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On special values of zeta functions
associated with a self-dual cone

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以下に掲げるのは松島与三氏還暦記念論文集(Birkhauser)のための原稿の一部である．京都の研究集会ではこの後半についてお話ししたので，その要約を提出する予定であったが，都合上原稿(の原稿)のまゝ出させて頂くことにした．本文で説明した通り，こゝに述べる方法は本質的に故新谷氏[11]のアイディアによるものである．$r=2$ (circular cone)の場合にはより精密な計算をすることができ，栗原氏も独立に結果を得ておられるが，これについてはまた別の機会に触れたいと思う．

§1. Introduction

To explain the main idea of this paper, and also to fix some notations, we start with reviewing the classical case of Riemann zeta function. As usual, we set

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (\mathrm{Re}\ s>1),$$

$$\Gamma(s) = \int_0^{\infty} x^{s-1}e^{-x}dx \qquad (\mathrm{Re}\ s>0).$$

Then, for $\mathrm{Re}\ s>1$, one obtains

$$\begin{aligned}\Gamma(s)\zeta(s) &= \sum_{n=1}^{\infty} n^{-s} \int_0^{\infty} x^{s-1}\, e^{-x}\, dx \\ &= \sum_{n=1}^{\infty} \int_0^{\infty} x'^{s-1}\, e^{-nx'}\, dx' \qquad (x = nx') \\ &= \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx .\end{aligned}$$

We put

$$b(x, y) = \frac{e^{xy}}{e^x - 1} = \sum_{\nu=0}^{\infty} \frac{B_\nu(y)}{\nu!}\, x^{\nu-1} \qquad (|x|<2\pi),$$

where

$$B_\nu(y) = \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} b_\mu\, y^{\nu-\mu}$$

is the Bernoulli polynomial, in which the $b_\mu$ are the Bernoulli numbers:

$$b_0 = 1, \quad b_1 = -\frac{1}{2},$$

$$b_\nu = \begin{cases} (-1)^{\frac{\nu}{2}-1} B_{\frac{\nu}{2}} & (\nu \text{ even}, \geqslant 2), \\ 0 & (\nu \text{ odd}, \geqslant 3). \end{cases}$$

Then the above integral can be transformed into a contour integral of the form

$$(1.1) \qquad \Gamma(s)\zeta(s) = (e^{2\pi i s} - 1)^{-1} \int_{I(\varepsilon,\infty)} x^{s-1}\, b(x, 0)\, dx,$$

where $I(\varepsilon, \infty)$ denotes the contour consisting of the half-line $[\varepsilon, \infty)$ taken twice in opposite directions and of a (small) circle of radius $\varepsilon$

about the origin taken in the counterclockwise direction. The contour integral is absolutely convergent for all $s \in \mathbf{C}$, so that the function $\Gamma(s)\zeta(s)$ can be analytically continued to a meromorphic function on $\mathbf{C}$. Moreover, in virtue of the functional equation of the gamma function:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} = 2\pi i \frac{e^{\pi i s}}{e^{2\pi i s} - 1}, \tag{1.2}$$

one obtains

$$\zeta(s) = e^{-\pi i s}\,\Gamma(1-s)\cdot\frac{1}{2\pi i}\int_{I(\varepsilon,\infty)} x^{s-1}\,b(x, 0)\,dx. \tag{1.3}$$

This shows that $\zeta(s)$ is holomorphic for $\mathrm{Re}\, s < 1$. In particular, for $s = 1 - m$, $m \in \mathbf{Z}^+$ (positive integers), the contour integral reduces to the residue of $x^{-m}b(x, 0)$ at $x = 0$, i.e., $b_m/m!$. Hence one obtains

$$\zeta(1-m) = (-1)^{m-1}(m-1)!\,\frac{b_m}{m!} = (-1)^{m-1}\frac{b_m}{m}. \tag{1.4}$$

Thus $\zeta(1-m)$ $(m \in \mathbf{Z}^+)$ is rational. In particular,

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(-1) = -\frac{1}{12},$$

$$\zeta(-2\mu) = 0, \quad \zeta(1-2\mu) = (-1)^{\mu}\frac{B_\mu}{2\mu} \quad (\mu \geqslant 1).$$

This result has been generalized by Hecke, Klingen and Siegel [13] to the case of Dedekind zeta functions of totally real number fields. More recently, Shintani [11] gave a proof based on a direct generaliztion of the classical method explained above. Zeta functions attached to self-dual homogeneous cones have been studied by Siegel [13] in a special case of quadratic cones, and by Sato-Shintani [8] in a more general context of "prehomogeneous spaces". (Cf. also Shintani [9], [10].) On the other hand, the gamma functions attached to self-dual homogeneous cones were studied by Koecher [5], Gindikin [3] and others (cf. e.g., Resnikoff [6]). In this paper, we try to extend Shintani's method (i.e., the classical method) to examine the rationality of the special values of zeta functions attached to self-dual homogeneous cones.

## §2. The gamma function of a self-dual homogeneous cone

2.1. Let $U$ be a real vector space of dimension $n$, endowed with a positive definite inner product $\langle\ \rangle$. By a "cone" in $U$ we always mean a non-degenerate open convex cone in $U$ with vertex at the origin, i.e., a non-empty open set $\mathcal{D}$ in $U$ such that

$$x, y \in \mathcal{D},\ \lambda, \mu \in \mathbb{R}^{+} \Longrightarrow \lambda x + \mu y \in \mathcal{D}$$

and such that $\mathcal{D}$ does not contain any straight line. A cone $\mathcal{D}$ in $U$ is called <u>homogeneous</u> if the group of linear automorphisms

$$G(\mathcal{D}) = \left\{ g \in GL(U) \,\middle|\, g(\mathcal{D}) = \mathcal{D} \right\}$$

is transitive on $\mathcal{D}$; and $\mathcal{D}$ is called <u>self-dual</u> if the "dual" of

$$\mathcal{D}^{*} = \left\{ x \in U \,\middle|\, \langle x, y \rangle > 0 \text{ for all } y \in \bar{\mathcal{D}} - \{0\} \right\}$$

coincides with $\mathcal{D}$.

Let $\mathcal{D}$ be a self-dual homogeneous cone in $U$ and $G = G(\mathcal{D})^{\circ}$. Then it is well-known (e.g., Satake [7]) that the Zariski closure of $G$ (in $GL(U)$) is a reductive algebraic group, containing $G(\mathcal{D})$ as a subgroup of finite index, and $g \longmapsto {}^{t}g^{-1}$ is a Cartan involution of $G$; the corresponding maximal compact subgroup $K = G \cap O(U)$ coincides with the isotropy subgroup of $G$ at a "base point" $e \in \mathcal{D}$ (which is not unique, but will be fixed once and for all). Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be the corresponding Cartan decomposition of $\mathfrak{g} = \text{Lie } G$. Then $\mathfrak{k} = \text{Lie } K$ and one has for $T \in \mathfrak{g}$

$$T \in \mathfrak{k} \iff {}^{t}T = -T \iff Te = 0. \tag{2.1}$$

It follows that, for each $u \in U$, there exists a uniquely determined element $T_u \in \mathfrak{p}$ such that $T_u e = u$. It is well-known that the vector space $U$ endowed with a product

$$u \circ u' = T_u u' \qquad (u, u' \in U)$$

becomes a formally real Jordan algebra (cf. Braun-Koecher [2], or Satake [7]).

