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Fractional Calculus
(Generalized integral and derivative)

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Chapter 1. Fractional derivative and integral of the function of single variable.

§ 1. Introduction

Bertram Ross (Prof. of Univ. of New Haven) made a very useful chronological list (with short comments for each paper), and this list is shown in the volume "The Fractional calculus" of K.B.Oldham and J.Spanier (Academic Press, 1974).

An international conference (Director is Prof. Bertram Ross) for fractional calculus was held June 15 and 16, 1974 at the University of New Haven, and this conference had 22 participants and 72 attendees. All reports in this conference were published as the Lecture Note in Mathematics Vol. 457 (Edited by Bertram Ross), the title of which is "Fractional Calculus and its Applications", by Springer-Verlag.

McBride, A.C. published a volume, the title of which is "Fractional Calculus and integral transforms of generalized functions" (Research Notes in Mathematics Series. Pitman Press), in 1979. Probably this volume is the newest book for fractional calculus at the present time (Oct. 20th, 1979) [36].

§ 2. Some definitions for fractional differintegration of the single variable.

There are many papers, in which the fractional derivative and integral of the function of single variable are discussed (see References). And many definitions on the fractional integral of the function of single variable are reported. Some of them are shown as follows.

By Riemann-Liouville [1]: Fractional integral of order α

$$f_\alpha^+(a, x) = \frac{1}{\Gamma(\alpha)}\int_a^x f(t)(x-t)^{\alpha-1}\,dt \quad \text{(right hand)}, \tag{1}$$

$$f_\alpha^-(x, b) = \frac{1}{\Gamma(\alpha)}\int_x^b f(t)(t-x)^{\alpha-1}\,dt \quad \text{(left hand)}, \tag{2}$$

where $a \leqslant x \leqslant b$, $\alpha > 0$ and Γ is the Gamma function.

Fractional derivative of order α

$$f_{-\alpha}^{+}(a, x) = \frac{d}{dx} f_{1-\alpha}^{+}(a, x) \qquad \text{(right hand)}, \tag{3}$$

$$f_{-\alpha}^{-}(x, b) = \frac{d}{dx} f_{1-\alpha}^{-}(x, b) \qquad \text{(left hand)}, \tag{4}$$

where $a \leqslant x \leqslant b$ and $1 \geqslant \alpha \geqslant 0$.

By Weyl [2]: Fractional integral of order α

$$f_{\alpha}^{+}(-\infty, x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} f(t)(x - t)^{\alpha-1}\, dt, \tag{5}$$

$$f_{\alpha}^{-}(x, +\infty) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} f(t)(t - x)^{\alpha-1}\, dt, \tag{6}$$

where $f(t)$ is a periodic function and it's mean value for one period is zero. But above formulae (5) and (6) are used as the definition of the α-th integral without any condition, at the present time.

By Erdélyi [3]: Fractional integral of order α

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t)\, dt, \quad I_x^0 f(x) = f(x) \tag{7}$$

$$K_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t - x)^{\alpha-1} f(t)\, dt, \quad K_x^0 f(x) = f(x) \tag{8}$$

By Kober [4]: Fractional integral of order α (using Erdélyi's notation)

$$I_x^{\eta,\alpha} f(x) = x^{-\eta-\alpha} I_x^{\alpha} x^{\eta} f(x), \quad I_x^{\eta,0} f(x) = f(x) \tag{9}$$

$$K_x^{\eta,\alpha} f(x) = x^{\eta} K_x^{\alpha} x^{-\eta-\alpha} f(x), \quad K_x^{\eta,0} f(x) = f(x) \tag{10}$$

By Okikiolu [5]: Fractional integral of order α

$$H_{\alpha}(f)(x) = \frac{1}{\phi(\alpha)} \int_{-\infty}^{\infty} f(t) \frac{|t - x|^{\alpha}}{t - x}\, dt, \tag{11}$$

$$K_{\alpha}(f)(x) = \frac{1}{\phi(\alpha)} \int_{-\infty}^{\infty} f(t)\, |t - x|^{\alpha-1}\, dt, \tag{12}$$

where $\phi(\alpha) = 2\Gamma(\alpha) \sin \frac{\pi\alpha}{2}$.

By Saxena [6]: Fractional integral of order α

$$I[f(x)] = I[\alpha, \beta, \gamma, m; f(x)]$$
$$= \frac{x^{-\gamma-1}}{\Gamma(1 - \alpha)} \int_0^x F(\alpha, \beta + m; \beta; \tfrac{t}{x}) t^{\gamma} f(t)\, dt, \tag{13}$$

$$R[f(x)] = R[\alpha, \beta, \delta, m; f(x)]$$
$$= \frac{x^{\delta}}{\Gamma(1 - \alpha)} \int_x^{\infty} F(\alpha, \beta + m; \beta; \tfrac{x}{t}) t^{-\delta-1} f(t)dt, \tag{14}$$

where $F(\alpha, \beta; \gamma; x)$ is the ordinary hypergeometric function, and α, β, γ, δ are complex parameters. And if $m = 0$, these are reduced to the Kober's fractional integral.

By Kalla and Saxena [7]: Fractional integral of order α

$$I[f(x)] = I[\alpha, \beta, \gamma; m, \mu, \eta, a; f(x)]$$

$$= \frac{\mu x^{-\eta-1}}{\Gamma(1-\alpha)} \int_0^x F(\alpha, \beta + m; \gamma; \frac{at^{\mu}}{x^{\mu}})\ t^{\eta} f(t)dt, \tag{15}$$

$$R[f(x)] = R[\alpha, \beta, \gamma; m, \mu, \delta, a; f(x)]$$

$$= \frac{\mu x^{\delta}}{\Gamma(1-\alpha)} \int_x^{\infty} F(\alpha, \beta + m; \gamma; \frac{ax^{\mu}}{t^{\mu}}) t^{-\delta-1} f(t)dt, \tag{16}$$

where α, β, γ, η, δ and a are complex parameters.

By M. Riesz [8]: Fractional integral of order α

$$f_{\alpha}(x) = \int_{-\infty}^{\infty} f(t)\ |x - t|^{\alpha-1}\ dt \qquad (0 < \alpha < 1) \tag{17}$$

By T. J. Osler [9]: Fractional derivative of order α of f(z) is

$$D_{z-a}^{\alpha} f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_a^{(z+)} f(\zeta)(\zeta - z)^{-\alpha-1} d\zeta \quad (\alpha \neq \text{negative integer}), \tag{18}$$

where he made a branch cut from z to a, and integral curve is an open contour which starts at a and encloses z in the positive sense, and return to a.

By B. Ross [10]: Fractional derivative order ν of f(z) is

$$\frac{d^{\nu}}{dz^{\nu}} f(z) = \frac{\Gamma(\nu + 1)}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}} d\zeta, \tag{19}$$

where he made a branch cut from z to infinity through the origin, and integral curve C is an open contour which encloses z in the positive sense and z does not on C (that is, C is an integral curve along that cut).

§ 3. Definitions of the fractional derivative and integral of the function of single variable

Definition 1. (Derivative): If f(z) is an analytic function and it has no branch point inside of C and on C, and

$$_{\underline{C}}f_{\nu} = {_{\underline{C}}f_{\nu}}(z) = \frac{\Gamma(\nu + 1)}{2\pi i} \int_{\underline{C}} \frac{f(\zeta)}{(\zeta - z)^{\nu+1}} d\zeta \tag{1}$$

$$\left(\begin{array}{l} \zeta \neq z,\ -\pi \leqslant \arg(\zeta - z) \leqslant \pi \\ \nu \neq -n,\ n = \text{integer} > 0 \\ \Gamma : \text{Gamma function} \end{array} \right)$$

$$= \frac{\Gamma(\nu + 1)}{2\pi i} \int_{-\infty}^{(0+)} \eta^{-(\nu+1)}\ f(z + \eta)d\eta, \quad (\zeta - z = \eta), \tag{2}$$

$$_{\underset{+}{C}}f_{\nu} = {_{C}f_{\nu}}(z) = \frac{\Gamma(\nu + 1)}{2\pi i} \int_{\underset{+}{C}} \frac{f(\zeta)}{(\zeta - z)^{\nu+1}} d\zeta \tag{3}$$

$$\left(\begin{array}{l} \zeta \neq z,\ 0 \leqslant \arg(\zeta - z) \leqslant 2\pi \\ \nu \neq -n,\ n = \text{integer} > 0 \end{array} \right)$$

$$= \frac{\Gamma(\nu + 1)}{2\pi i} \int_{\infty}^{(0+)} \eta^{-(\nu+1)}\ f(z + \eta)d\eta, \ (\zeta - z = \eta), \tag{4}$$

$$f_{-n} = {_{C}f_{-n}} = \lim_{\nu \to -n} {_{C}f_{\nu}} \qquad (C = \{\underset{+}{C}, \underline{C}\}), \tag{5}$$

where $\underline{C}$ and $\underset{+}{C}$ are the integral curve which are shown in Fig. 1 and Fig. 2 (that is, $\underline{C}$ is a curve along the cut joining points z and $-\infty + i\mathrm{Im}(z)$, and $\underset{+}{C}$ is a curve along the cut joining points z and ∞ +

iIm(z)), then $f_\nu = {}_C f_\nu(z)$ $(\nu > 0,\ C = \{\underset{-}{C},\ \underset{+}{C}\})$ is the fractional derivative of order ν.

