily-Borel [30] for example), which is smooth except finite number of quotient singularities corresponding to elliptic fixed points on $H\times H$ with respect to Γ. The surface S is callled the Hilbert modular surface attached to Γ, or to F.

Let $\overline{S}$ be the standard compactification, which is a union of S and $SL_2(O_F)\backslash P^1(f)$ as a set. Here $P^1(F)=F\cup\{\infty\}$ is the 1-dimensional projective line over F. As is well known, the cardinality of the finite set $SL_2(O_F)\backslash P^1(F)$ is equal to the class number h_F of

F. An element of $SL_2(O_F)\backslash P^1(F)$ is called an equivalence class of cusps, or by an abuse of languages, simply called a cusp. $\bar{S}$ has singularities at these cusps, whose resolution is precisely studied by Hirzebruch [5].

(1.1) <u>Cohomology groups of Hilbert modular surfaces</u>.

Let us recall some basic facts on the homology and cohomology groups of S and $\bar{S}$. Let $\pi:\tilde{S} \longrightarrow \bar{S}$ be a resolution of the quotient and cusp singularities of $\bar{S}$. Then the cohomology group $H^2(\tilde{S},Q)$ is a direct sum of the image of $\pi^*:H^2(\bar{S},Q) \longrightarrow H^2(\tilde{S},Q)$, and the subspace of $H^2(\tilde{S},Q)$ generated by algebraic cycles which are obtained as irreducible components of the inverse image loci of singularities by . Restricting the intersection form on $H^2(\tilde{S},Q)$ to $\pi^*H^2(\bar{S},Q)$, we can define a nondegenerate pairing on $\pi^*H^2(\bar{S},Q)$.

Lemma. <u>Let $(\tilde{S}',\pi')$ be another desingularization of</u> $\bar{S}$. <u>Then</u> $\pi^*H^2(\bar{S},Q)$ <u>and</u> $\pi'^*H^2(\bar{S},Q)$ <u>are canonically isomorphic as vector spaces with inner products defined by intersection forms</u>.

Proof. Routine. Q.E.D.

By this lemma we can define a unique natural intersection form on the coimage of $\pi^*:H^2(\tilde{S},Q) \longrightarrow H^2(\bar{S},Q)$. Now let us investigate the kernel of π^*. Let $r_1,r_2,\dots,r_k$ (resp. $c_1,c_2,\dots,c_h$) be the set of quotient (resp. cusp) singularities on $\bar{S}$. Put $R_i=\pi^{-1}(r_i)$ and $C_i=\pi^{-1}(c_i)$. Then, considering the exact sequences of the relative cohomology groups, we have

$$\begin{tikzcd}
{} \arrow[r] & 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H^2(S \bmod \{r_1,\dots,r_k\}\cup\{c_1,\dots,c_h\},Q) \arrow[r, "j"] \arrow[d, "\# \cong"] \arrow[dr, phantom, "\circlearrowleft"] & H^2(\bar{S},Q) \arrow[r] \arrow[d] & 0 \arrow[d] \\
{} \arrow[r, dash] & H^1(\bigcup_{i=1}^{k} R_i \cup \bigcup_{i=1}^{h} C_i,Q) \arrow[r, hook] & H^2(\tilde{S} \bmod \bigcup_i R_i \cup \bigcup_i C_i,Q) \arrow[r] & H^2(\tilde{S},Q) \arrow[r] & * \arrow[r, dash] & {}
\end{tikzcd}$$

Here the isomorphism $\#$ is obtained by the excision theorem. By means of this diagram, we have $\mathrm{Image}\,\partial = \mathrm{Ker}\,\pi^*$. Since r_i ($1\leqq i\leqq k$) are quotient singularities, $H^1(R_i,Q)=0$. Therefore, we have the following proposition.

Proposition. <u>The sequence</u>

$$\bigoplus_{i=1}^{h} H^1(C_i,Q) \xrightarrow{j\#^{-1}\partial} H^2(\overline{S},Q) \longrightarrow H^2(\tilde{S},Q)$$

<u>is exact</u>.

We apply this exact sequence for calculation of the mixed Hodge structure of $H^2(\overline{S},Q)$. From now on we consult with Deligne [4].

(1.2) <u>The mixed Hodge structures of Hilbert modular surfaces.</u>

Let us consider the mixed Hodge structure of $H^2(S,Q)$, $H_c^2(S,Q)$ and $H^2(\overline{S},Q)$. By means of the previous proposition, we have an exact sequence of cohomological descent (cf. Saint-Donat [17]), we can calculate the mixed Hodge structure of $H^2(\overline{S},Q)$ by this(Deligne[4]). By results of Hirzebruch [5], we can take as C_i a stable curve of genus 1, which is not an elliptic curve (the socalled Neron's N-polygon). Therefore, $H^1(C_i,Q)\cong Q$, and the weight filtration of the mixed Hodge structure of $H^1(C_i,Q)$ is given by

$$W_kH^1(C_i,Q)=0 \quad (k<0),\ W_0H^1(C_i,Q)\cong Q \ (k=0),\ W_kH^1(C_i,Q)=H^1(C_i,Q) \ (k\geqq 0).$$

Therefore the weight filtration of $H^2(\overline{S},Q)$ is given by

$$W_kH^2(\overline{S},Q)=0 \quad (k<0),$$
$$W_0H^2(\overline{S},Q)\cong \oplus Q \quad (k=0),$$
$$W_1H^2(\overline{S},Q)=W_0H^2(\overline{S},Q) \quad (k=1),$$
$$W_2H^2(\overline{S},Q)=H^2(\overline{S},Q) \quad (k\geqq 2).$$

Moreover

$$Gr_2WH^2(\overline{S},Q)=W_2H^2(\overline{S},Q)/W_1H^2(\overline{S},Q)$$

is a polarized homogeneous sub-Hodge structure of the polarized Hodge structure of $H^2(S,\mathbb{Q})$.