We define the (regular) trace on $U$ by

(2.2) $$\tau(u) = tr(T_u).$$

For the given $(\mathcal{D}, e)$, one may assume (by Schur's lemma) that the inner product $\langle\ \rangle$ is so normalized that one has

(2.3) $$\langle u, u' \rangle = \tau(u \circ u') \qquad (u, u' \in U).$$

Next, let $u \in \mathcal{D}$. Then, since $G$ is transitive on $\mathcal{D}$, there exists $g_1 \in G$ such that $u = g_1 e$. We define the (regular) norm $N(u)$ by

$$N(u) = \det(g_1),$$

which is clearly independent of the choice of $g_1$. There exists a unique element $u_1 \in U$ such that $u = \exp u_1$ (which is defined to be $(\exp T_{u_1})e$); then by definition one has

(2.4) $$N(u) = \det(\exp T_{u_1}) = e^{\tau(u_1)}.$$

In terms of the "quadratic multiplication" $P(u) = 2\, T_u^{\,2} - T_{u^2}$, one can also write $N(u) = \det(P(u))^{\frac{1}{2}}$. By the definition, it is clear that

(2.5) $$N(e) = 1, \quad N(gu) = \det(g)\, N(u) \quad (g \in G(\mathcal{D}),\ u \in \mathcal{D}),$$

which characterizes the norm uniquely. Denoting the Euclidean measue on $U$ by $du$, we see that $d_{\mathcal{D}}(u) = N(u)^{-1} du$ is an invariant measure on $\mathcal{D}$.

Example. Let $U = Sym_r(\mathbb{R})$ (the space of real symmetric matrices of degree $r$) and $\mathcal{D} = \mathcal{P}_r(\mathbb{R})$ (the cone of positive definite elements in $U$). Then one has

$$T_u(u') = \frac{1}{2}(uu' + u'u)$$

and so

$$\tau(u) = \frac{r+1}{2}\, tr(u), \qquad N(u) = \det(u)^{\frac{r+1}{2}}.$$

2.2. We define the gamma function of the cone $\mathcal{L}$ by

$$\Gamma_{\mathcal{L}}(s) = \int_{\mathcal{L}} N(u)^{s-1} e^{-\tau(u)} du \tag{2.6}$$

which converges absolutely for Re s sufficiently large (actually for Re s $> 1 - \frac{r}{n}$ as we will see later).

LEMMA 2.1. Suppose that the inner product $\langle\ \rangle$ is normalized by (2.3). Then one has for any $v \in \mathcal{L}$

$$\int_{\mathcal{L}} N(u)^{s-1} e^{-\langle u, v \rangle} du = \Gamma_{\mathcal{L}}(s)\, N(v)^{-s}. \tag{2.7}$$

Proof. Let $v = g_1 e$ with $g_1 \in G$ and put $u' = {}^t g_1 u$. Then one has

$$\langle u, v \rangle = \langle u, g_1 e \rangle = \langle u', e \rangle = \tau(u').$$

Hence by (2.5) the left-hand side of (2.7) is equal to

$$\int_{\mathcal{L}} N(u)^s e^{-\langle u, v \rangle} d_{\mathcal{L}}(u)$$

$$= \int_{\mathcal{L}} (\det(g_1)^{-1} N(u'))^s e^{-\tau(u')} d_{\mathcal{L}}(u')$$

$$= N(v)^{-s}\, \Gamma_{\mathcal{L}}(s), \text{ q.e.d.}$$

It is known that the function $\Gamma_{\mathcal{L}}(s)$ can be expressed as a product of ordinary gamma functions (cf. e.g., Resnikoff loc. cit.). For the sake of completeness, we sketch a proof . First, it is clear that, if

$$\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_m$$

is the decomposition of $\mathcal{L}$ into the direct product of irreducible (self-dual homogeneous) cones, then one has

$$\Gamma_{\mathcal{L}}(s) = \Gamma_{\mathcal{L}_1}(s) \cdots \Gamma_{\mathcal{L}_m}(s).$$

Hence, for our purpose, we may assume that $\mathcal{L}$ is irreducible.

We need the root structure of $\mathfrak{g}$ , which can be determined as follows. Let

$$e = \sum_{i=1}^{r} e_i, \qquad e_i e_j = \delta_{ij}\, e_i \tag{2.8}$$

be a decomposition of $e$ (in the Jordan algebra $U$) into the sum of mutually orthogonal primitive idempotents. ("Primitive" means that each $e_i$ can not be decomposed into the sum of mutually orthogonal idempotents any more.) Then we obtain the following decomposition of $U$ into the direct sum of subspaces ("Peirce decomposition").

$$U = \bigoplus_{1 \leq i \leq j \leq r} U_{ij} , \tag{2.9}$$

where

$$U_{ii} = \left\{ u \in U \;\middle|\; e_i u = u \right\},$$

$$U_{ij} = \left\{ u \in U \;\middle|\; e_i u = e_j u = \tfrac{1}{2} u \right\} \qquad (i \neq j).$$

Then one has $e_k u = 0$ for $u \in U_{ij}$, $k \neq i, j$. Moreover

$$\dim U_{ii} = 1, \quad \dim U_{ij} = d \quad (i \neq j), \tag{2.10}$$

where $d$ is a positive integer depending on the irreducible cone $\mathcal{R}$. (For instance, one has $d = 1$ for $\mathcal{R} = \mathcal{P}_r(R)$.) From (2.9), (2.10) one has the relation

$$n = r + \frac{1}{2} r(r-1)d, \qquad \text{i.e.,} \qquad d = \frac{2(n-r)}{r(r-1)} . \tag{2.11}$$

It follows that

$$\tau(e_i) = tr(T_{e_i}) = 1 + \frac{1}{2}(r-1)d = \frac{n}{r} . \tag{2.12}$$

Put

$$\mathfrak{a} = \left\{ T_{e_i} \quad (1 \leq i \leq r) \right\} . \tag{2.13}$$

Then $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{g}$ of dimension $r$ contained in $\mathfrak{p}$. We denote by $(\lambda_i)$ the basis of $\mathfrak{a}^*$ (the dual space of $\mathfrak{a}$) dual to $(T_{e_i})$, i.e., one has the relation

$$T = \sum_{i=1}^{r} \lambda_i(T)\, T_{e_i} \qquad (T \in \mathfrak{a}).$$

We put $\alpha_{ij} = \frac{1}{2}(\lambda_i - \lambda_j)$ $(i \neq j)$.