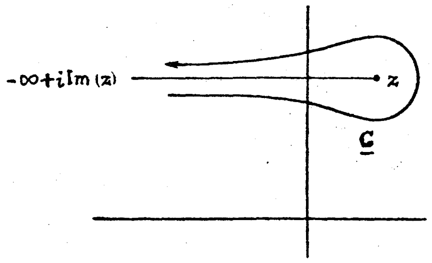

Fig. 1.

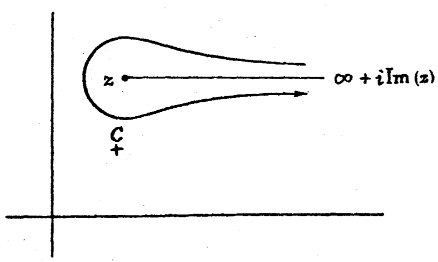

Fig. 2.

Definition 2 (Integral). $f_\nu (\nu < 0)$ is the fractional integral of order $|\nu|$. That is, the derivative of fractional order $-\nu (\nu > 0)$ is the fractional integral of order ν [11] [12].

§ 4. Formal unification of derivative and integral of the function of single variable

Theorem 1. If f(z) is the analytic function and it has no branch point inside of C and on C $(C = \{\underset{-}{C},\ \underset{+}{C}\})$, and

$$f_\nu = {}_C f_\nu = \{ {}_{\underset{-}{C}} f_\nu,\ {}_{\underset{+}{C}} f_\nu \} \tag{1}$$

then

$$f_\nu \text{ is } \begin{cases} \text{derivative} & \text{for} \quad \nu > 0 \\ \text{original} & \text{for} \quad \nu = 0 \\ \text{integral} & \text{for} \quad \nu < 0 \end{cases} \tag{2}$$

for real ν, and

$$f_\nu \text{ is } \begin{cases} \text{derivative} & \text{for} \quad \mathrm{Re}(\nu) > 0 \\ \text{original} & \text{for} \quad \nu = 0 \\ \text{integral} & \text{for} \quad \mathrm{Re}(\nu) < 0 \end{cases} \quad , \tag{3}$$

for complex ν.

And in case of $\mathrm{Re}(\nu) = 0$, f_ν is only formal differintegration regardless of $\mathrm{Im}(\nu) \gtrless 0$. That is, we have no derivative and integral for ν = pure imaginary.

Proof: It is clear by Definition 1 and 2 in §3.

Chapter 2. Fractional derivative and integral of the function of many variables

§ 1. Introduction

On the fractional derivative and integral of the function of many variables, there are few papers ([13]~[18]). And some definitions of fractional integral of the function of many variables are shown as follows.

Riesz's fractional integral $f_\nu(P)$ of order ν is given by

$$f_\nu(P)=H_n^{-1}\int_{E_n} r_{PQ}^{\nu-n}\, f(Q)\, dQ, \tag{1}$$

where
$$H_n=\pi^{n/2}\, 2^\nu\, \Gamma(\tfrac{\nu}{2})\left[\Gamma(\tfrac{n-\nu}{2})\right]^{-1} \tag{2}$$

E_n denotes all of Euclidean n-space, and r_{PQ} denotes the distance between P and Q ([8], [18]).

And by G. V. Welland, following representation is used.

$$f_\nu(x)=\int_{E_n}\frac{f(t)}{|x-t|^{n-\nu}}\,dt \qquad (0<\nu<1), \tag{3}$$

where x and t denote points $(x_1, x_2, \ldots, x_n)$ and $(t_1, t_2, \ldots, t_n)$ of n-dimensional Euclidean space, and $|x|=(\sum_{k=1}^{n} x_k^2)^{1/2}$ [17]

§ 2. Definitions of the fractional derivative and integral of the function of many variables

Definition 1 (Derivative): If $f(z_1, z_2, \ldots, z_n)$ is an analytic function and it has no branch point inside of C_k and on C_k for all z_k, and

$$f_{\nu_1,\nu_2,\ldots,\nu_n}= f_{C_n, C_{n-1}\ldots, C_1}\, f_{\nu_1, \nu_2, \ldots, \nu_n}(z_1, z_2, \ldots, z_n) \tag{1}$$

$$=\frac{\prod_{k=1}^{n}\Gamma(\nu_k+1)}{(2\pi i)^n}\int_{C_1}\cdots\int_{C_n}\frac{f(\zeta_1, \zeta_2, \cdots, \zeta_n)}{\prod_{k=1}^{n}(\zeta_k-z_k)^{\nu_k+1}}\, d\zeta_1\, d\zeta_2 \cdots d\zeta_n, \tag{2}$$

where
$$f_{-m_k}=\lim_{\nu_k\to -m_k} f_{\nu_k} \qquad (m_k=\text{integer}>0,\ K=1,2,3,\cdots,n) \tag{3}$$

$$C_k=\{\underset{-}{C},\ \underset{+}{C}\}, \qquad \zeta_k\neq z_k \tag{4}$$

and
$$-\pi\leqslant\arg(\zeta_k-z_k)\leqslant\pi \quad \text{for}\quad C_k=\underset{-}{C}, \tag{5}$$

$$0\leqslant\arg(\zeta_k-z_k)\leqslant 2\pi \quad\text{for}\quad C_k=\underset{+}{C}, \tag{6}$$

then

$f_{\nu_1, \nu_2, \cdots, \nu_n}$ $(\nu_k>0)$ is the fractional partial derivative of order ν_k for z_k.

Definition 2 (Integral): $f_{\nu_1, \nu_2, \cdots, \nu_n}$ $(\nu_k<0,\ K=1,2,\cdots n)$ is the fractional integral of order $|\nu_k|$ for z_k. That is, fractional derivative of order $-\nu_k(\nu_k>0)$ is the fractional integral of order ν_k for z_k [11].

Chapter 3. General properties of ${}_C f_\nu$

§ 1. On convergence of $f_\nu={}_C f_\nu$

Theorem 1. If f(z) is an analytic function, and if M(= const.>0) and α(= const. $\geqslant$0) such that

$$|f(z \pm r)| < Me^{-\alpha r} \tag{1}$$

exist, then ${}_{C}f_{\nu}(C=\{\underline{C},\ \underset{+}{C}\})$ converges absolutely for $\mathrm{Re}(\nu)<0$, where $+$ sign for $\underset{+}{C}$ and $-$ sign for $\underline{C}$.

Theorem 2. If $f(z)$ is an analytic and periodic function, we have then

$$\text{(i)} \qquad {}_{\underline{C}}f_{\nu} = \frac{1}{\Gamma(-\nu)}\int_0^T \left\{ \sum_{k=0}^{\infty} \frac{1}{(s+kT)^{\nu+1}} \right\} f(z-s)ds \tag{2}$$

$$\text{(ii)} \qquad {}_{\underset{+}{C}}f_{\nu} = \frac{e^{-i2\pi\nu}}{\Gamma(-\nu)}\int_0^T \left\{ \sum_{k=0}^{\infty} \frac{1}{(s+kT)^{\nu+1}} \right\} f(z-s)ds \tag{3}$$

for $-1<\mathrm{Re}(\nu)<0$, where T is the period of function $f(z)$.

§ 2. On $(f_\mu)_\nu = f_{\mu+\nu}$

Theorem 1. If $f(z)$ is an analytic and one valued function, we have then

$$(f_\mu)_\nu = f_{\mu+\nu} \tag{1}$$

for μ and ν such that $\left|\dfrac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\,\Gamma(\nu+1)}\right| \leqslant M(\text{const.})$.

And (1) is called as "index law" [11] [12].

Proof: $$f_\mu = {}_{\underline{C}}f_\mu(z) = \frac{\Gamma(\mu+1)}{2\pi i}\int_{\underline{C}} \frac{f(\zeta)}{(\zeta-z)^{\mu+1}}d\zeta \qquad (\text{for } C=\underline{C}) \tag{2}$$

Therefore we have

$$\left((f_\mu)\right)_\nu = {}_{\underline{C}}\left({}_{\underline{C}}f_\mu(z)\right)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_{\underline{C}} \frac{{}_{C}f_\mu(\eta)}{(\eta-z)^{\nu+1}}d\eta \qquad (\text{using } (2))$$

$$= \frac{\Gamma(\nu+1)}{2\pi i}\cdot\frac{\Gamma(\mu+1)}{2\pi i}\int_{\underline{C}} f(\zeta)d\zeta \int_{\underline{C}} \frac{1}{(\zeta-\eta)^{\mu+1}(\eta-z)^{\nu+1}}d\eta \tag{3}$$

Putting $\eta - z = w$ and $\zeta - z = p \equiv \delta e^{i\phi}$ (δ,ϕ: real, $\delta \neq 0$), we have then

$$\int_{\underline{C}} \frac{1}{(\zeta-\eta)^{\mu+1}(\eta-z)^{\nu+1}}d\eta \quad \int_{-\infty}^{(0+)} \frac{1}{(p-w)^{\mu+1}w^{\nu+1}}dw \quad \left(\begin{array}{l}\text{put } w=pu,\\ \text{then } u=\frac{1}{\delta}e^{-i\phi}w\end{array}\right) \tag{4}$$

$$= -p^{-(\mu+\nu+1)}\int_{-\infty}^{(0+)} u^{-(\nu+1)}(1-u)^{-(\mu+1)}du \qquad (\text{for } |\phi|<\pi/2) \tag{5}$$

$$= -p^{-(\mu+\nu+1)}\frac{2\pi i\,\Gamma(\mu+\nu+1)}{\Gamma(\nu+1)\,\Gamma(\mu+1)} \tag{6}$$

for $|\phi|<\pi/2$ and $-\mathrm{Re}(\mu+1)<\mathrm{Re}(\nu)<0$.