By means of the trivial isomorphism $H^2_c(S,Q)\cong H^2(\overline{S},Q)$, we can transcribe the mixed Hodge structure on $H^2(\overline{S},\mathbb{Q})$ to $H^2_c(S,Q)$.

Since S is rationally smooth, we have the Poincare duality

$$H^2(S,\mathbb{Q})\times H^2_c(S,\mathbb{Q}) \longrightarrow Q[-2].$$

Therefore the weight filtration of the mixed Hodge structure on $H^2(S,\mathbb{Q})$ is given by

$$W_kH^2(S,\mathbb{Q})=0 \quad (k\leqq 1),$$
$$W_2H^2(S,\mathbb{Q})=W_3H^2(S,\mathbb{Q}),$$
$$W_kH^2(S,\mathbb{Q})=H^2(S,\mathbb{Q}) \qquad (k\geqq 4),$$

and $W_2H^2(S,Q)$ is a homogeneous polarized Hodge structure of "poids" 2, and $Gr_4WH^2(S,\mathbb{Q})\cong Q[-2]$.

(1.3) <u>The Hodge decomposition of the pure part</u> $W_2H^2(S,\mathbb{Q})$ <u>of</u> $H^2(S,\mathbb{Q})$.

From now on, for simplicity, we assume that the class number h_F of F is 1. Let $\mathcal{H}=\mathcal{H}(\Gamma,\Delta)$ be the Hecke algebra $\Gamma=SL_2(O_F)$ with respect to the commensurator $\Delta=\{\alpha\in M_2(O_F) \mid \det\alpha\in O_F-\{0\}\}$. Since we can naturally regard any element of $\mathcal{H}$ as an algebraic correspondence of S, the Hecke algebra $\mathcal{H}$ acts on $H^2(S,\mathbb{Q})$, $H^2_c(S,\mathbb{Q})$, and $H^2(S,\mathbb{Q})$.

<u>Remark</u>. We can write the cohomology groups $H^2(S,\mathbb{Q})$, and $H^2_c(S,\mathbb{Q})\cong H^2(\overline{S},\mathbb{Q})\cong H^2(\overline{S}$ mod cusps,$\mathbb{Q})$ as relative cohomology groups of the discrete group $\Gamma=SL_2(O_F)$. Therefore these cohomology groups are given as cohomology groups of certain complex of Γ-equivariant cochains on $H\times H$. Therefore, the commensurator Δ acts naturally on these cohomology groups.

The Hecke algebra $\mathcal{H}$ acts also on the space $S_2(\Gamma)$ of holomorphic cusp forms on $H\times H$ with respect to

By means of the mapping

$$f(z_1,z_2) \in S_2(\Gamma) \longmapsto (2\pi i)^2 f(z_1,z_2)dz_1\wedge dz_2,$$

we can regard any element of $S_2(\Gamma)$ as a holomorphic 2-form on S. And it is known that these 2-forms are prolonged to holomorphic 2-forms on a smooth model $\tilde{S}$, and moreover by this mapping $S_2(\Gamma)$ is canonically isomorphic to $\Gamma(\tilde{S},\Omega^2_{\tilde{S}})$ (cf. Freitag [3]). Especially, we have

$$\dim_C S_2(\Gamma)=p_g(\tilde{S}).$$

Now the Hodge decomposition of $W_2H^2(S,Q)$ is given by the following theorem.

Theorem (1.1). (Hirzebruch [6]).

<u>Let</u> F <u>be a real quadratic field with has units with negative norm</u>. <u>Let</u> ε <u>be such a unit</u>, <u>whose two embeddings into</u> R <u>satisfies</u> $\varepsilon_1>0$, $\varepsilon_2<0$, $\varepsilon_1\varepsilon_2=-1$. <u>Then as</u> Γ<u>-modules</u>, <u>we have a natural isomorphism</u>

$$\begin{aligned} W_2H^2(S,Q)\otimes_Q C \cong &\{(2\pi i)^2 f(z_1,z_2)dz_1\, dz_2 \mid f\in S_2(\Gamma)\} \\ &\oplus (2\pi i)^2 f(\varepsilon_1 z_1,\varepsilon_2\bar{z}_2)dz_1\wedge d\bar{z}_2 \mid f\in S_2(\Gamma)\} \\ &\oplus\{(2\pi i)^2 f(\varepsilon_2\bar{z}_2,\varepsilon_1 z_1)d\bar{z}_1\wedge dz_2 \mid f\in S_2(\Gamma)\} \\ &\oplus\{(2\pi i)^2 f(-\bar{z}_1,-\bar{z}_2)d\bar{z}_1\wedge d\bar{z}_2 \mid f\in S_2(\Gamma)\} \\ &\oplus \mathbb{C}\,\frac{dz_1\wedge d\bar{z}_1}{y_1^2} \oplus \mathbb{C}\,\frac{dz_2\wedge d\bar{z}_2}{y_2^2}. \end{aligned}$$

<u>Here</u> $z_i=x_i+\sqrt{-1}y_i$ $(i=1,2)$.