PROPOSITION 1. The root system of $\mathfrak{g}$ relative to $\mathfrak{a}$ is given by $\Phi = \{\alpha_{ij}\ (i \neq j)\}$. The root space $\mathfrak{g}(\alpha_{ij})$ corresponding to $\alpha_{ij}$ is given by

$$\mathfrak{g}(\alpha_{ij}) = \{T_u + [T_{e_i - e_j}, T_u] \mid u \in U_{ij}\}. \tag{2.14}$$

This can be verified by a straightforward computation; see e.g., Ash et al. [1] Ch. II, §3. Proposition 1 implies that the R-rank of $\mathfrak{g}$ is equal to $r$ and the root system $\Phi$ is of type $(A_{r-1})$.

2.3. Next we determine the Haar measure of $G$. Put

$$\mathfrak{n} = \sum_{i<j} \mathfrak{g}(\alpha_{ij})$$

and let $A$, $N$ be the analytic subgroups of $G$ corresponding to $\mathfrak{a}$, $\mathfrak{n}$, respectively. Then one has an Iwasawa decomposition $G = NA \cdot K (\approx N \times A \times K)$, which gives rise to the following formula for (the volume element of) a (biinvariant) Haar measure on $G$:

$$dg = c_1\, e^{-2\rho(\log a)} d\underline{n}\, da\, dk \tag{2.15}$$

for $g = \underline{n}ak$ with $\underline{n} \in N$, $a \in A$, $k \in K$, where $d\underline{n}$, $da$, $dk$ denote Haar measures on $N$, $A$, $K$, respectively, $c_1$ is a positive constant depending on the normalization of the Haar measures, and $\rho$ is a linear form on $\mathfrak{a}$ defined by

$$\rho(T) = \frac{1}{2}\operatorname{tr}(\operatorname{ad} T \mid \mathfrak{n}) \qquad (T \in \mathfrak{a});$$

by Proposition 1 one has

$$\rho = \frac{d}{2}\sum_{i<j}\alpha_{ij} = \frac{d}{2}\sum_{i=1}^{r}(r - 2i + 1)\lambda_i. \tag{2.16}$$

The Haar measure of $K$ is always normalized by $\int_K dk = 1$. We make an identification $A = (\mathbf{R}^+)^r$ by the correspondence $a \longleftrightarrow (t_i)$ defined by the relation $a = \exp(\sum \lambda_i T_{e_i})$, $t_i = e^{\lambda_i}$; then one has $da = \prod (dt_i/t_i)$. Moreover one has

$$\begin{aligned} \det(a) &= e^{\tau(\sum \lambda_i e_i)} = e^{\frac{n}{r}\sum\lambda_i} = \Big(\prod_{i=1}^{r} t_i\Big)^{\frac{n}{r}}, \\ a\cdot e &= \sum e^{\lambda_i} e_i = \sum_{i=1}^{r} t_i e_i, \end{aligned} \tag{2.17}$$

$$e^{2\rho(\log a)} = \prod_{i=1}^{r} t_i^{\frac{d}{2}(r-2i+1)}.$$

Since $\mathcal{D} = G/K$, we can normalize the Haar measure of $G$ by the relation $dg = d_{\mathcal{D}}(u)\cdot dk$ where $u = ge$. Then by (2.15), (2.16), (2.17) one has

$$\begin{aligned}
\Gamma_{\mathcal{D}}(s) &= \int_G N(ge)^s e^{-\tau(ge)}\, dg \\
&= c_1 \int_A \det(a)^s e^{-2\rho(\log a)}\, da \int_N e^{-\tau(\underline{n}\cdot ae)}\, d\underline{n} \qquad (2.18)\\
&= c_1 \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} (t_i^{\frac{n}{r}s-\frac{d}{2}(r-2i+1)-1}\, dt_i) \\
&\qquad \times \int_N e^{-\tau(\underline{n}(\sum t_i e_i))}\, d\underline{n}
\end{aligned}$$

To compute the integral over $N$, we introduce some notations. For $u = \sum_{i<j} u_{ij} \in U$ with $u_{ij} \in U_{ij}$, we put

$$\begin{aligned}
T_u^{(+)} &= \tfrac{1}{2}(T_u + \sum_{i<j} [T_{e_i - e_j}, T_{u_{ij}}]), \qquad (2.19)\\
\varepsilon^{(+)}(u) &= \sum_{i<j} \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \sum_{i<k_1<\cdots<k_{\nu-1}<j} u_{ik_1} u_{k_1k_2} \cdots u_{k_{\nu-1}j}.
\end{aligned}$$

Then one has The $U_{ij}$-component of $\varepsilon^{(+)}(u)$ is denoted by $\varepsilon_{ij}^{(+)}(u)$.

LEMMA 2. The notation being as above, one has

$$\begin{aligned}
(2.20)\qquad (\exp T_u^{(+)})(\sum_{i=1}^{r} t_i e_i) &= \sum_{i=1}^{r} (t_i + \tfrac{1}{4} \sum_{k>i} t_k\, \varepsilon_{ik}^{(+)}(u)^2) e_i \\
&\quad + \tfrac{1}{2} \sum_{i<j} (t_j\, \varepsilon_{ij}^{(+)}(u) + \sum_{k>j} t_k\, \varepsilon_{ik}^{(+)}(u)\, \varepsilon_{jk}^{(+)}(u))
\end{aligned}$$

This may be regarded as a generalization of the so-called "Jacobi transformation". The proof is again straightforward. It follows that, if $\underline{n} = \exp T_u^{(+)}$ $(u \in \sum_{i<j} U_{ij})$, one has

$$(2.21)\qquad \tau(\underline{n}(\sum_i t_i e_i)) = \frac{n}{r} \sum_i t_i + \frac{1}{8} \sum_{i<k} \tau(\varepsilon_{ik}^{(+)}(u)^2) t_k.$$

We denote the Euclidean measure on $U_{ij}$ $(i<j)$ (relative to the inner product $\langle\ \rangle$) by $du_{ij}$ and define the Haar measure on $N$ by

$$d\underline{n} = \prod_{i<j} du_{ij} \qquad \text{for } \underline{n} = \exp T_u^{(+)}.$$

Since the map $\varepsilon^{(+)}$ is a bijection of $\sum_{i<j} U_{ij}$ onto itself with jacobian

equal to one, one has

$$du = \prod_{i<j} du_{ij} = \prod_{i<j} du'_{ij},$$

where $u' = \mathcal{E}^{(+)}(u)$. Hence by (2.21) one has

$$\int_N e^{-\tau(\underline{n}\,\sum t_i e_i)}\, d\underline{n} = e^{-\frac{n}{r}\sum t_i} \prod_{i<j} \int_{U_{ij}} e^{-\frac{t_j}{8}\tau(u'^2_{ij})} du'_{ij}$$

$$= e^{-\frac{n}{r}\sum t_i} \prod_{i<j} \left(\frac{8\pi}{t_j}\right)^{\frac{d}{2}}$$

$$= (8\pi)^{\frac{n-r}{2}} \prod_j (t_j^{-\frac{d}{2}(j-1)}\, e^{-\frac{n}{r}t_j}).$$