Substituting (6) into (3), we have then

$$ {}_{\underline{C}}\left({}_{\underline{C}}f_\mu(z)\right)_\nu = \frac{\Gamma(\mu+\nu+1)}{2\pi i}\int_{\underline{C}} \frac{f(\zeta)}{(\zeta-z)^{\mu+\nu+1}}d\zeta = {}_{C}f_{\mu+\nu}(z) \tag{7}$$

for $|\phi|<\pi/2$ and $-\mathrm{Re}(\mu+1)<\mathrm{Re}(\nu)<0$.

In the same way, we have

$$ {}_{\underset{+}{C}}\left({}_{\underset{+}{C}}f_\mu(z)\right)_\nu = {}_{\underset{+}{C}}f_{\mu+\nu}(z) \tag{8}$$

for $\pi/2<|\phi|\leqslant\pi$ and $-\mathrm{Re}(\mu+1)<\mathrm{Re}(\nu)<0$.

Therefore we have

$$ {}_c\left({}_cf_\mu\right)_\nu = {}_cf_{\mu+\nu}, \qquad c=\{\underline{c},\ \underset{+}{c}\} \tag{9}$$

in general for $-\operatorname{Re}(\mu+1) < \operatorname{Re}(\nu) < 0$, from (7) and (8).

And this relationship of (9) holds good for μ and ν such that

$$|\Gamma(\mu+\nu+1)/\Gamma(\mu+1)\Gamma(\nu+1)| \leqslant M(\text{const.}),$$

by the analytical continuation. Consequently we have this Theorem 1.

Theorem 2. If f(z) is an analytic and one valued function, we have then

$$(f_\mu)_\nu = (f_\nu)_\mu \tag{10}$$

§ 3. Linearity

Theorem 1. If f(z) is an analytic and one valued function, we have then

$$(af)_\nu = af_\nu \qquad (f \equiv f(z)), \tag{1}$$

where a is a constant.

Theorem 2. If u(z) and v(z) are analytic and one valued functions respectively, we have then

$$(au + bv)_\nu = au_\nu + bv_\nu, \tag{2}$$

where a and b are constants.

§ 4. Fractional differintegration of product uv

Theorem 1. If u(z) and v(z) are analytic and one valued functions respectively, then

$$\{uv\}_\nu \quad \text{is} \begin{cases} \text{fractional derivative of product uv for } \operatorname{Re}(\nu) > 0. \\ \text{original of product uv for } \nu = 0. \\ \text{fractional integral of product uv for } \operatorname{Re}(\nu) < 0. \end{cases} \tag{1}$$

where

$$\{u\ v\}_\nu = \{(u\ v)_\nu,\ (v\ u)_\nu\}, \tag{2}$$

$$(u\ v)_\nu = \sum_{n=0}^{\infty} P(\nu\ ,\ n)u_{\nu-n}\ v_n\ , \tag{3}$$

$$(v\ u)_\nu = \sum_{n=0}^{\infty} P(\nu\ ,\ n)v_{\nu-n}\ u_n\ , \tag{4}$$

and

$$P(\nu\ ,\ n) = \Gamma(\nu+1)/\Gamma(\nu-n+1)\Gamma(n+1) \quad [19] \tag{5}$$

Proof: Through the author's definition, we have

$$\{uv\}_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{u(\zeta)\ v(\zeta)\ d\zeta}{(\zeta-z)^{\nu+1}} \qquad (\zeta \neq z, \quad C=\{\underline{C},\ \underset{+}{C}\}) \tag{6}$$

$$= \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{u(\zeta)\ d\zeta}{(\zeta-z)^{\nu+1}}\cdot\frac{1}{2\pi i}\int_{C_1}\frac{v(\xi)\ d\xi}{\xi-\zeta} \tag{7}$$

$(\xi \neq \zeta,\ C_1=\{\underline{C}_1,\ \underset{+}{C}_1\},\ -\pi \leqslant \arg(\xi-\zeta) \leqslant \pi$ for $\underline{C}_1$, $0 \leqslant \arg(\xi-\zeta) \leqslant 2\pi$ for $\underset{+}{C}_1)$

$$= \frac{\Gamma(\nu+1)}{(2\pi i)^2}\sum_{n=0}^{\infty}\int_C \frac{u(\zeta)\ d\zeta}{(\zeta-z)^{\nu-n+1}}\int_{C_1}\frac{v(\xi)\ d\xi}{(\xi-z)^{n+1}} \tag{8}$$

$$= \sum_{n=0}^{\infty}\frac{\Gamma(\nu+1)}{\Gamma(\nu-n+1)\Gamma(n+1)}u_{\nu-n}\cdot v_n \qquad \left(\text{for } \left|\frac{\zeta-z}{\xi-z}\right|<1\right) \tag{9}$$

$$= \sum_{n=0}^{\infty} P(\nu, n) u_{\nu-n} \cdot v_n \equiv (u\ v)_\nu , \tag{10}$$

In the same way, we have

$$\{u\ v\}_\nu = \{v\ u\}_\nu = \sum_{n=0}^{\infty} P(\nu, n) v_{\nu-n}\ u_n \equiv (v\ u)_\nu \qquad \left(\text{for} \left|\frac{\zeta - z}{\xi - z}\right| < 1\right) \tag{11}$$

But, in case of $\left|\frac{\xi - z}{\zeta - z}\right| < 1$, we have

$$\{u\ v\}_\nu = - \sum_{n=-1}^{-\infty} P(\nu, n) u_{\nu-n} \cdot v_n \tag{12}$$

and

$$\{u\ v\}_\nu = \{v\ u\}_\nu = - \sum_{n=-1}^{-\infty} P(\nu, n) v_{\nu-n}\ u_n . \tag{13}$$

Formulae (12) and (13) become zero because of existence of $\Gamma(n + 1)$ in their denominator. This means the fractional and integer order's derivative of all products uv are zero. Therefore we can not adopt them, we have then

$$\{u\ v\}_\nu = \{v\ u\}_\nu = \{(u\ v)_\nu,\ (v\ u)_\nu\} \tag{14}$$

as the fractional differintegration of product uv, through (10) and (11). This result holds good for $\nu = m =$integer, of course, and if $\nu = m =$integer >0, (14) coincide with the Leibniz rule for products. And in case of $\nu = -m$($m =$integer >0), $\lim_{\nu \to -m} \{u\ v\} = \{u\ v\}_{-m}$ means m times integral of product uv.

§ 5. Convergence of $(u\ v)_\nu$

Theorem 1: If $u_{\nu-n} \leqslant L$(const.), $|v_n| \leqslant M$(const.) and $|\nu| \leqslant N$ (const.) then $(u\ v)_\nu$ converges for

$$\left| u_{\nu-n} v_n / u_{\nu-n-1} v_{n+1} \right| > 1. \qquad [21] \tag{1}$$

§ 6. Properties of fractional differintegration of products

Theorem 1. We have following relations.

(i) $\quad (u\ v)_\nu = (v\ u)_\nu \quad$ for $\quad \nu = m =$integer, (1)

(ii) $\quad (u\ v)_\nu \neq (v\ u)_\nu \quad$ for $\quad \nu \neq m =$integer, (2)

where $u \neq v$.

Theorem 2. We have

$$f_\nu = (f \cdot 1)_\nu \qquad (f \equiv f(z)) \tag{3}$$

for $\nu \neq$integer.

Note 1. Some interesting papers concerning the fractional derivative of products are reported by Professor T.J.Osler [22]. And in his paper, a brief historical survey on the fractional derivative of products and a new proof for the formula

$$D_z^\alpha\ uv = \sum_{n=-\infty}^{\infty} \frac{\Gamma(\alpha + 1)\ D^{\alpha-\gamma-n}\ u\ D^{\gamma+n}\ v}{\Gamma(\alpha - \gamma - n + 1)\,\Gamma(\gamma + n + 1)} , \tag{4}$$

(where $\alpha \neq$integer<0) which is fractional derivative of product uv are shown.