(1.4) <u>The primitive part</u> $W_2H^2(S,\mathbb{Q})_{pr}$ <u>of</u> $W_2H^2(S,\mathbb{Q})$.

Let L_1 and L_2 be the invertible sheaves corresponding to the automorphic factors $j_i(z,g)=(c_iz_i+d_i)(i=1,2)$, respectively. Here

$$z=(z_1,z_2)\in H\times H,\ \text{and}\ \ g=(\begin{pmatrix}a_1 & b_1\\ c_1 & d_1\end{pmatrix},\begin{pmatrix}a_2 & b_2\\ c_2 & d_2\end{pmatrix})\in SL_2(R)\times SL_2(R).$$

We can prolong these two sheaves to two invertible sheaves L_1 and L_2 on a smooth proper model of S by the results of Ueno-Van der Geer [24]. The Chern calsses of theseinvertible sheaves define two elements of $H^2(S,\mathbb{Q})$, and accordingly two elements of $W_2H^2(S,\mathbb{Q})$. Let $W_2H^2(S,\mathbb{Q})_{pr}$ be the orthogonal complement of the space spanned by these two elements, with respect to the intersection form. Then the Hodge decomposition of $W_2H^2(S,\mathbb{Q})_{pr}$ is given by the right hand side of the isomorphism of Theorem (1.1), without the last two direct factors

$$\mathbb{C}\,\frac{dz_1\wedge d\bar{z}_1}{y_1^2}\ \oplus\ \mathbb{C}\,\frac{dz_2\wedge d\bar{z}_2}{y_2^2}\ .$$

Because the Chern forms of L_1 and L_2 are given the above two (1,1)-type 2-forms.

§2. Polarized Hodge structures attached to primitive cusp forms of weight 2.

Definition. A Hilbert modular cusp form f ($\neq 0$) of weight 2 with respect to Γ is called primitive, if it is a common eigenfunction of all Hecke operators: $T_{\mathfrak{a}}f=a_{\mathfrak{a}}f$ ($\mathfrak{a}\subset O_F$).
We denote by K_f a subfield of $\mathbb{C}$, generated by eigenvalues $a_{\mathfrak{a}}$ over $\mathbb{Q}$.

Lemma. K_f is contained in $\mathbb{R}$.

Proof. Well-known. Recall that in our case all Hecke operators $T_{\mathfrak{a}}$ are self-adjoint with respect to the Petersson metric (,) on $S_2(\Gamma)$. Q.E.D.

Let ϕ_f be the homomorphism of $\mathcal{H}$ into K_f, defined by the mapping $\phi_f: T_{\mathfrak{a}} \longrightarrow a_{\mathfrak{a}}$.

Put $\tilde{H}_2(S,\mathbb{Q})_{pr}=\mathrm{Coimage}(j_*:H_2(S,\mathbb{Q}) \longrightarrow H_2(\tilde{S},\mathbb{Q}))/\{$the space generated by the homology classes correponding to the line bundles L_1 and $L_2\}$,

and define $H_2(f)$ by

$$H_2(f)=\tilde{H}_2(S,\mathbb{Q})_{pr}\otimes_{(\mathcal{H},\phi_f)}K_f.$$

Then the dual space of $H_2(f)$ over K_f is canonically isomorphic to

$$H^2(f)=\{\eta\in W_2H^2(S,\mathbb{Q})_{pr}\otimes_{\mathbb{Q}}K_f \mid T_{\mathfrak{a}}\eta=a_{\mathfrak{a}}\eta\}.$$

Theorem (2.1). Let f be aprimitive form of weight 2.

(i) The restriction $\psi_f:H^2(f)\times H^2(f) \longrightarrow K_f$ of the extension of scalars

$$\psi\otimes_{\mathbb{Q}}K_f:W_2H^2(S,\mathbb{Q})_{pr}\otimes_{\mathbb{Q}}K_f\times W_2H^2(S,\mathbb{Q})_{pr}\otimes_{\mathbb{Q}}K_f \longrightarrow K_f$$

of the intersection form ψ, *is a nondegenerate symmetric bilinear form.*

(ii) *The extension of scalars of* ψ_f *with respect to the injection* $K_f \hookrightarrow \mathbb{R}$

$$\psi_f \otimes_{K_f} \mathbb{R} : H^2(f) \otimes_{K_f} \mathbb{R} \times H^2(f) \otimes_{K_f} \mathbb{R} \longrightarrow \mathbb{R}$$

is of signature (2+,2−), *and a polarization of a homogeneous Hodge structure of "poids" 2 given by* (iii)

(iii) $H^2(f) \otimes_{K_f} \mathbb{C}$ *has a Hodge decomposition*:

$$H^2(f) \otimes_{K_f} \mathbb{C} \cong \mathbb{C}(2\pi i)^2 f(z_1, z_2) dz_1 \wedge dz_2 \oplus \mathbb{C}(2\pi i)^2 f(\varepsilon_1 z_1, \varepsilon_2 \bar{z}_2) dz_1 \wedge d\bar{z}_2$$
$$\oplus\, \mathbb{C}(2\pi i)^2 f(\varepsilon_2 \bar{z}_1, \varepsilon_1 z_2) d\bar{z}_1 \wedge dz_2 \oplus \mathbb{C}(2\pi i)^2 f(-\bar{z}_1, -\bar{z}_2) d\bar{z}_1 \wedge d\bar{z}_2 .$$

Especially $\dim_{K_f} H^2(f) = 4$.