Inserting this in (2.18), one obtains

$$\Gamma_{\mathcal{L}}(s) = c_1 (8\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \left( \int_0^\infty t_j^{\frac{n}{r}s - \frac{d}{2}(r-j)-1}\, e^{-\frac{n}{r}t_j}\, dt_j \right)$$

$$= c_1 (8\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \left(\frac{n}{r}\right)^{-\frac{n}{r}s + \frac{d}{2}(r-j)} \Gamma\left(\frac{n}{r}s - \frac{d}{2}(r-j)\right)$$

$$= c_1 (8\pi)^{\frac{n-r}{2}} \left(\frac{n}{r}\right)^{-ns + \frac{n-r}{2}} \prod_{j=1}^{r} \Gamma\left(\frac{n}{r}s - \frac{d}{2}(j-1)\right).$$

The constant $c_1$ can be determined by the following observation. We set

$$U_0 = \sum_{i=1}^{r} U_{ii} = \{e_1, \dots, e_r\}_R$$

and denote by $du_0$ the Euclidean measure on $U_0$ (relative to $\langle\ \rangle$). Then, since $\langle e_i, e_j\rangle = \frac{n}{r}\delta_{ij}$, the bijection $A \longrightarrow U_0$ defined by $a = \exp T_{u_0}$, or equivalently by $ae = \exp u_0$, gives the relation

$$du_0 = \left(\frac{n}{r}\right)^{\frac{r}{2}} da.$$

Hence, when

$$u = (\underline{n}a)e = \underline{n}\left(\sum_i t_i e_i\right),$$

$$\underline{n} = \exp T_x^{(+)}, \quad x \in \sum_{i<j} U_{ij}, \quad x' = \mathcal{E}^{(+)}(x),$$

one has by Lemma 2

$$\frac{\partial(u)}{\partial(t, x)} = \frac{\partial(u_0, u_{ij})}{\partial(t_i, x'_{ij})} = \left(\frac{n}{r}\right)^{\frac{r}{2}} \prod_{j=1}^{r} \left(\frac{t_j}{2}\right)^{(j-1)d}$$

$$= 2^{r-n}\left(\frac{n}{r}\right)^{\frac{r}{2}} \prod_{j=1}^{r} t_j^{(j-1)d}.$$

It follows that

$$d_{\mathcal{L}}(u) = 2^{r-n}\left(\frac{n}{r}\right)^{\frac{r}{2}} \prod_{j=1}^{r} \left(t_j^{(j-1)d-\frac{n}{r}} dt_j\right) dx,$$

which, in view of (2.11) and (2.16), implies (2.15) and the relation

(2.22) $$c_1 = 2^{r-n}\left(\frac{n}{r}\right)^{\frac{r}{2}} .$$

Thus we obtain the formula

(2.23) $$\Gamma_{\mathcal{L}}(s) = (2\pi)^{\frac{n-r}{2}} \left(\frac{n}{r}\right)^{n(\frac{1}{2}-s)} \prod_{j=1}^{r} \Gamma\left(\frac{n}{r}s - \frac{d}{2}(j-1)\right).$$

Our computation also shows that the integral for $\Gamma_{\mathcal{L}}(s)$ converges absolutely for $\mathrm{Re}\, s > 1 - \frac{r}{n}$.

From the relation (1.2) one obtains

$$\Gamma_{\mathcal{L}}(s)\, \Gamma_{\mathcal{L}}(1-s) = (2\pi)^{n-r} \prod_{j=1}^{r} \Gamma\left(\frac{n}{r}s - \frac{d}{2}(j-1)\right) \Gamma\left(\frac{n}{r}(1-s) - \frac{d}{2}(r-j)\right)$$

$$= (2\pi)^{n-r}(2\pi i)^{r} \prod_{j=1}^{r} \frac{e^{\pi i(\frac{n}{r}s-\frac{d}{2}(j-1))}}{e^{2\pi i(\frac{n}{r}s-\frac{d}{2}(j-1))} - 1}$$

Since one has by (2.11)

$$n - r = d\,\frac{r(r-1)}{2} \equiv \begin{cases} 0 \pmod 2 & \text{for } d \text{ even} \\ [\frac{r}{2}] \pmod 2 & \text{for } d \text{ odd,} \end{cases}$$

one has

$$\prod_{j=1}^{r} e^{-\pi i \frac{d}{2}(j-1)} = (-i)^{d\frac{r(r-1)}{2}} = \begin{cases} i^{n-r} & \text{for } d \text{ even} \\ i^{n-r}(-1)^{[\frac{r}{2}]} & \text{for } d \text{ odd.} \end{cases}$$

Hence one obtains the following functional equation:

(2.24) $$\Gamma_{\mathcal{L}}(s)\, \Gamma_{\mathcal{L}}(1-s) = (2\pi i)^{n} e^{n\pi i s} \begin{cases} \left(e^{2\pi i \frac{n}{r}s} - 1\right)^{-r} & (d \text{ even}) \\ \left(e^{2\pi i \frac{n}{r}s} - 1\right)^{-[\frac{r+1}{2}]}\left(e^{2\pi i \frac{n}{r}s}+1\right)^{-[\frac{r}{2}]} & (d \text{ odd}). \end{cases}$$

§3. Zeta functions of a self-dual homogeneous cone.

3.1. We fix a $\mathbb{Q}$-structure on $U$ and assume that (the Zariski closure of) $G$ is defined over $\mathbb{Q}$ and $e \in U_{\mathbb{Q}}$; then (the Zariski closure of) $K$ is also defined over $\mathbb{Q}$. We also fix a lattice $L$ in $U$ compatible with that $\mathbb{Q}$-structure, i.e., such that $U_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$, and an arithmetic subgroup $\Gamma$ fixing $L$, i.e., a subgroup of $G_L = \{ g \in G \mid gL = L \}$ of finite index; for simplicity we assume that $\Gamma$ has no fixed point in $\mathcal{L}$. We then define the zeta function associated with $\mathcal{L}$, $\Gamma$, $L$ as follows:

$$\zeta_{\mathcal{L}}(s; \Gamma, L) = \sum_{u \in \Gamma \backslash \mathcal{L} \cap L} N(u)^{-s}, \tag{3.1}$$

the summation being taken over a complete set of representatives of $\mathcal{L} \cap L$ modulo $\Gamma$. It can be shown easily that this series is absolutely convergent for $\operatorname{Re} s > 1$.