But, in T.J.Osler's definition, the starting and end point of integral curve have finite distance from the origin (his all results are obtained by the integral curve which has the origin as starting and end point), that is,his definition is a type of Riemann-Liouville's definition, therefore by his treatment we can not obtain the above author's results. Author's definition is a type of Weyl.

Note 2. We must notice that

$$(u\ v)_\nu \neq (v\ u)_\nu \quad \text{for} \quad \nu \neq \text{integer},$$
$$(u\ v)_\nu = (v\ u)_\nu \quad \text{for} \quad \nu = \text{integer}$$

and
$$f_\nu = (f\cdot 1)_\nu \quad \text{for} \quad \nu \neq \text{integer},$$

which are mentioned in § 6.

§ 7. Some properties of regular function (I)

Theorem 1. If $|f_\nu(z)| \leqslant M$(const.), we have then

(i) $$(f\cdot z)_{\nu+1} - \frac{\nu+1}{z}(f\cdot z)_\nu = f_{\nu+1}\ z - \frac{\nu(\nu+1)}{z} f_{\nu-1} \tag{1}$$

(ii) $$(f\cdot z)_{\nu+2} - \frac{\nu+2}{z}(f\cdot z)_{\nu+1} + \frac{(\nu+1)(\nu+2)}{z^2}(f\cdot z)_\nu$$
$$= f_{\nu+2}\cdot z + \frac{\nu(\nu+1)(\nu+2)}{z^2} f_{\nu-1} \tag{2}$$

for arbitrary ν and $z \neq 0$, where $(f\cdot z)_\nu = (f(z)\cdot z)_\nu$ and $f_\nu = f_\nu(z)$.

Theorem 2. If $|f_\nu(z)| \leqslant M$(const.) , we have then following relationships for arbitrary ν, $f_\nu \neq 0$ and for $z \neq 0$.

(i) $$\frac{d}{dz}\left(\frac{(f\cdot z)_\nu}{f_\nu}\right) = \left(\frac{(f\cdot z)_\nu}{f_\nu}\right)_1 = (\nu+1) - \frac{\nu f_{\nu-1}\ f_{\nu+1}}{(f_\nu)^2} \tag{3}$$

(ii) $$\left(\frac{(f\cdot z)_\nu}{f_\nu}\right)_{\mu+1} = -\nu\left(\frac{f_{\nu-1}\ \ f_{\nu+1}}{(f_\nu)}\right)_\mu \qquad (\text{for } \mu \neq \text{integer} \leqslant 0) \tag{4}$$

(iii) $$\int \frac{f_{\nu-1}\ f_{\nu+1}}{(f_\nu)^2}\, dz = \left(\frac{f_{\nu-1}\ f_{\nu+1}}{(f_\nu)^2}\right)_{-1} = \frac{\nu+1}{\nu} z - \frac{(f\cdot z)_\nu}{\nu\cdot f_\nu} \tag{5}$$

(for $\nu \neq 0$. Omitting the constant of integration.)

(iv) $$\frac{d}{dz}\left((f\cdot z)_\nu\cdot f_\nu\right) = \left((f\cdot z)_\nu\cdot f_\nu\right)_1$$
$$= 2zf_\nu f_{\nu+1} + \nu f_{\nu-1}f_{\nu+1} + (\nu+1)(f_\nu)^2 \tag{6}$$

(v) $$\int \left(2zf_\nu f_{\nu+1} + \nu f_{\nu-1}f_{\nu+1} + (\nu+1)(f_\nu)^2\right) dz = (f\cdot z)_\nu\cdot f_\nu$$

(Omitting the constant of integration) (7)

Theorem 3. If $|f_\nu(z)| \leqslant M$(const.), we have then following equality and inequalities for real $z(\neq 0)$, ν and for $f_\nu \neq 0$.

(i) $$f_{\nu-1}f_{\nu+1} = (f_\nu)^2 \iff \left(\frac{(f\cdot z)_\nu}{f}\right)_1 = 1 \quad \text{for} \quad \nu \neq 0 \tag{8}$$

(ii) $$f_{\nu-1}f_{\nu+1} > (f_\nu)^2 \iff \left(\frac{(f\ z)_\nu}{f_\nu}\right) < 1 \quad \text{for} \quad \nu > 0 \tag{9}$$

(ii) $f_{\nu-1}f_{\nu+1} > (f_\nu)^2 \iff \left(\frac{(f\cdot z)_\nu}{f_\nu}\right)_1 > 1$ for $\nu < 0$ (10)

(iii) $f_{\nu-1}f_{\nu+1} < (f_\nu)^2 \iff \begin{cases} \left(\frac{(f\cdot z)_\nu}{f_\nu}\right)_1 > 1 & \text{for } \nu > 0 \quad (11) \\ \left(\frac{(f\cdot z)_\nu}{f_\nu}\right)_1 < 1 & \text{for } \nu < 0 \quad (12) \end{cases}$

Theorem 4. If $|f_\nu(z)| \leqslant M$(const.), we have then following inequatities

(i) $\left(\frac{f_{\nu-1}f_{\nu+1}}{(f_\nu)^2}\right)_\mu > 0$ $\quad \left(\frac{(f\cdot z)_\nu}{f_\nu}\right)_{\mu+1} \begin{matrix} <0 \\ >0 \end{matrix}$ $\begin{matrix} \text{for } \nu > 0 & (13) \\ \text{for } \nu < 0 & (14) \end{matrix}$

(ii) $\left(\frac{f_{\nu-1}f_{\nu+1}}{(f_\nu)^2}\right)_\mu < 0$ $\quad \left(\frac{(f\cdot z)_\nu}{f_\nu}\right)_{\mu+1} \begin{matrix} >0 \\ <0 \end{matrix}$ $\begin{matrix} \text{for } \nu > 0 & (15) \\ \text{for } \nu < 0 & (16) \end{matrix}$

for real ν, $z \neq 0$, $f_\nu \neq 0$ and for $\mu \neq$ integer $\leqslant 0$.

Theorem 5. We have following relationships for $|f_\nu(z)| \leqslant M$(const.)

(i) $\lim_{\nu\to 0} \frac{(f\cdot z)_\nu}{f_\nu} = z$ for $\lim_{\nu\to 0}\left|\frac{f_{\nu-1}}{f_\nu}\right| \leqslant M$(const.), (17)

(ii) $\lim_{\nu\to 0}\left(\frac{(f\cdot z)_\nu}{f_\nu}\right)_1 = 1$ for $\lim_{\nu\to 0}\left|\frac{f_{\nu-1}f_{\nu+1}}{(f_\nu)^2}\right| \leqslant M$(const.), (18)

(iii) $\lim_{|\nu|\to\infty}\left(\frac{f_{\nu-1}f_{\nu+1}}{(f_\nu)^2}\right)_{-1} = z$ for $\lim_{|\nu|\to\infty}\left|\frac{(f\cdot z)_\nu}{\nu f_\nu}\right| = M$(const.). (19)

Where $z \neq 0$, $\nu \neq 0$ and $f_\nu \neq 0$.

§ 8. Some properties of regular function (II)

In this paragraph, some properties of the regular function which are obtained through the fractional differintegration of product are described.

Theorem 1. If $u = u(z)$ and $v = v(z)$ are one valued regular functions respectively, we have then

(i) $\int (u_1 \cdot v)_\nu\, dz = (uv)_\nu - \int (uv_1)_\nu\, dz$ (1)

(ii) $\int \left(u_1 \cdot \frac{1}{v}\right)_\nu dz = \left(u \cdot \frac{1}{v}\right)_\nu + \int \left(u \cdot \frac{v_1}{v^2}\right)_\nu dz$ $\quad (v \neq 0)$. (2)

Theorem 2. If $u = u(z)$ and $v = v(z)$ are one valued regular functions respectively and $uv \neq 0$, we have then

(i) $\lim_{\nu\to 0} \frac{(u\ v)_\nu}{u_\nu} = v$ for $\lim_{\nu\to 0}\left|\frac{u_{\nu-k}}{u_\nu}\right| \leqslant M$(const.) (k=integer > 0) (3)

(ii) $\lim_{\nu\to 0} \frac{(u\ v)_\nu}{v} = \lim_{\nu\to 0} u_\nu = u$ for $\lim_{\nu\to 0}|u_{\nu-k}| \leqslant M$(const.) (k=integer > 0) (4)

Chapter 4. Fractional differintegration of constant and of transcendental functions

§ 1. Exponential function [12]

Theorem 1. $(e^{-az})_\nu = e^{-i\pi\nu}\, a^\nu\, e^{-az}$ $\quad$ (for $a \neq 0$). (1)

Theorem 2. $(e^{az})_\nu = a^\nu \, e^{az}$ (for $a \neq 0$). (2)

§ 2. Hyperbolic function

Theorem 1. $(\cosh az)_\nu = (-ia)^\nu \cosh(az + i\frac{\pi}{2}\nu)$ (for $a \neq 0$). (1)

Theorem 2. $(\sinh az)_\nu = (-ia)^\nu \sinh(az + i\frac{\pi}{2}\nu)$ (for $a \neq 0$). (2)