Proof. We first recall the following basic fact.

Multiplicity One Theorem. *Let* f *and* g *be two primitive forms of weight* 2. *If* $T_{\mathfrak{n}} f = a_{\mathfrak{n}} f$ *and* $T_{\mathfrak{n}} g = a_{\mathfrak{n}} g$ *for all integral ideal of* O_F. *Then* f *is a constant multiple of* g.

Evidently, we can canonically identify $H^2(f) \otimes_{K_f} C$ with

$$\{ \eta \in W_2 H^2(S,Q)_{pr} \otimes_Q C \quad T_{\mathfrak{n}} \eta = a_{\mathfrak{n}} \eta \} .$$

By Multiplicity One Theorem
and Theorem (1.1), this sapce is isomorphic to the right hand side of the statement (iii). So (iii) is proved.

Let $f_1, f_2, \ldots\ldots, f_{p_g}$ be basis of $S_2(\Gamma)$ consisting of primitive forms. Then we have

$$W_2 H^2(S,Q)_{pr} \otimes_Q \mathbb{C} = \bigoplus_{i=1}^{p_g} H^2(f_i) \otimes_{K_{f_i}} C .$$

Hence $\qquad W_2H^2(S,Q)_{pr}\otimes_{Q}\mathbb{R}=\bigoplus_{i=1}^{p_g} H^2(f_i)\otimes_{K_{f_i}}\mathbb{R}.$

Let us consider $\psi\otimes\mathbb{R}$. Let f_1 and f_2 be two primitive form such that $f_1\neq cf_2$ for any $c\in\mathbf{C}$, and denote by $a_{\mathfrak{a},1}$ and $a_{\mathfrak{a},2}$ the eigenvalues of $T_{\mathfrak{a}}$ for f_1 and f_2, respectively. Then, for $\eta_1\in H^2(f_1)\otimes_{K_{f_1}}\mathbb{R}$ and $\eta_2\in H^2(f_2)\otimes_{K_{f_2}}\mathbb{R}$, we have

$$a_{\mathfrak{a},1}\langle\eta_1,\eta_2\rangle=\langle a_{\mathfrak{a},1}\eta_1,\eta_2\rangle=\langle T_{\mathfrak{a}}\eta_1,\eta_2\rangle=\langle\eta_1,T_{\mathfrak{a}}\eta_2\rangle=\langle\eta_1,a_{\mathfrak{a},2}\eta_2\rangle=a_{\mathfrak{a},2}\langle\eta_1,\eta_2\rangle$$

where $\langle\ ,\ \rangle$ is the intersection form $\psi\otimes\mathbb{R}$.

Since $f_1\neq cf_2$ for any $c\in\mathbf{C}$, by Multiplicity One Theorem, there exist some ideals $\mathfrak{a}$ such that $a_{\mathfrak{a},1}\neq a_{\mathfrak{a},2}$. Hence $\langle\eta_1,\eta_2\rangle=0$

Therefore the bilinear form $\psi\otimes_{Q}\mathbb{R}$ is a direct sum of $\psi_{f_i}\otimes\mathbb{R}$ $(i=1,2,\dots,p_g)$, and consequently each ψ_{f_i} is nondegenerate.

Thus (i) is proved.

Because of the index theorem of Hodge, the bilinear form $\psi\otimes_{Q}\mathbb{R}$ has signature $(2p_g+,2p_g-)$, where $p_g=\dim_{\mathbf{C}}S_2(\Gamma)$. For we took the "primitive part" of $W_2H^2(S,\mathbb{Q})$. The Hodge decomposion of $H^2(f)\otimes_{K_f}\mathbf{C}$ and the fact that Hecke operator commute with the C operator of Weil imply (ii). Q.E.D.

3. Q-basis theorem.

Let $S_2(\Gamma;\mathbb{Q})$ be the space of Hilbert cusp forms, whose Fourier coefficients are rational numbers. Then the following theorem is known.

Q-Basis Theorem (3.1). (Shimura [14], or Rapoport [10]).
(i) $S_2(\Gamma;\mathbb{Q})$ spans $S_2(\Gamma)$. In other words, there is a natural isomorphism $S_2(\Gamma;\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C} \cong S_2(\Gamma)$.
(ii) Let $f(z_1,z_2)$ be an element of $S_2(\Gamma)$ with Fourier expansion

$$f(z_1,z_2)=\sum_{\nu\in\delta_+^{-1}} a(\nu)\exp[2\pi i(\nu_1 z_1+\nu_2 z_2)],$$

where δ_+^{-1} is the intersection of the codifferent and the set of totally positive elements.
For any element $\sigma\in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, we define $f^{\sigma}(z_1,z_2)$ by a formal Fourier expansion

$$f^{\sigma}(z_1,z_2)=\sum_{\nu\in\delta_+^{-1}} a(\nu)^{\sigma}\exp[2\pi i(\nu_1 z_1+\nu_2 z_2)].$$

Then $f^{\sigma}(z_1,z_2)$ belongs to $S_2(\Gamma)$.