By the reduction theory, $\Gamma$ has a fundamental domain in $\mathcal{L}$ which is a rational polyhedral cone. More precisely, there exists a finite set of simplicial cones

$$\begin{aligned} C^{(i)} &= \{v_1^{(i)}, \ldots, v_{\ell_i}^{(i)}\}_{\mathbb{R}_+} \\ &= \Big\{ \sum_{j=1}^{\ell_i} \lambda_j v_j^{(i)} \;\Big|\; \lambda_j \in \mathbb{R}_+ \Big\} \qquad (1 \leq i \leq m), \end{aligned}$$

where $v_1^{(i)}, \ldots, v_{\ell_i}^{(i)}$ are linearly independent elements in $\overline{\mathcal{L}} \cap L$, such that

$$\mathcal{L} = \coprod_{\substack{\gamma \in \Gamma \\ 1 \leq i \leq m}} \gamma C^{(i)}.$$

It follows that

$$\zeta(s; \Gamma, L) = \sum_{i=1}^{m} \sum_{u \in C^{(i)} \cap L} N(u)^{-s}.$$

For a set of linearly independent vectors $v_1, \ldots, v_\ell \in L$, we put

$$R((v_j), L) = \Big\{ \sum_{j=1}^{\ell} \lambda_j v_j \;\Big|\; 0 < \lambda_j \leq 1 \Big\} \cap L,$$

which is finite. Then $u \in C^{(i)} \cap L$ can be written uniquely in the form

$$u = v_0 + \sum_{j=1}^{\ell_i} m_j v_j^{(i)}, \quad v_0 \in R((v_j^{(i)}), L),\ m \in \mathbb{Z},\ m \geq 0.$$

For a set of linearly independent vectors $v_1, \ldots, v_\ell \in \mathcal{D} \cap V_Q$ and $v_0 = \sum_j \alpha_j v_j \in \mathcal{D}$ $(\alpha_j \in Q_+)$, we define a "partial zeta function" by

(3.2) $$\zeta_{\mathcal{D}}(s;\, (v_j),\, v_0) = \sum_{m_j \geq 0} N(v_0 + \sum_{j=1}^{\ell} m_j v_j)^{-s},$$

which will also be written as $\zeta_{\mathcal{D}}(s;\, (v_j),\, (\alpha_j))$. Then the zeta function (3.1) can be written as a finite sum of partial zeta functions as follows:

(3.3) $$\zeta_{\mathcal{D}}(s;\, \Gamma,\, L) = \sum_{i=1}^{m} \sum_{v_0 \in R((v_j^{(i)}), L)} \zeta_{\mathcal{D}}(s;\, (v_j^{(i)}),\, v_0).$$

Hence the study of special values of $\zeta_{\mathcal{D}}(s;\, \Gamma,\, L)$ is reduced to that of the partial zeta functions of the form (3.2).

3.2. Let $(v_j)$ and $v_0$ be as above. Then by (2.7) one obtains

$$\begin{aligned}
\Gamma_{\mathcal{D}}(s)\zeta_{\mathcal{D}}(s;\, (v_j),\, v_0) &= \sum_{\substack{m_j \geq 0 \\ 1 \leq j \leq \ell}} \Gamma_{\mathcal{D}}(s) N(v_0 + \sum_{j=1}^{\ell} m_j v_j)^{-s} \\
&= \sum_{\substack{m_j \geq 0 \\ 1 \leq j \leq \ell}} \int_{\mathcal{D}} N(u)^{s-1} e^{-\sum_j (\alpha_j + m_j) \langle v_j,\, u \rangle} du \\
&= \int_{\mathcal{D}} N(u)^{s} \prod_{j=1}^{\ell} b(\langle v_j,\, u \rangle,\, 1 - \alpha_j)\, d_{\mathcal{D}}(u) \\
&= \int_{G} \det(g)^{s} \prod_{j=1}^{\ell} b(\langle v_j,\, ge \rangle,\, 1 - \alpha_j)\, dg.
\end{aligned}$$

In the notation of § 2, but this time using the decomposition $G = KAK$, one has

(3.4) $$dg = c\, \Delta(a)\, dk \cdot da \cdot dk'$$

for $g = kak'$, $k, k' \in K$, $a \in A$. Here $c$ is a positive constant and

$$\begin{aligned}
\Delta(a) &= \prod_{\alpha \in \Phi_+} (e^{\alpha(\log a)} - e^{-\alpha(\log a)})^{d} \\
&= (\prod_{i=1}^{r} t_i)^{-\frac{d}{2}(r-1)} \, |\Delta(t_1, \ldots, t_r)|^{d},
\end{aligned}$$

where $\Delta(t_1, \ldots, t_r) = \prod_{i<j} (t_i - t_j)$ (cf. Helgason [4], Ch. X, § 1). Hence in view of (2.11) and (2.17) one has

(3.5) $$\Gamma_{\mathcal{D}}(s)\, \zeta_{\mathcal{D}}(s;\, (v_j),\, (\alpha_j)) = c \int_0^{\infty} \cdots \int_0^{\infty} (\prod_{i=1}^{r} t_i)^{\frac{n}{r}(s-1)} |\Delta(t)|^{d} F(t) \prod_{i=1}^{r} dt_i,$$

where

$$F(t_1, \ldots, t_r) = \int_K \prod_{j=1}^{\ell} b(\langle v_j, k \sum t_i e_i \rangle, 1 - \alpha_j)\, dk.$$

It is clear that $F(t_1, \ldots, t_r)$ is holomorphic for $\operatorname{Re} t_i > 0$ $(1 \leq i \leq r)$.

Since $K$ contains an element which induces any given permutation of $e_1, \ldots, e_r$, the function $F$ is symmetric. Hence, denoting by $B_1$ an open simplicial cone in $R^r$ defined by $t_1 > \ldots > t_r > 0$, one has

$$(3.5') \qquad \Gamma_{\mathcal{L}}(s)\, \zeta_{\mathcal{L}}(s; (v_j), (\alpha_j)) = c\, r! \int_{B_1} (\prod t_i)^{\frac{n}{r}(s-1)} \Delta(t)^d F(t) \prod dt_i .$$

3.3. Still following Shintani [11], we make a change of variables $(t_i) \longrightarrow (t_1, \tau_2, \ldots, \tau_r)$ with $\tau_i = t_i/t_{i-1}$ $(2 \leq i \leq r)$. Then $B_1$ can be expressed as

$$B_1 = \left\{ (t_i) \,\middle|\, t_i = t_1 \prod_{j=2}^{i} \tau_j ,\ 0 < t_1 < \infty,\ 0 < \tau_i < 1 \right\}.$$

Putting $\tau_1 = t_1$, one has

$$\frac{\partial(t_1, \ldots, t_r)}{\partial(t_1, \tau_2, \ldots, \tau_r)} = \prod_{i=1}^{r} \tau_i^{r-i},$$

$$\prod t_i = \prod \tau_i^{r-i+1},$$

$$\Delta(t) = \prod \tau_i^{\frac{1}{2}(r-i+1)(r-i)} \prod_{2 \leq i < j \leq r} (1 - \tau_i \ldots \tau_j).$$

It follows that the exponent of $\tau_i$ in the integrand in (3.5') is equal to

$$(r-i+1)\tfrac{n}{r}(s-1) + \tfrac{d}{2}(r-i+1)(r-i) + r - i$$
$$= (r-i+1)\left\{\tfrac{n}{r} s - \tfrac{d}{2}(i-1)\right\} - 1.$$

Hence one has

$$(3.6) \qquad \Gamma_{\mathcal{L}}(s)\, \zeta_{\mathcal{L}}(s; (v_j), (\alpha_j)) = c\, r! \int_0^\infty t^{ns-1}\, dt \int_0^1 \!\!\cdots\! \int_0^1 \prod \tau_i^{(r-i+1)\{\frac{n}{r}s - \frac{d}{2}(i-1)\} - 1}\, \tilde{F}(t_1, \tau) \prod_{i=2}^{r} d\tau_i,$$

where

$$(3.7) \qquad \tilde{F}(t_1, \tau) = \prod_{2 \leq i < j \leq r} (1 - \tau_i \ldots \tau_j)^d\, F(t_1, t_1\tau_2, \ldots, t_1\tau_2 \cdots \tau_r).$$