§ 3. Trigonometric function [12]

Theorem 1. $(\cos az)_\nu = a^\nu \cos(az + \frac{\pi}{2}\nu)$ (for $a \neq 0$) (1)

Theorem 2. $(\sin az)_\nu = a^\nu \sin(az + \frac{\pi}{2}\nu)$ (for $a \neq 0$) (2)

Proof: We have

$$(\cos az)_\nu = \left(\frac{1}{2}(e^{iaz} + e^{-iaz})\right)_\nu = \frac{1}{2}\left\{(e^{iaz})_\nu + (e^{-iaz})_\nu\right\}$$

$$= a^\nu \cos(az + \frac{\pi}{2}\nu)$$

with using the Theorem 1 and 2 of § 1.

$$(\sin az)_\nu = \frac{1}{2i}\left\{(e^{iaz})_\nu - (e^{-iaz})_\nu\right\} = a^\nu \sin(az + \frac{\pi}{2}\nu)$$

Another proof of Theorem 1.

$$(\cos az)_\nu = {}_c(\cos az)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_c \frac{\cos a\zeta}{(\zeta - z)^{\nu+1}}\,d\zeta \qquad (c = \{\underline{C},\ \underset{+}{C}\})$$

$$= \frac{\Gamma(\nu+1)}{2\pi i}\int_{\mp\infty}^{(0+)} \eta^{-(\nu+1)}\cos(az + a\eta)\,d\eta \qquad \left(\begin{array}{l}\zeta - z = \eta\text{, and we have}\\ -\infty,\ -\pi \leqslant \arg\eta \leqslant \pi \text{ for } \underline{C}\\ +\infty,\ 0 \leqslant \arg\eta \leqslant 2\pi \text{ for } \underset{+}{C}\end{array}\right)$$

$$= \frac{\Gamma(\nu+1)}{2\pi i}\left\{(\cos az)\int_{\mp\infty}^{(0+)} \eta^{-(\nu+1)}\cos a\eta\,d\eta\right.$$

$$\left. - (\sin az)\int_{\mp\infty}^{(0+)} \eta^{-(\nu+1)}\sin a\eta\,d\eta\right\} \qquad (3)$$

Using $\underline{C}$ for $\mathrm{Re}(a) > 0$ (that is, $|\arg a| < \frac{\pi}{2}$) and $\underset{+}{C}$ for $\mathrm{Re}(a) < 0$ (that is, $\frac{\pi}{2} < |\arg a| < \pi$), we have

$$\int_{\mp\infty}^{(0+)} \eta^{-(\nu+1)}\cos a\eta\,d\eta = \frac{2\pi i}{\Gamma(\nu+1)}\,a^\nu \cos\frac{\pi}{2}\nu \qquad \left(\begin{array}{l}\text{for } -1 < \mathrm{Re}(\nu) < 0\\ \text{for } a \neq 0\end{array}\right) \qquad (4)$$

$$\int_{\mp\infty}^{(0+)} \eta^{-(\nu+1)}\sin a\eta\,d\eta = \frac{2\pi i}{\Gamma(\nu+1)}\,a^\nu \sin\frac{\pi}{2}\nu \qquad \left(\begin{array}{l}\text{for } -1 < \mathrm{Re}(\nu) < 1\\ \text{for } a \neq 0\end{array}\right) \qquad (5)$$

by contour integration.

Substituting (4) and (5) into (3), we obtain

$$(\cos az)_\nu = {}_c(\cos az)_\nu = a^\nu\left\{\cos az \cdot \cos\frac{\pi}{2}\nu - \sin az \cdot \sin\frac{\pi}{2}\nu\right\}$$

$$= a^\nu \cos(az + \frac{\pi}{2}\nu) \qquad \left(\begin{array}{l}\text{for } -1 < \mathrm{Re}(\nu) < 0\\ \text{for } a \neq 0\end{array}\right) \qquad (6)$$

In the same way we can obtain Theorem 2.

And these theorems hold good for arbitrary ν by the analytical continuation.

§ 4. Power function [20]

Theorem 1. If $\left|\dfrac{\Gamma(\nu - a)}{\Gamma(-a)}\right| \leqslant M(\text{const.})$, we have then

$$(z^a)_\nu = e^{-i\pi\nu} \frac{\Gamma(\nu - a)}{\Gamma(-a)} z^{a-\nu} \tag{1}$$

§ 5. Fractional differintegration of 1

Theorem 1. We have $\quad 1_\nu = (1)_\nu = 0 \qquad (1)$

for $\nu \neq -m$, where $m = \text{integer} \geqslant 0$.

Proof: In case of $a = 0$, we have $\quad z^a = z^0 = 1$.

Therefore we have above Theorem by Theorem 1 of §4, for $\nu \neq -m$.

Consequently, we see that the constant 1 has the integer order's integral only, by the above Theorem.

And by direct computation we have $(1)_\nu = 0$, for $\nu > 0$.

§ 6. Logarithmic function (I) [19][20]

We have the following theorem for fractional derivative of the function log az.

Theorem 1. If $|\Gamma(\nu)| \leqslant M(\text{const.})$, we have then

$$(\log az)_\nu = - e^{-i\pi\nu} \Gamma(\nu) z^{-\nu} \qquad (\text{for } a \neq 0). \tag{1}$$

Proof: $\displaystyle \log az = \int_1^{az} \frac{1}{t} dt = \int_1^{az} dt \int_0^\infty e^{-ts} ds \qquad (\mathrm{Re}(t) > 0,\ a \neq 0)$

$$= \int_0^\infty \int_1^{az} e^{-ts} dt\, ds = \int_0^\infty \frac{e^{-s} - e^{-azs}}{s} ds \qquad (\text{for } \mathrm{Re}(az) > 0) \tag{2}$$

Therefore we have

$$(\log az)_\nu = {}_{C}(\log az)_\nu = \frac{\Gamma(\nu + 1)}{2\pi i} \int_C \frac{\log a\zeta}{(\zeta - z)^{\nu+1}} d\zeta \tag{3}$$

$$= \frac{\Gamma(\nu + 1)}{2\pi i} \int_C \frac{\displaystyle\int_0^\infty \frac{e^{-s} - e^{-a\zeta s}}{s} ds}{(\zeta - z)^{\nu+1}} d\zeta \tag{4}$$

$$= \int_0^\infty \frac{e^{-s}}{s} ds \left\{ \frac{\Gamma(\nu + 1)}{2\pi i} \int_C \frac{1}{(\zeta - z)^{\nu+1}} d\zeta \right\}$$

$$- \int_0^\infty \frac{1}{s} ds \left\{ \frac{\Gamma(\nu + 1)}{2\pi i} \int_C \frac{e^{-as\zeta}}{(\zeta - z)^{\nu+1}} d\zeta \right\} \tag{5}$$

And we have seen (for $\mathrm{Re}(\nu) > 0$)

$${}_{C}(1)_\nu = \frac{\Gamma(\nu + 1)}{2\pi i} \int_C \frac{1}{(\zeta - z)^{\nu+1}} d\zeta = 0 \tag{6}$$

$$ {}_{\underset{=}{c}}(1)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{=}{c}}\frac{1}{(\zeta - z)^{\nu+1}}d\zeta = 0 \tag{7}$$

in the previous section, and

$$\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{+}{c}}\frac{e^{-as\zeta}}{(\zeta - z)^{\nu+1}}d\zeta = e^{-i\pi\nu}(as)^\nu e^{-asz} \quad \begin{pmatrix}\text{for } |\arg a| < \pi/2\\ \text{see section } 1\end{pmatrix} \tag{8}$$

$$\frac{\Gamma(\nu+1)}{2\pi i}\int_{\underset{=}{c}}\frac{e^{-as\zeta}}{(\zeta - z)^{\nu+1}}d\zeta = e^{-i\pi\nu}(as)^\nu e^{-asz} \quad \begin{pmatrix}\text{for } \pi/2 < |\arg a| \leqq \pi\\ \text{see section } 1\end{pmatrix} \tag{9}$$

Substituting (6) and (8) into (5), we have then

$$(\log az)_\nu = {}_{\underset{+}{c}}(\log az)_\nu = -\int_0^\infty \frac{1}{s}\, e^{-i\pi\nu}(as)^\nu e^{-azs}ds \qquad (\text{put } azs = v)$$

$$= -e^{-i\pi\nu}a^\nu(az)^{-\nu}\int_0^\infty v^{\nu-1}e^{-v}dv \qquad (\text{for } |\arg az| < \pi/2)$$

$$= -e^{-i\pi\nu}z^{-\nu}\,\Gamma(\nu) \qquad \begin{pmatrix}\text{for } |\arg a| < \pi/2\\ |\arg az| < \pi/2\end{pmatrix} \tag{10}$$

In the same way, we have (using (7) and (9))

$$(\log az)_\nu = {}_{\underset{=}{c}}(\log az)_\nu = -e^{-i\pi\nu}z^{-\nu}\,\Gamma(\nu) \quad \begin{pmatrix}\text{for } \pi/2 < |\arg a| \leqq \pi\\ \text{for } |\arg az| < \pi/2\end{pmatrix}. \tag{11}$$

Consequently we have Theorem 1, for $\mathrm{Re}(\nu) > 0$.