Corollary (3.2). Let f be a primitive form of weight 2 such that $T_{\mathfrak{a}}f=a_{\mathfrak{a}}f$. And let $\{\sigma_1=\mathrm{id}_{K_f},\ \sigma_2,\ \ldots,\sigma_d\}$ be the set of all embeddings of K_f into $\mathbb{R}$, where $d=[K_f:\mathbb{Q}]$. Then there exist primitive forms in $S_2(\Gamma)$, such that

$$T_{\mathfrak{a}}f_i=\sigma_i(a_{\mathfrak{a}})f_i \qquad (i=2,3,\ldots,d)$$

for all integral ideal $\mathfrak{a}$. Especially we have $K_{f_i}=\sigma_i(K_f)$ $(1\leq i\leq d)$, and K_f is a totally real field.

§ 4. Clifford algebras and abelian varieties.

Let us recall the data given in the previous section.

Data (4.1)

K_f: a totally real algebraic number field of degreee d over $\mathbb{Q}$.

$\{\sigma_1, \sigma_2, \ldots, \sigma_d\}$ be the set of all embeddings of K_f to $\mathbb{R}$.

$H(f)$: a K_f-module of rank 4 on which a symmetric non-denerate bilinear form ψ_f with values in K_f is defined, which satisfies the following condition:

(i) For any σ_i ($i=1,\ldots,d$), $H(f)\otimes_{K_f,\sigma_i}\mathbb{R}$ is equipped with a homogeneous Hodge structure of "poids" 2, and $\psi\otimes_{K_f,\sigma_i}\mathbb{R}$ gives a polarization with respect to this Hodge structure.

Let start from this data, and construct an abelian variety attached to this data. First we choose an integral structure. From now on, until the last part of this section, we omit the subscript f to simplify the notation. Now let O_K be an order of K, and let H_Z be an O_K-module of rank 4 in H. And moreover we can choose H_Z such that the values of ψ on $H_Z \times H_Z$ is contained in O_K.

Let $C^+(H_Z)$ be the even Clifford alegebra attached to over O_K. Thus this algebra is of rank $2^3=8$ over O_K. There is a sophisticated description to attach abelian variety by Deligne [2]. We recall here more naive definition of [8].

First, we consider a real torus of dimension 8d

$$C^+(H_Z)\otimes_Z R/C^+(H_Z).$$

Clearly on this real torus, the Clifford algebra $C^+(H_Z)$ acts by left multiplication. Next, we define a complex structure on this real torus.

Let us note the natural isomorphism

$$C^+(H_Z)\otimes_Z R \simeq \bigoplus_{i=1}^{d} C^+(H_Z)\otimes_{O_K,\epsilon_i} \mathbb{R} \simeq \bigoplus_i C^+(H_Z \otimes_{O_K,\epsilon_i} R).$$

Let us consider the intersection

$$H_Z \otimes_{O_K,\epsilon_i} \mathbb{R} \bigcap \left(H^{2,0} \oplus H^{0,2}\right),$$

for each i (i=1,2,...,d). Choose an orthognal basis e_i^+, e_i^- of this intersection subspace with respect to $\psi \otimes_{\epsilon_i} \mathbb{R}$. Then,

$J_i = e_i^+ e_i^-$ (i=1,2,...,d) defines a complex structure on each factor $C^+(H_Z \otimes_{O_K,\epsilon_i} \mathbb{R})$ by means of right multiplication. Therefore the direct sum $J = \bigoplus_i J_i$ of J_i defines a complex structure on our real torus by right multiplication.

Theorem (4.2) <u>This complex torus has a structure of abelian variety</u>.

Proof. It is well-known that a complex torus with sufficient many endomorphisms becomes automatically an abelian variety, as in our case. ~~And the homogene~~ See also Kuga-Satake [8], or Deligne [2] for K=Q. Q.E.D.

<u>Remark</u>. The dimension of this abelian variety is equal to 4d. The isogenie class of this abelian variety does not depend upon the choice of the integral structure H_Z in H.

§ 5. Involutions of Hilbert modular surfaces and decomposition of abelian varieties.

Let ε be a unit such that $\varepsilon_1>0$ and $\varepsilon_2<0$. Let G_∞, H_∞, and F_∞ be involutions on H H defined by

$$G_\infty:\ (z_1,z_2)\in H\times H \longmapsto (\varepsilon_1 z_1,\varepsilon_2\bar{z}_2)\in H\times H,$$
$$H_\infty:\ (z_1,z_2)\in H\times H \longmapsto (\varepsilon_2\bar{z}_1,\varepsilon_1 z_2)\in H\times H,$$
$$F_\infty:\ (z_1,z_2)\in H\times H \longmapsto (-z_1,-z_2)\in H\times H.$$

Clearly, we have $G_\infty H_\infty=H_\infty G_\infty=F_\infty$.

On passing to the quotient S, we obtain involutuions on H H, which we denote by the same symbol

$$G_\infty:\ S\longrightarrow S,\ H_\infty:S\longrightarrow S,\ F_\infty=H_\infty G_\infty=G_\infty H_\infty:S\longrightarrow S.$$

Evidently these involutuion acts on the homology groups and cohomology groups $H_2(S,Q)$, $H^2(S,Q)$, $W_2H^2(S,Q)$ etc..