3.4. We now assume that all $v_j$'s are in $\mathcal{D}$ (not on the boundary of $\mathcal{D}$). (In the situation explained in 3.1, this means that the $\mathbb{Q}$-rank of $G$ is equal to 1.) Then for any $v \in \bar{\mathcal{D}} - \{0\}$, one has $< v_j , v > > 0$; in particular,

(3.8) $$< v_j , ke_i > > 0 \quad \text{for all } k \in K,\ 1 \leqslant i \leqslant r.$$

Put

(3.9) $$\begin{aligned} \xi_j &= < v_j , k \sum t_i e_i > \\ &= t_1 < v_j , k(e_1 + \sum_{i=2}^{r} \tau_2 \cdots \tau_i e_i) > \\ &= t_1 < v_j , ke_1 > (1 + \sum_{i=2}^{r} \tau_2 \cdots \tau_i \frac{< v_j , ke_i >}{< v_j , ke_1 >}). \end{aligned}$$

For the fixed $e_i$, $v_j$, choose $\rho$, $\rho_1 > 0$ in such a way that

(3.10) $$\begin{cases} \displaystyle\sum_{i=2}^{r} \rho^{i-1} \frac{< v_j , ke_i >}{< v_j , ke_1 >} < 1 & \text{for all } k \in K,\ 1 \leqslant j \leqslant \ell, \\ \displaystyle\rho_1 < \frac{\pi}{< v_j , ke_1 >} & \text{for all } 1 \leqslant j \leqslant \ell . \end{cases}$$

Then for

(3.11) $$0 < |t_1| < \rho_1 , \quad |\tau_i| < \rho \qquad (2 \leqslant i \leqslant r),$$

one has $0 < |\xi_j| < 2\pi$ and so $b(\xi_j , 1 - \alpha_j)$ is holomorphic. Hence the function $F(t) = F(t_1 , t_1\tau_2 , \ldots , t_1\tau_2 \cdots \tau_r)$ has a Laurent expansion in $t_1, \tau_2 , \ldots , \tau_r$ in the domain defined by (3.11). The coefficients in this expansion is a Q-linear combination of the integrals of the form

(3.12) $$I((\nu_{ij})) = \int_K \prod_{\substack{1\leqslant i \leqslant r \\ 1 \leqslant j \leqslant \ell}} < v_j , ke_i >^{\nu_{ij}} dk$$

where $\nu_{ij} \geqslant 0$ for $2 \leqslant i \leqslant r$ and $\nu_{ij} \in \mathbb{Z}$ for all $i, j$.

3.5. Let $I(\varepsilon, 1)$ denote the contour consisting of the line segment $[\varepsilon, 1]$ taken twice in opposite directions and of a (small) circle of radius $\varepsilon$ about the origin taken in the counterclockwise direction. When the $\tau_i$ $(2 \leq i \leq r)$ are on $I(\varepsilon, 1)$, one has by (2.12)

$$\left|< v_j , k(e_1 + \sum_{i=2}^{r} \tau_2 \cdots \tau_i e_i) >\right| \leq |v_j| \cdot \sum_{i=1}^{r} |e_i| = \sqrt{nr}\, |v_j|$$

and

$$\begin{aligned} \mathrm{Re} < v_j , k(e_1 + \sum_{i=2}^{r} \tau_2 \cdots \tau_i e_i) > &= < v_j , ke_1 > + \sum_{i=2}^{r} \mathrm{Re}(\tau_2 \cdots \tau_i) < v_j , ke_i > \\ &\geq < v_j , ke_1 > - \varepsilon |v_j| \sum_{i=2}^{r} |e_i| \\ &= < v_j , ke_1 > - \varepsilon (r-1) \sqrt{\tfrac{n}{r}}\, |v_j| . \end{aligned}$$

We choose $\varepsilon$ so that one has

(3.13) $\quad \varepsilon \sqrt{rn}\, |v_j| < \mathrm{Min} \left\{ 2\pi, < v_j , ke_1 > \ (k \in K) \right\}$ for all $1 \leq j \leq \ell$,

Then the above inequalities show that $< v_j , k(e_1 + \sum_{i=2}^{r} \tau_2 \cdots \tau_i e_i) >$ belongs to the domain

$$\left\{ z \in \mathbf{C} \;\middle|\; |z| < \frac{2\pi}{\varepsilon} , \ \mathrm{Re}\ z > \varepsilon \sqrt{\tfrac{n}{r}}\, |v_j| \right\} .$$

It follows that, if $t_1$ is on the contour $I(\varepsilon, \infty)$, one has

$$|\xi_j| < 2\pi \quad \text{or} \quad \mathrm{Re}\ \xi_j > 0,$$

so that the function $b(\xi_j , 1 - \alpha_j)$ is holomorphic.

From this observation, it is clear that the integral on the r.h.s. of (3.6) is equal to the contour integral

$$(e^{2\pi ins} - 1)^{-1} \int_{t_1 \in I(\varepsilon, \infty)} \cdot \prod_{i=2}^{r} (e^{2\pi i \frac{r-i+1}{r} ns} - 1)^{-1} \int_{\tau_i \in I(\varepsilon, 1)}$$

which is independent of the choice of $\varepsilon$ satisfying (3.13). As is easily seen, the contour integral converges for all $s \in \mathbf{C}$. Hence the above integral ~~$I(\alpha(1), \ldots, \alpha(n))$~~, viewed as a function in $s$, can be continued to a meromorphic function on the whole plane; the possible poles are of the form

$$\frac{r\nu}{(r-i+1)n} \quad (\nu \in \mathbf{Z}).$$

§ 4. The special values of the zeta functions.

4.1. As a preliminary, we check the rationality of the constant $c$ in (3.4). For that purpose, we compute $\Gamma_{\mathcal{D}}(s)$ by using the decomposition $G = KAK$.

$$(4.1)\qquad \begin{aligned} \Gamma_{\mathcal{D}}(s) &= \int_{\mathcal{D}} N(u)^s\, e^{-\tau(u)}\, d_{\mathcal{D}}(u) \\ &= \int_{G} N(ge)^s\, e^{-\tau(ge)}\, dg \\ &= c\int_{A} \det(a)^s\, e^{-\tau(ae)}\, \Delta(a)\, da \\ &= c\int_0^\infty\!\!\cdots\!\int_0^\infty (\prod t_i)^{\frac{n}{r}(s-1)}\, |\Delta(t)|^d\, e^{-\frac{n}{r}\sum t_i} \prod dt_i . \end{aligned}$$