This theorem gives fractional derivative of order $\nu(\mathrm{Re}(\nu) > 0)$ of the function log az, because $(1)_\nu = 0$ for $\mathrm{Re}(\nu) > 0$. But this theorem holds good for ν such that $|\Gamma(\nu)| \leqq M$(const.), by the analytical continuation. That is, we can not obtain $(\log az)_{-m}$ for m=integer >0, by this theorem.

§ 7. Logarithmic function (II) [21]

(I) Theorem 1:

$$(\log z)_\nu = \sum_{n=0}^\infty \frac{\Gamma(\nu+2)}{\Gamma(\nu-n+2)\,\Gamma(n+1)}(\log z)_{\nu+1-n}\cdot(z)_n + e^{-i\pi\nu}\frac{\Gamma(\nu)}{\Gamma(-1)}z^{-\nu} \tag{1}$$

Proof: In this case, we must start from the following relation.

$$(\log z)_{-1} = z\log z - z. \tag{2}$$

We have then $\quad (\log z)_{\mu-1} = (\log z\cdot z)_\mu - (z)_\mu, \hfill (3)$

and

$$(\log z\cdot z)_\mu = \sum_{n=0}^\infty \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)\,\Gamma(n+1)}(\log z)_{\mu-n}\cdot(z)_n, \qquad [20] \tag{4}$$

by the theorem on the fractional differintegration of products (see Chap. 3 §4).

And we have $\quad (z)_\mu = e^{-i\pi\mu}\dfrac{\Gamma(\mu-1)}{\Gamma(-1)}z^{1-\mu} \hfill (5)$

by Theorem 1 of § 4. Substituting (4) and (5) into (3), we have then

$$(\log z)_{\mu-1} = \sum_{n=0}^\infty \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)\,\Gamma(n+1)}(\log z)_{\mu-n}\cdot(z)_n - e^{-i\pi\mu}\frac{\Gamma(\mu-1)}{\Gamma(-1)}z^{1-\mu} \tag{6}$$

Putting $\mu - 1 = \nu$ in (6), we obtain

$$(\log z)_\nu = \sum_{n=0}^{\infty} \frac{\Gamma(\nu+2)}{\Gamma(\nu-n+2)\Gamma(n+1)} (\log z)_{\nu+1-n} \cdot (z)_n + e^{-i\pi\nu} \frac{\Gamma(\nu)}{\Gamma(-1)} z^{-\nu} \quad (1)$$

Corollary 1. If $\nu \neq -m$ (m=integer$\geqslant 0$), $(\log z)_\nu$ satisfies following fractional differintegral equation.

$$(\log z)_{\nu+1} + \frac{\nu}{z}(\log z)_\nu = 0. \quad (2)$$

Corollary 2. If m=integer>0, $(\log z)_{-m}$ satisfies following differintegral equation.

$$(\log z)_{-m+1} - \frac{m}{z}(\log z)_{-m} - \frac{1}{m!} z^{m-1} = 0 \ . \quad (3)$$

Note 1. Solving (2) and (3), we have then

$$(\log z)_\nu = -e^{-i\pi\nu}\Gamma(\nu) z^{-\nu} \qquad (\text{for } \nu \neq \text{integer} \leqslant 0), \quad (4)$$

and

$$(\log z)_{-m} = \frac{1}{m!} z^m \log z + z^m \sum_{k=0}^{m-1} \frac{(-1)^{m-k}}{k! \cdot (m-k)! \cdot (m-k)} \quad (\text{for } m=\text{integer} \geqslant 1) \quad (5)$$

respectively.

Note 2. We can't find above results for $(\log z)_\nu$ in the volume of A.Erdélyi et-al [23], but in the volume of K.B.Oldham and J.Spanier [24] we find

$$\frac{d^q \ln(x)}{dx^q} = \frac{x^{-q}}{\Gamma(1-q)} \left[\ln(x) - \gamma - \psi(1-q) \right] .$$

And the same result is shown by Bertram Ross through the another treatment from that of K.B.Oldham and J.Spanier, in the Lecture Note vol 457 [25]. However, the starting point of this result is the definition of Riemann-Liouville.

And in the paper of K.S.Miller (Riverside Research Institute), the title of which is "The Weyl fractional calculus" [26], we can't find $(\log z)_\nu$

Chapter 5. Special function and fractional differintegration

§ 1. Application of fractional differintegration to the solution of Legendre's differential equation [27]

In this chapter, some results which are obtained through the application of Nishimoto's definition for fractional differintegration to the solution of Legendre's differential equation are reported.

Legendre's differential equation is

$$(1-z^2)w'' - 2zw' + \nu(\nu+1)w = 0, \quad (1)$$

and a solution of this equation is

$$P_n(z) = \frac{1}{2^n (n!)} \frac{d^n}{dz^n}(z^2-1)^n \qquad (\text{Rodrigue's formula}) \quad (2)$$

$$= \frac{1}{2^n \cdot 2\pi i} \oint \frac{(\zeta^2-1)^n}{(\zeta - z)^{n+1}} d\zeta \qquad (\text{Schläfli's integral}) \quad (3)$$

for $\nu = n =$ integer $\geqslant 0$.
Now we denote

$$ {}_{C}L_{\nu}(z) \equiv \frac{1}{2^{\nu}\Gamma(\nu+1)}\frac{d^{\nu}}{dz^{\nu}}(z^{2}-1)^{\nu} \quad \left(\begin{array}{l}\text{here, we use Nishimoto's defi-}\\ \text{nition for fractional order}\end{array}\right) $$

$$ = \frac{1}{2^{\nu}\cdot 2\pi i}\int_{C}\frac{(\zeta^{2}-1)^{\nu}}{(\zeta-z)^{\nu+1}}\,d\zeta \qquad (\zeta \neq z,\ C=\{\underline{C},\ \underset{+}{C}\}), \tag{4} $$

and $\underline{C}=\{{}_1\underline{C}, {}_2\underline{C}, {}_3\underline{C}\}$, $\underset{+}{C}=\{{}_1\underset{+}{C}, {}_2\underset{+}{C}, {}_3\underset{+}{C}\}$, where ${}_kC=\{{}_k\underline{C},\ {}_k\underset{+}{C}\}$(k=1, 2, 3) is the integral curves which are shown in Fig. 1. That is, all ${}_k\underline{C}$ are the integral curve along the cut joining z and $-\infty + i\mathrm{Im}(z)$, and along the cut joining z and $\infty + i\mathrm{Im}(z)$ for ${}_k\underset{+}{C}$. And ${}_1C$ surrounds only one point z, ${}_2C$ surrounds two points z and 1, and ${}_3C$ surrounds three points z, 1 and -1.

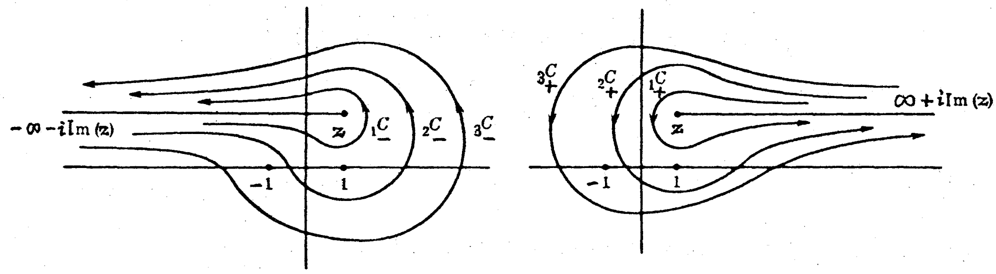

Fig. 1.

(I) We have then

$$ {}_{1C}L_{\nu}(z) = \frac{1}{2^{\nu}\cdot 2\pi i}\int_{{}_1C}\frac{(\zeta^{2}-1)^{\nu}}{(\zeta-z)^{\nu+1}}\,d\zeta \qquad \left(\begin{array}{l}{}_1C=\{{}_1\underline{C},\ {}_1\underset{+}{C}\}.\quad \mathrm{Re}(z)>0\\ |z+1|>|z-1|>\rho>0\end{array}\right) $$

$$ = g(\nu)\int_{\rho}^{\infty}\left\{\frac{(z\mp r)^{2}-1}{2r}\right\}^{\nu}\frac{1}{r}\,dr + \frac{1}{2\pi}\int_{a}^{b}\left\{\frac{(z+\rho e^{i\theta})^{2}-1}{2\rho e^{i\theta}}\right\}^{\nu}d\theta \qquad (\nu \neq -n) \tag{5} $$

where $g(\nu)=1/\{\Gamma(\nu+1)\Gamma(-\nu)\}$, $a=-\pi$, $b=\pi$ and - (for double sign $\mp$) for ${}_1\underline{C}$ and $g(\nu)=e^{-i\pi\nu}/\{\Gamma(\nu+1)\Gamma(-\nu)\}$, a=0, $b=2\pi$ and + (for double sign $\mp$) for ${}_1\underset{+}{C}$.(n=integer$\geqslant 0$)

Putting $r=\sqrt{z^{2}-1}\ e^{\varphi - i\phi}$ ($\sqrt{z^{2}-1}=\delta e^{i\phi}\neq 0$. δ, φ, ϕ: real), we have then

$$ {}_{1\underline{C}}L_{\nu}(z) = \frac{1}{\Gamma(\nu+1)\Gamma(-\nu)}\int_{-\infty}^{\infty}\{-z+\sqrt{z^{2}-1}\cosh(\varphi - i\phi)\}^{\nu}d\varphi \tag{6} $$

$$ {}_{1\underset{+}{C}}L_{\nu}(z) = \frac{e^{-i\pi\nu}}{\Gamma(\nu+1)\Gamma(-\nu)}\int_{-\infty}^{\infty}\{z+\sqrt{z^{2}-1}\cosh(\varphi - i\phi)\}^{\nu}d\varphi \tag{7} $$

from (5), for $z^{2}\neq 1$ and $\nu < 0$.