Note here that G_∞ and H_∞ changes the orientation. To check this apply G_∞ and H_∞ to $dz_1\wedge d\bar{z}_1\wedge dz_2\wedge d\bar{z}_2$.

Remark. F_∞ is the Frobenius at infinity. Namely let S_R be a canonical model of S defined over the real number field $\mathbb{R}$. Then F_∞ coincides with the action of the nontrivial element of Gal(C/R) on $\mathbb{C}$-valued points S of S_R.

Definition.- Proposition (5.1) Put

$$H_{++}(f)=\{\alpha\in H(f)\ |\ G_\infty\alpha=\alpha\ ,\ H_\infty\alpha=\alpha\},$$
$$H_{+-}(f)=\{\alpha\in H(f)\ |\ G_\infty\alpha=\alpha\ ,\ H_\infty\alpha=-\alpha\},$$
$$H_{-+}(f)=\{\alpha\in H(f)\ |\ G_\infty\alpha=-\alpha\ ,\ H_\infty\alpha=\alpha\},$$
$$H_{--}(f)=\{\alpha\in H(f)\ |\ G_\infty\alpha=-\alpha\ ,\ H_\infty\alpha=-\alpha\}.$$

Then we have a direct sum decomposition

$$H(f)=H_{++}(f)\oplus H_{+-}(f)\oplus H_{-+}(f)\oplus H_{--}(f).$$

Moreover $H_{++}(f)$, $H_{+-}(f),\dots,H_{--}(f)$ are all of rank one over K_f.

Proof. Note that the Hecke operators commutes with G_∞, H_∞, and F_∞. And use the Hodge decomposition (Theorem (1.1)).

Noting that G_∞ and H_∞ change the orientation of S, we have the following proposition.

Proposition (5.2). For the intersection form ψ_f, we have

$$\psi_f(H_{++}(f),H_{++}(f))=\psi_f(H_{++}(f),H_{+-}(f))=\psi_f(H_{++}(f),H_{-+}(f))$$
$$=\psi_f(H_{+-}(f),H_{--}(f))=\psi_f(H_{-+}(f),H_{--}(f))=\psi_f(H_{--}(f),H_{--}(f)=0.$$

And $H_{++}(f)$ and $H_{--}(f)$, and $H_{+-}(f)$ and $H_{-+}(f)$ are mutually dual with respect to ψ_f each other.

Corollary (5.3). ψ_f is a kernel form.

Corollary (5.4) $C^+(H(f))$ the even Clifford algebra over $H(f)$ with respect to ψ_f is a direct product of two copies of 2 2 matrices with entries in K_f; $C^+(H(f)) \cong M_2(K_f) \oplus M_2(K_f)$.

Corollary (5.4) Let $A(f)$ be an abelian variety constructed in the previous section. Then we have a decomposition upto isogenie:

$$A(f) \underset{\text{isogenie}}{\sim} A_{f,1}\times A_{f,1}\times A_{f,2}\times A_{f,2}.$$

Here $A_{f,i}$ (i=1,2) are both of dimension $d=[K_f:Q]$, with endomorphism rings $K_f \hookrightarrow \mathrm{End}_C(A_{f,i})\otimes_{\mathbb{Z}} Q$.

6. Explicite calculation of period lattices of $A_{f,1}$ and $A_{f,2}$.

In this section, we give the period lattices of the isogenie classes of abelian varieties represented by $A_{f,1}$ and $A_{f,2}$.

Let f be a primitive form of weight 2, and let K_f be the field of the eigenvalues. Let $\{\sigma_1=id_{K_f}, \sigma_2, \ldots, \sigma_d\}$ be the set of embeddings of K_f into R, where $d=[K_f:Q]$. Let f^{σ_i} be the primitive forms obtained from f by applying σ_i to the Fourier coefficients of f (We normalize f). Let $H_2(f^{\sigma_i})$ be the homology group attached to f^{σ_i} $(i=1,2,\ldots,d)$. Let consider the action of G_∞ and H_∞ on $H_2(f^{\sigma_i})$. And define $H_2^{++}(f)$ etc. similarly as the case of cohomology groups. We normalize f such that the "first" Fourier coefficient of f is a rational number.

Put

$$\omega_f=(2\pi i)^2 f(z_1,z_2)dz_1\wedge dz_2$$

for such form f. Then we have the following lemma.

Lemma (6.1)

If $\gamma^{++}\in H_2^{++}(f)$, and $\gamma^{--}\in H_2^{--}(f)$. Then the period integrals

$$\int_{\gamma^{++}}\omega_f \quad \text{and} \quad \int_{\gamma^{--}}\omega_f \quad \text{are real numbers,}$$

and if $\gamma^{+-}\in H_2^{+-}(f)$, and $\gamma^{-+}\in H_2^{-+}(f)$, then ~~we have~~ the period integrals

$$\int_{\gamma^{+-}}\omega_f \quad \text{and} \quad \int_{\gamma^{-+}}\omega_f \quad \text{are purely imaginary numbers.}$$

Definition. Fix the above four cycles γ^{++}, γ^{+-}, γ^{-+}, γ^{--} in $H_2(f)=H_2(S,Q)\otimes_{(\mathcal{H},\phi)} K_f$. And let γ_i^{++}, etc be the element of $H_2(f^{\sigma_i})$, obtained from γ^{++} etc. with respect to conjugation of K_f over Q.