We make another change of variables:

$$t = \sum_{i=1}^{r} t_i\,, \qquad t_i' = t_i/t.$$

Then

$$\frac{\partial(t_1, \ldots, t_r)}{\partial(t, t_1', \ldots, t_{r-1}')} = (-1)^{r-1}\, t^{r-1},$$

and the exponent of $t$ in the integrand in the last member of (4.1) is equal to

$$n(s-1) + \frac{d}{2} r(r-1) + r - 1 = ns - 1.$$

Hence one has

$$(4.2)\qquad \Gamma_{\mathcal{D}}(s) = c\cdot\gamma(s)\cdot\beta(s),$$

where

$$(4.3)\qquad \left\{ \begin{aligned} \gamma(s) &= \int_0^\infty t^{ns-1}\, e^{-\frac{n}{r}t}\, dt = (\tfrac{r}{n})^{ns}\, \Gamma(ns), \\ \beta(s) &= \int_{\substack{t_i'>0 \\ \sum t_i'<1}} \{t_1' \cdots t_{r-1}'(1-\textstyle\sum t_i')\}^{\frac{n}{r}(s-1)} \times \\ &\qquad |\Delta(t_1', \ldots, t_{r-1}', 1-\textstyle\sum t_i')|^d \prod dt_i' . \end{aligned} \right.$$

For $s = 1$, one has

$$\Gamma_{\mathcal{D}}(1) = c\cdot\gamma(1)\cdot\beta(1) = c\,(\tfrac{r}{n})^n\,(n-1)!\,\beta(1),$$

$$\beta(1) = \int_{\substack{t_i' > 0 \\ \sum t_i' < 1}} |\Delta(t_1', \ldots, t_{r-1}', 1-\sum t_i')|^d \prod dt_i' \in \mathbb{Q}.$$

By (2.23) one has

$$(4.4) \qquad \Gamma_{\mathcal{D}}(1) = (2\pi)^{\frac{n-r}{2}} \left(\frac{r}{n}\right)^{\frac{n}{2}} \prod_{j=1}^{r} \Gamma\left(1 + \frac{d}{2}(j-1)\right)$$

$$\underset{\mathbb{Q}}{\sim} \begin{cases} \pi^{\frac{n-r}{2}} & (d \text{ even}) \\ \pi^{\frac{1}{2}(n-[\frac{r+1}{2}])} & (d \text{ odd}), \end{cases}$$

where $a \underset{\mathbb{Q}}{\sim} b$ means that $a/b \in \mathbb{Q}$. Thus one has

$$(4.5) \qquad c = \frac{(2\pi)^{\frac{n-r}{2}} \left(\frac{n}{r}\right)^{\frac{n}{2}} \prod_{j=1}^{r} \Gamma\left(1 + \frac{d}{2}(j-1)\right)}{(n-1)! \ \beta(1)} \underset{\mathbb{Q}}{\sim} \Gamma_{\mathcal{D}}(1).$$

Since $\Gamma_{\mathcal{D}}(1) \underset{\mathbb{Q}}{\sim} \Gamma_{\mathcal{D}}(1 + \frac{r}{n}\nu)$ for $\nu \in \mathbb{Z}$, one obtains

$$(4.6) \qquad c\,\Gamma_{\mathcal{D}}(1 + \tfrac{r}{n}\nu) \underset{\mathbb{Q}}{\sim} \Gamma_{\mathcal{D}}(1)^2 \underset{\mathbb{Q}}{\sim} \begin{cases} \pi^{n-r} & (d \text{ even}) \\ \pi^{n-[\frac{r+1}{2}]} & (d \text{ odd}). \end{cases}$$

4.2. We first consider the case where $d$ is even. Then by (2.24) one has

$$\Gamma_{\mathcal{D}}(s)\,\Gamma_{\mathcal{D}}(1-s) = (2\pi i)^n e^{\pi i n s}(e^{2\pi i \frac{n}{r} s} - 1)^{-r}.$$

Hence

$$(4.7) \qquad \zeta_{\mathcal{D}}(s; (v_j), (\alpha_j)) = \frac{c\,\Gamma_{\mathcal{D}}(1-s)}{(2\pi i)^{n-r} e^{\pi i n s}} \times R(s),$$

where

$$R(s) = \left(\frac{e^{2\pi i \frac{n}{r} s} - 1}{2\pi i}\right)^r \cdot r! \int_{B_1} \left(\prod t_i\right)^{\frac{n}{r}(s-1)} \Delta(t)^d F(t) \prod dt_i$$

$$= \prod_{j=1}^{r} \frac{e^{2\pi i \frac{n}{r} s} - 1}{e^{2\pi i \frac{r-j+1}{r} ns} - 1} \times \frac{1}{(2\pi i)^r} \int_{I(\varepsilon,\infty)} t_1^{ns-1}\, dt_1$$

$$\Big(\prod_{r=2}^{r} \int_{\tau_i \in I(\varepsilon,1)}\Big) \prod_{i=2}^{r} \tau_i^{(r-i+1)\{\frac{n}{r} s - \frac{d}{2}(i-1)\}-1}\, r!\, \tilde{F}(t_1, \tau_i) \prod d\tau_i .$$

We are interested in the values of $\zeta_{\mathcal{D}}$ at $s = -\frac{r}{n}\nu$ ($\nu = 0, 1, \ldots$). The first factor in the right hand side of (4.7) is holomorphic for $\mathrm{Re}\, s < \frac{r}{n}$ and by (4.6) the value at $s = -\frac{r}{n}\nu$ is rational:

$$(4.9)\qquad \frac{c\,\Gamma_{\mathcal{D}}(1+\frac{r}{n}\nu)}{(2\pi i)^{n-r}\,e^{-r\nu\pi i}} = (-1)^{\frac{n-r}{2}+r\nu}\,\frac{c\,\Gamma_{\mathcal{D}}(1+\frac{r}{n}\nu)}{(2\pi)^{n-r}} \in \mathbb{Q}.$$

On the other hand, it is clear that

$$\frac{e^{2\pi i \frac{n}{r} s} - 1}{e^{2\pi i \frac{r-i+1}{r} ns} - 1} \longrightarrow \frac{1}{r-i+1} \qquad \text{when } s \longrightarrow -\frac{r}{n}\nu .$$

Hence we see that $R(-\frac{r}{n}\nu)$ is equal to the coefficient of

$$t_1^{r\nu} \prod_{i=2}^{r} \tau_i^{(r-i+1)\{\nu + \frac{d}{2}(i-1)\}}$$

in the Laurent expansion of $\tilde{F}(t_1, \tau)$,

which is a Q-linear combination of $I((\nu_{ij}))$.