And Heine's integral representation for Legendre's function of the second kind is

$$ Q_{\nu}(z) = \frac{1}{2}\int_{-\infty}^{\infty}\{z+\sqrt{z^{2}-1}\cdot\cosh\varphi\}\,d\varphi, \tag{8} $$

therefore we have

$$ {}_{1\underline{C}}L_{\nu}(-z) = \frac{1}{\Gamma(\nu+1)\Gamma(-\nu)}\cdot 2Q_{\nu}(z), \qquad (\text{for } \phi = 0) \tag{9} $$

$$ {}_{2\underset{+}{C}}L_{\nu}(z) = \frac{e^{-i\pi\nu}}{\Gamma(\nu+1)\Gamma(-\nu)}\cdot 2Q_{\nu}(z). \qquad (\text{for } \phi = 0) \tag{10} $$

and $$ {}_{1\underline{C}}L_\nu(-z) = e^{i\pi\nu}\, {}_{1\underset{+}{C}}L_\nu(z)\ . \tag{11}$$

That is, ${}_{1C}L_\nu(z)$ contains Legendre function of the second kind.

(II) Put $$\zeta = z + \sqrt{z^2-1}\,\xi\ ,\quad \sqrt{z^2-1} \equiv s e^{i\phi} \tag{12}$$

(s, ϕ ; real, $z^2 \neq 1$) for Re(z) > 0, we have then

$$ {}_{2C}L_\nu(z) = \frac{1}{2\pi i}\int_{{}_2C} \frac{(\zeta^2-1)^\nu}{2^\nu(\zeta - z)^{\nu+1}}\, d\zeta \qquad (|z-1| < s < |z+1|)$$

$$= \frac{1}{2\pi}\int_a^b \left\{ z + \sqrt{z^2-1}\cos(\theta - \phi)\right\}^\nu d\theta\ , \tag{13}$$

where $a = -\pi$, $b = \pi$ for ${}_2\underline{C}$, and $a = 0$, $b = 2\pi$ for ${}_2\underset{+}{C}$.

And Laplace's first integral for Legendre function of the first kind is

$$P_\nu(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{ z + \sqrt{z^2-1}\cos\theta\right\}^\nu d\theta, \tag{14}$$

therefore we obtain $${}_{2\underline{C}}L_\nu(z) = P_\nu(z) = {}_{2\underset{+}{C}}L_\nu(z) \qquad (\text{for } \phi = 0), \tag{15}$$

for Re(z) > 0.

(III) In case z is a pure imaginary, if we put

$\zeta = z + s\xi$ ($s = e^{\varphi}\sqrt{y^2+1}$, where $0 \leq \varphi < \infty$ and $y = \mathrm{Im}(z)$)

for integral curve ${}_3C = \{{}_3\underline{C},\ {}_3\underset{+}{C}\}$, we have

$${}_{3C}L_\nu(z) = \frac{1}{2\pi}\int_a^b \left\{ z + \sqrt{z^2-1}\sin(\theta - i\varphi)\right\}^\nu d\theta \tag{16}$$

where $a = -\pi$, $b = \pi$ for ${}_3\underline{C}$ and $a = 0$, $b = 2\pi$ for ${}_3\underset{+}{C}$.

(IV) And the author want to make it clear that we must use

(i) $$\left((z^2-1)^\nu\right)_\nu = \frac{2^\nu\Gamma(\nu+1)}{2\pi}\int_{-\pi}^{\pi}\left\{\sqrt{z^2-1}\cos(\theta-\phi)+z\right\}^\nu d\theta \tag{17}$$

$(\nu > 0\ (\mathrm{Re}(\nu) > 0).\ \ \mathrm{Re}(z) > 0)$

$$= 2^\nu\Gamma(\nu+1)P_\nu(z) \qquad (\text{for } \phi = 0)$$

for(fractional) derivative of the function$(z^2-1)^\nu$, because in case of $\nu < 0$, $z = \pm 1$ change to singular point (since the author's definition does not allow of the existence of singular point of the function in the integral curve C and on C), and we must use

(ii) $$\left((z^2-1)^\nu\right)_\nu = \frac{2^\nu}{\Gamma(-\nu)}\int_{-\infty}^{\infty}\left\{-z + \sqrt{z^2-1}\cosh(\varphi - i\phi)\right\}^\nu d\varphi \tag{18}$$

$$= \frac{2^\nu}{\Gamma(-\nu)}\cdot 2Q(-z) \qquad \left(\begin{matrix}\text{for } z(\text{real}) < -1\\ \text{and for } \phi = 0\end{matrix}\right) \qquad (\nu < 0\ (\mathrm{Re}(\nu) < 0))$$

for (fractional) integral of the function $(z^2-1)^\nu$, by the author's definition for fractional differintegration (we must use (7) for z(real) > 1).

And Heine's integral representation for Legendre's function of the second kind is shown as

$$Q_\nu(z)=\frac{1}{2}\int_{-\infty}^{\infty}\{z+\sqrt{z^2-1}\cosh\varphi\}^\nu d\varphi \qquad (\text{for } z>1)$$

in general. Therefore the form

$$Q_\nu(z)=\frac{1}{2}\int_{-\infty}^{\infty}\{z+\sqrt{z^2-1}\cosh(\varphi-i\phi)\}^\nu d\varphi \qquad (\text{for } z\neq\pm1) \tag{19}$$

which is obtained by the author in (I) is the more general form of Heine's integral representation.

§ 2. Some comments on treatment of § 1

(I) If we put $\qquad w=\int_C(\zeta-z)^\lambda v(\zeta)d\zeta \qquad (1)$

as a solution to the Legendre's differential equation, we have

$$\left[\frac{(\nu+1)(\zeta^2-1)^{\nu+1}}{(\zeta-z)^{\nu+2}}\right]_C=0 \tag{2}$$

as a equation to determine the integral curve C. The results mentioned in §1 mean that we can choose the integral curves which are shown in Fig. 1 for unrestricted ν.

(II) Only ${}_{1C}L_\nu$(of the group ${}_{1C}L_\nu$, ${}_{2C}L_\nu$ and ${}_{3C}L_\nu$)contains the Nishimoto's fractional differintegration of order ν of the function $f(z)=(z^2-1)^\nu$. Next, we have

$${}_{1C}L_\nu={}_{2C}L_\nu={}_{3C}L_\nu=P_\nu(z)=\frac{1}{2^\nu\Gamma(\nu+1)}\left((z^2-1)^\nu\right)_\nu$$

for $\nu=n=$ integer $\geqslant 0$. And ${}_{2C}L_\nu$ contains Nishimoto's fractional derivative for $\nu>0$.

(III) $${}_{1C}L_\nu=\frac{1}{2^\nu\cdot 2\pi i}\int_{{}_1C}\frac{(\zeta^2-1)^\nu}{(\zeta-z)^{\nu+1}}d\zeta \qquad \left(\begin{array}{l}{}_1C=\{{}_1\underline{C},\ {}_1\bar{C}\}.\quad \mathrm{Re}(z)>0\\ |z+1|>|z+1|>\rho>0\end{array}\right)$$

$$=\frac{1}{2^\nu\Gamma(\nu+1)}\cdot\frac{\Gamma(\nu+1)}{2\pi i}\int_{-\infty+i\,\mathrm{Im}(z)}^{(z+)}\frac{(\zeta^2-1)^\nu}{(\zeta-z)^{\nu+1}}d\zeta=\frac{1}{2^\nu\Gamma(\nu+1)}\left((z^2-1)^\nu\right)_\nu$$

is a solution to the Legendre's differential equation

$$(1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\nu(\nu+1)w=0. \tag{3}$$

That is, we have $\qquad w=\dfrac{1}{2^\nu\Gamma(\nu+1)}\left((z^2-1)^\nu\right)_\nu \qquad (4)$

Putting $\qquad u=(z^2-1)^\nu \qquad (5)$

we have then $\qquad w=\dfrac{1}{2^\nu\Gamma(\nu+1)}u_\nu \qquad (6)$

and substituting this into (3), we obtain

$$(1-z)^2u_{\nu+2}-2zu_{\nu+1}+\nu(\nu+1)u_\nu=0\ . \tag{7}$$

This is a fractional order's differintegral equation. This result is an important one.