Then we put

$$W_{++}(f^{\sigma_i}) = \int_{\gamma_i^{++}} \omega_{f^{\sigma_i}} \ , \ W_{+-}(f^{\sigma_i}) = \dots\dots\dots\dots\dots\dots$$

Theorem (6.2) <u>The period lattices of</u> (isogenie classes of) $A_{f,1}$ <u>and</u> $A_{f,2}$ <u>are given by</u>

$$L_1 = \{ v=(v_1,v_2,v_3,\dots,v_d) \in \mathbb{C}^d \mid v_i = \sigma_i(\alpha) + \sigma_i(\beta) W_{+-}(f^{\sigma_i})/W_{++}(f^{\sigma_i}) \quad (i=1,2,\dots,d), \text{ for some } \alpha, \beta \text{ in } K_f. \},$$

<u>and</u>

$$L_2 = \{ v=(v_1,v_2,\dots,v_d) \in \mathbb{C}^d \mid v_i = \sigma_i(\alpha) + \sigma_i(\beta) W_{-+}(f^{\sigma_i})/W_{++}(f^{\sigma_i}) \quad (i=1,2,\dots,d), \text{ for some } \alpha, \beta \text{ in } K_f \}.$$

In the calculation of these periods, we need the following theorem.

Theorem (6.3) <u>(The period relation of Riemann-Hodge).</u>

<u>If</u> $\hat{\psi}_f(\gamma^{++},\gamma^{--}) = \hat{\psi}_f(\gamma^{+-},\gamma^{-+})$ <u>for the intersection form</u> $\hat{\psi}_f$ <u>of</u> $H_2(f)$, <u>we have</u>

$$W_{++}(f)W_{--}(f) = W_{+-}(f)W_{-+}(f).$$

Proof of Theorem (6.2). A very easy computation of Clifford algebras. We omit it here, because of a short of time.

以下，締切時期がせまってきているので日本語で．

前節までで，Abel多様体の構成に関する，general nonsenseはおしまいである．

注．さて京都での研究集会のときに質問された，Shimura[23]，Mountjoy[22]，Hida[21]等によって定義されている，アーベル多様体と，上に構成したものとの関係についてであるが，これについては次のように考えている：上の3論文は，本質的には，ある条件をみたすrational polarized Hodge structureがあれば，それよりアーベル多様体が構成できるというものである（その条件はMountjoy[22]に書かれている）．最初にとる，rational polarized Hodge structureの"poids"をnとするとき，これらの構成法はある意味で，これを，rational polarized Hodge structure of "poids" 1と見做すことになる．従って$n \neq 1$のとき，このアーベル多様体は，"自然な"ものとは思われない．少なくとも，Grothendieck－Deligneのmotivesの考え方では説明不可能である．

さて，特に，やはりHilbert modular cusp formsにアーベル多様体をattachしたHida[21]との関係であるが，この場合"poids" $n = [F:\mathbb{Q}]$（但しFは考えている総実な体）で，$n \neq 1$である．これで納得できない人は，前節，定理（6.2）の周期と[21]での周期を調べてみれば，この二つは全くちがったものであることが容易にcheckできるはずである．

§7. Self-conjugate forms に付随するアーベル多様体.

序文に書かれた定理 A を証明するのがこの節の目標である。この証明は容易であるが長い。概略のみ書く。

定理 (7.1) (= 定理 A)

φ を reellen Neben typus (elliptic modular) cusp form of weight 2, で φ は primitive とする。f をその lifting to Hilbert modular forms とする。A_φ を Shimura [20] のアーベル多様体, $A_\varphi \sim B_\varphi \times B_\varphi'$ をその分解とする。$B_\varphi \sim B_\varphi'$ (isogenie) は知られている。

さてこのとき, $\mathbb{C}$ 上で B_φ と $A_{f,1}, A_{f,2}$ とは isogenie である。▲

(証明の概略) 定理 (6.2) による period lattices の計算による。

$W_{+-}(f)/W_{-+}(f) \in K_f$ はすぐにわかる。よって $A_{f,1}$ と $A_{f,2}$ とは K_f-isogenous. 問題は $W_{++}(f)$, $W_{+-}(f)$, $W_{-+}(f)$, $W_{--}(f)$ の計算である。$k_\varphi = K_f$ はよく知られている。まず次のことがわかる。

命題 (7.2) $\quad W_{++}(f)/W_{+}(\varphi)^2 \in K_f, \quad W_{--}(f)/W_{-}(\varphi)^2 \in K_f$. ▲

(証明の概略) $W_+(\varphi), W_-(\varphi)$ は Shimura [15] [16] で定義されたものとする。この命題は lifting の本来の定義 (Doi-Naganuma [11] Naganuma [12]) から容易に出てくる。▲

問題は $W_{+-}(f)$, $W_{-+}(f)$ の計算である。$W_{+-}(f)=W_{-+}(f)$ としてよい。Riemann-Hodge の period relation（定理 (6.3)）より

$$\{W_{+-}(f)\}^2/\{W_{++}(f)\,W_{--}(f)\} = 1$$

よって

$$\left\{\frac{W_{+-}(f)}{W_{+}(\varphi)\,W_{-}(\varphi)}\right\}^2 \in K_f$$

であることはすぐわかる。だがこれだけでは $W_{+-}(f)/W_{+}(\varphi)W_{-}(\varphi)\in K_f$ はわからない。$W_{+-}(f)/W_{+}(\varphi)W_{-}(\varphi) \in K_f$ は別の方向で証明する。