4.3. From now on we assume that $d$ is odd. By the classification theory, it is known that this assumption implies that $r = 2$ $(n = d + 2)$ or $d = 1$ $(n = \frac{1}{2}r(r+1))$. By (2.24) one has

$$\Gamma_{\mathcal{D}}(s)\,\Gamma_{\mathcal{D}}(1-s) = (2\pi i)^n\, e^{n\pi i s}\,(e^{2\pi i \frac{n}{r} s} - 1)^{-[\frac{r+1}{2}]}\,(e^{2\pi i \frac{n}{r} s} + 1)^{-[\frac{r}{2}]}.$$

Hence

$$(4.11)\qquad \zeta_{\mathcal{D}}(s; (v_j), (\alpha_j)) = \frac{c\,\Gamma_{\mathcal{D}}(1-s)}{(2\pi i)^{n-[\frac{r+1}{2}]}\,e^{n\pi i s}} \times R^{(1)}(s)\, R^{(2)}(s),$$

where

$$R^{(1)}(s) = (2\pi i)^{[\frac{r}{2}]}\, r!\, \frac{(e^{2\pi i \frac{n}{r} s} - 1)^{[\frac{r+1}{2}]}\,(e^{2\pi i \frac{n}{r} s} + 1)^{[\frac{r}{2}]}}{\prod_{k=1}^{r} (e^{2\pi i (r-k+1)\{\frac{n}{r} s - \frac{d}{2}(k-1)\}} - 1)},$$

$$R^{(2)}(s) = (2\pi i)^{-r} \int_{I(\varepsilon,\infty)} t_1^{ns-1}\,dt_1 \int\cdots\int_{I(\varepsilon,1)^{r-1}} \prod \tau_i^{(r-i+1)\{\frac{n}{r} s - \frac{d}{2}(i-1)\}-1}\,\tilde{F}(t_1,\tau) \prod d\tau_i .$$

The first factor in the right hand side of (4.11) is holomorphic for $\mathrm{Re}\, s < \frac{r}{n}$ and by (4.6) the value at $s = -\frac{r}{n}\nu$ $(\nu \geq 0)$ is rational:

$$(4.12) \qquad \frac{c\, \Gamma_{\mathcal{L}}(1+\frac{r}{n}\nu)}{(2\pi i)^{n-[\frac{r+1}{2}]}\, e^{-\pi i r\nu}} = (-1)^{\frac{1}{2}(n-[\frac{r+1}{2}])+r\nu}\, \frac{c\, \Gamma_{\mathcal{L}}(1+\frac{r}{n}\nu)}{(2\pi)^{n-[\frac{r+1}{2}]}} \in \mathbb{Q}.$$

Note that one has

$$n \equiv [\tfrac{r+1}{2}] \pmod 2,$$

since

$$n = d+2 \equiv 1 = [\tfrac{3}{2}] \pmod 2 \qquad \text{if} \quad r = 2, \text{ and}$$

$$n = \tfrac{1}{2}r(r+1) \equiv [\tfrac{r+1}{2}] \pmod 2 \quad \text{if} \quad d = 1.$$

4.4. To compute $R^{(1)}(s)$, we first note

$$e^{\pi i d(k-1)(r-k+1)} = \begin{cases} -1 & \text{if } k \equiv r \equiv 0 \pmod 2, \\ 1 & \text{otherwise.} \end{cases}$$

We put

$$[\tfrac{r}{2}] = r_1, \qquad \zeta = e^{2\pi i \frac{n}{r} s}.$$

<u>The case r is odd</u>. One has

$$R^{(1)}(s) = (2\pi i)^{r_1}\, r!\, \frac{(\zeta - 1)^{r_1+1}(\zeta+1)^{r_1}}{\prod_{k=1}^{r}(\zeta^k - 1)}$$

$$= \frac{r!}{\prod_{k=1}^{r}(\zeta^{k-1}+\dots+\zeta+1)}\, (2\pi i \cdot \frac{\zeta+1}{\zeta-1})^{r_1}.$$

Hence, when $s \longrightarrow -\frac{r}{n}\nu$, one has

$$(4.13) \qquad (s + \tfrac{r}{n}\nu)^{r_1} R^{(1)}(s) \longrightarrow (2\tfrac{r}{n})^{r_1}.$$

Thus $R^{(1)}(s)$ has a pole of order $r_1$ at $s = -\frac{r}{n}\nu$.

<u>The case r is even</u>. One has

$$R^{(1)}(s) = (2\pi i)^{r_1}\, r!\, \frac{(\zeta - 1)^{r_1}(\zeta+1)^{r_1}}{\prod_{k=1}^{r}\{(-1)^k\zeta^k - 1\}}$$

$$= (-2\pi i)^{r_1} \frac{r!}{\prod\limits_{\substack{1\le k\le r \\ k \text{ even}}} (\zeta^{k-1}+\ldots+\zeta+1) \prod\limits_{\substack{1\le k\le r \\ k \text{ odd}}} (\zeta^{k-1}-\ldots-\zeta+1)}$$

Hence $R^{(1)}$ is holomorphic at $s = -\frac{r}{n}\nu$ and

$$(4.14) \quad R^{(1)}(-\tfrac{r}{n}\nu) = (-2\pi i)^{r_1} \frac{r!}{(2r_1)!!} = (-\pi i)^{r_1} \frac{r!}{r_1!} .$$

4.5. When $r$ is odd (hence $d = 1$, $n = \frac{1}{2}r(r+1)$), $R^{(2)}(s)$ for $s = -\frac{r}{n}\nu = -\frac{2\nu}{r+1}$ is given by the coefficient of

$$t_1^{r\nu} \prod_{i=2}^{r} \tau_i^{(r-i+1)(\nu+\frac{i-1}{2})}$$

in the Laurent expansion of $\widetilde{F}(t_1, \tau)$. Hence $\zeta_{\mathcal{D}}(s; (v_j), (\alpha_j))$ has at most a pole of order $r_1 = \frac{r-1}{2}$ at $s = -\frac{2\nu}{r+1}$ and one has

$$(4.15) \quad \lim_{s\to -\frac{2\nu}{r+1}} (s + \frac{2\nu}{r+1})^{r_1} \zeta_{\mathcal{D}}(s; (v_j), (\alpha_j)) \underset{Q}{\sim} R^{(2)}(-\frac{2\nu}{r+1}).$$

To treat the case $r$ is even, we use the formula

$$\int_{I(\varepsilon,1)} t^{\frac{m}{2}-1}\, dt = -\frac{4}{m} \qquad (m \text{ odd}),$$

which can be verified easily. When $r$ is even, the value of $R^{(2)}(s)$ for $s = -\frac{r}{n}\nu$ is given by

$$(4.16) \quad (-\pi i)^{-r_1} \sum_{m_1,\ldots,m_{r_1}\in\mathbb{Z}} \frac{a_{(m_j)}}{\prod\limits_{j=1}^{r_1} (m_j - (r-2j+1)\{\nu + \frac{d}{2}(2j-1)\})}$$

where $a_{(m_j)}$ is the coefficient of

$$t_1^{r\nu} \prod_{j=2}^{r_1} \tau_{2j-1}^{(r-2j+2)(\nu+d(j-1))} \prod_{j=1}^{r_1} \tau_{2j}^{m_j}$$

in $\widetilde{F}(t_1, \tau)$. Hence for the value of $\zeta_{\mathcal{D}}$, one has

$$(4.17) \quad \zeta_{\mathcal{D}}(-\tfrac{r}{n}\nu ; (v_j), (\alpha_j)) \underset{Q}{\sim} (2\pi i)^{r_1} R^{(2)}(-\tfrac{r}{n}\nu).$$

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