基本的な考え方は、Oda [ ] の定理 1 を使って、Doi-Naganuma map の adjoint map を考える。すると周期積分を Fourier 係数にもつ，reellen Nebentypus elliptic cusp forms が表われる。（Hirzebruch-Zagier は、これらの周期積分を交点数と考えた。）

Typical な Fourier 係数を c とする。$W_{+-}(f)/c\sqrt{-1} \in K_f$ は自明にちかい。次は、$\sqrt{-1}\,c/W_{+}(\varphi)W_{-}(\varphi)$ を示すことである。

Petersson metric を $(\ ,\ )_{\mathrm{ellipt}}$ $(\ ,\ )_{\mathrm{Hil}}$ で表わすことにする。（適当な normalization が必要であるが）

__定理__ (7.3)　(Oda [25] + α)

$$c\,(\varphi,\varphi)_{\mathrm{ellipt}}\big/(f,f)_{\mathrm{Hil}} \in K_f.$$ ▲

__命題__ (7.4)

$$(\varphi,\varphi)\big/W_{+}(\varphi)W_{-}(\varphi)\sqrt{-1} \in K_\varphi = K_f,$$
$$(f,f)\big/W_{++}(f)W_{--}(\varphi) \in K_f$$ ▲

系 (7.5) $$c\, W_+(\varphi) W_-(\varphi) \sqrt{-1} \Big/ W_{++}(f) W_{--}(f) \in K_f\,.$$ ▲

さて $W_{++}(f) W_{--}(f) \big/ W_+(\varphi)^2 W_-(\varphi)^2 \in K_f$ であった。よって $c\sqrt{-1} \big/ W_+(\varphi) W_-(\varphi) \in K_f$。さて $W_{+-}(f) \big/ c\sqrt{-1} \in K_f$ であったから，これらより

$$W_{+-}(f) \Big/ W_+(\varphi) W_-(\varphi) \in K_f,$$

よって定理 (7.1) は証明された. ▲

§8. 定理 B の証明.

これも全く初等的な周期の計算である. 但し定理 B の系の証明では, Shimura [15] [16] の結果を少し一般化する必要がある (Nebentypus elliptic modular の場合に)。ある種の周期の消えないことをいう必要がある. ここでも再び [25] の Theorem 1 を使う。そして次の命題を得る.

命題 (8.1) f を self-conjugate primitive form of weight 2 で ~~[illegible]~~ Nebentypus elliptic cusp form φ より lifting で得られるとする. このとき $W_{++}(f) = W_+(\varphi)^2$, $W_{+-}(f) = W_{-+}(f) = W_+(\varphi) W_-(\varphi)$, $W_{--}(f) = W_-(\varphi)^2$ は K_f 上 linear independent な数である.

(証明) これは次の Ribet [18], Momose [19] の結果に帰着される.

定理 (8.2) (cf [18], [19]) φ Neben-Typus elliptic modular cusp form of weight 2. Suppose that φ primitive. Let B_φ be the abelian variety attached to φ by Shimura [20]. Then B_φ has no complex multiplication.

命題 (8.1) の否定 $\Longrightarrow$ B_φ は CM type abelian variety となる.

§9. l-adic realization of the tensor product conjecture と問題.

Tensor product conjecture の l-adic realization は S が K3 曲面と双有理同値のときは, Deligne [2] の結果によって, 非常に弱い形であるが O.K. である. S 自身でなく symmetric Hilbert modular surfaces が K3 のときも同様なことがいえる.

問題. $A_{f,1}$, $A_{f,2}$ の定義体を代数体にまで下げること.

予想は F まで下がることを期待している. Level が 1 でない Hilbert modular surface については, これは容易に解ける場合がいろいろとありそうである. Level が 1 の場合は難しく見える. Deligne [2] の証明の論法をそのまま一般化しようとすれば,

Hilbert modular surfaces の変形を考える必要がある.
F が $\mathbb{Q}(\sqrt{53})$, $\mathbb{Q}(\sqrt{61})$, $\mathbb{Q}(\sqrt{73})$ のとき S は elliptic fibre space over $\mathbb{P}^1$ と双有理同値で, これについては Akao[20] の結果がある.
F が $\mathbb{Q}(\sqrt{89})$, $\mathbb{Q}(\sqrt{97})$ のときは S は $p_g=3$ $c_1^2=2$, よって $c_1^2=2p_g-4$ の general type の曲面と双有理同値である. これの変形についても Horikawa[27] の結果がある. これらの結果から出発して少くとも特別な場合に, 定義体を下げることと, Tensor product conjecture の l-adic realization の弱い形での証明は可能かも知れない. しかし証明に至るいくつかの step はまだ霧の中である. いずれにせよ, 定理 A によって lifting からくる Hilbert modular forms f に対しては $A_{f,1} \sim A_{f,2}$ の定義体は, これが B_φ と isogenous なことから自然に下がっている.

$A_{f,1}$, $A_{f,2}$ の定義は一般化された Birch-Swinnerton-Dyer Conjecture とも compatible であることは容易にわかる. とにかく $A_{f,1}$, $A_{f,2}$ の定義が自然なものであることはまちがいないようである.

Tensor product conjecture の l-adic realization を考えることが次の問題である.

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