(IV) If the author's above treatment for Legendre's differential equation was found in old time, perhaps the problem of fractional calculus has been solved already. In this sense, the application which is shown in § 1 is very important.

(V) Keith B.Oldham and Jerome Spanier [24] Bertram Ross [28] and M. A.Al - Bassam [29] [30] applied their definition to the solution of Legendre's differential equation. The author hopes all readers compare author's above treatment with those of Oldham and Spanier, Ross and Al - Bassam etc.

§ 3. Representation of the Bessel function of the first kind with use of fractional differintegration

Theorem 1. Bessel function of the first kind $J_\nu(z)$ is represented as follows

$$J_\nu(z)=\sum_{m=o}^{\infty}\frac{(-1)^m}{m!\,\Gamma(\nu+m+1)}\left(e^{-\frac{z}{2}t}\right)_o\cdot{}_t\left(e^{\frac{z}{2}t}\right)_{(\nu+2m)} \qquad (|\arg z|<\tfrac{\pi}{2}) \qquad (1)$$

where ${}_t\left(e^{\frac{z}{2}t}\right)_{(\nu+2m)}$ means fractional differintegration of order $\nu+2m$ (with respect to t) of the function $e^{\frac{z}{2}t}$.

§ 4. A representation of the Hermite function with use of fractional differintegration

Theorem 1. The Hermite function $H_\nu(z)$ is represented as follows.

$$H_\nu(z)=e^{i\pi\nu}\sum_{m=o}^{\infty}\frac{(-1)^m\,\Gamma(\nu+1)}{m!\,\Gamma(\nu+1-2m)}\left(e^{2zt}\right)_o\cdot{}_t\left(e^{-2zt}\right)_{(\nu-2m)} \qquad (|\arg z|<\tfrac{\pi}{2}). \qquad (1)$$

§ 5. A representation of the Laguerre function with use of fractional differintegration

Theorem 1. The Laguerre function $L_\nu(z)$ is represented as follows

$$L_\nu(z)=e^{z}(e^{-z}\,z^{\nu})_\nu \qquad (\nu:\ \text{arbitrary}) \qquad (1)$$

where $(e^{-z}\,z^{\nu})_\nu$ means fractional differintegration of order ν(with respect to z) of the function $e^{-z}\,z^{\nu}$.

§ 6. Beta function and fractional differintegration of the functions z^{p-1} and $(1-z)^{q-1}$

For the Beta function, we have following representations.

Theorem 1. If $\left|\left[(z^{p-1})_{-q}\right]_{z=1}\right|\leqslant M$(const.), we have then

$$B(p,\ q)=\frac{e^{-i\pi q}\,\Gamma(q)\,\sin\pi(p+q)}{\sin\pi p}\left[(z^{p-1})_{-q}\right]_{z=1} \qquad (1)$$

for $\qquad D_{p,q}=\{(p\ ,q)\,|\,p\neq m,\ q=m\leqslant 0\}$ $\qquad$ (m = integer)

where $(z^{p-1})_{-q}$ means fractional differintegration of order -q of function z^{p-1}.

Theorem 2. If $\left|\left[((1-z)^{q-1})_{-p}\right]_{z=o}\right|\leqslant M$(const.), we have then

$$B(p,\ q)=\frac{\Gamma(p)\ \sin\pi(p+q)}{\sin\pi q}\left[\left((1-z)^{q-1}\right)_{-p}\right]_{z=o} \qquad (2)$$

for $\qquad D_{p,q}=\left\{(p,q \quad p\neq m\leqslant 0,\ q\neq m\right\} \qquad (m=\text{integer}).$

§ 7. Some contour integral representations for Hypergeometric function

We have following contour integral representation for Hypergeometric function.

Theorem 1. If $|z|>1$ and $\left|\frac{\Gamma(\beta)\ \Gamma(\alpha-\beta+1)}{\Gamma(\alpha)}\right|\leqslant M(\text{const.})$,

we have then

$$\begin{aligned}&{}_2F_1(\alpha,\ \alpha-\gamma+1:\alpha-\beta+1:\ 1/z)\\ &=e^{i\pi\{\alpha\pm(\beta-\gamma)\}}\ \frac{\Gamma(\beta)\ \Gamma(\alpha-\beta+1)}{2\pi i\,\Gamma(\alpha)}\ z^{\alpha}\int_{\infty+i\mathrm{Im}(z)}^{(z+)}\frac{\zeta^{\beta-\gamma}(1-\zeta)^{\gamma-\alpha-1}}{(\zeta-z)^{\beta}}\,d\zeta, \qquad (1)\end{aligned}$$

where ${}_2F_1(\alpha,\ \alpha-\gamma+1:\ \alpha-\beta+1:\ 1/z)$ means ordinary Hypergeometric function and we choose + sign for $|\arg z|<\pi/2$, -sign for $\pi/2<|\arg z|<\pi$, for double sign $\pm$.

§ 8. Relationship $(z^{\beta-\gamma}\ (1-z)^{\gamma-\alpha-1})_{\beta-1}$ and Hypergeometric function

(I) We have following Theorem.

Theorem 1: If $|z|>1$ and $\left|\frac{\Gamma(\alpha)}{(\beta-1)\ \Gamma(\alpha-\beta+1)}\right|\leqslant M(\text{const.})$, we have then

$$\begin{aligned}&(z^{\beta-\gamma}\ (1-z)^{\gamma-\alpha-1})_{\beta-1}\\ &=e^{i\pi\{-\alpha\pm(\gamma-\beta)\}}\frac{\Gamma(\alpha)}{(\beta-1)\ \Gamma(\alpha-\beta+1)}\ z^{-\alpha}\ {}_2F_1\left(\alpha,\ \alpha-\gamma+1:\ \alpha-\beta+1:\ \frac{1}{z}\right) \qquad (1)\end{aligned}$$

where we choose (+) sign for $|\arg z|<\pi/2$ and (-) sign for $\pi/2<|\arg z|<\pi$ for double sign $\pm$.

(II) Some comments

(i) Through the above results we see that the fractional differintegrated function $(z^{\beta-\gamma}(1-z)^{\gamma-\alpha-1})_{\beta-1}$ is a solution to Gauss' Hypergeometric differential equation

$$z(1-z)\frac{d^2w}{dz^2}+\left\{\gamma-(\alpha+\beta+1)z\right\}\frac{dw}{dz}-\alpha\beta w=0. \qquad (2)$$

Therefore, putting $\qquad u=z^{\beta-\gamma}\ (1-z)^{\gamma-\alpha-1}, \qquad (3)$

we obtain following fractional order's differintegral equation for u

$$z(1-z)u_{\beta+1}+\left\{\gamma-(\alpha+\beta+1)z\right\}u_{\beta}-\alpha\beta u_{\beta-1}=0 \qquad (4)$$

from (2), because β is arbitrary and

$$w=u_{\beta-1}=(z^{\beta-\gamma}(1-z)^{\gamma-\alpha-1})_{\beta-1}. \qquad (5)$$

That is, $u=z^{\beta-\gamma}(1-z)^{\gamma-\alpha-1}$ is a solution of the fractional order's differintegral equation (4).
And we have the relationship $\quad {}_2F_1(\alpha,\beta:\gamma:z)={}_2F_1(\beta,\alpha:\gamma:z)$,
consequently we see that $\quad v=z^{\alpha-\gamma}(1-z)^{\gamma-\beta-1}$ (6)
is a solution to fractional order's differintegral equation

$$z(1-z)v_{\alpha+1}+\{\gamma-(\alpha+\beta+1)z\}v_{\alpha}-\alpha\beta v_{\alpha-1}=0, \tag{7}$$

too.

(ii) If we put

$$w=\int_L(\zeta-z)^{\lambda}v(\zeta)d\zeta \tag{8}$$

as a solution to the Hypergeometric differential equation (2), we have

$$\left[\zeta^{\beta-\gamma+1}(1-\zeta)^{\gamma-\alpha}(\zeta-z)^{-\beta-1}\right]_L=0. \tag{9}$$

as a equation to determine the integral curve L. The results mentioned in §7 mean that we can choose the integral curve $L=C=\{\underline{C},\,\underset{+}{C}\}$.

Note.

$$w=(-z)^{-\alpha}{}_2F_1(\alpha,\ \alpha-\gamma+1;\ \alpha-\beta+1;\ \tfrac{1}{z}) \tag{10}$$

is a solution to Gauss' Hypergeometric differential equation (2).

Chapter 6. Some fractional order's linear ordinary differintegral equations with constant coefficients

§1. Solutions of fractional order's differintegral equations of type $w_{\nu}(z)+\lambda w(z)=0$ (ν: real)

There are some papers ([24] [28] [31] [32]) on the solutions of fractional order's differintegral equation of type

$$w_{\nu}(z)+\lambda w(z)=F(z) \qquad (\lambda:\ \text{const.}\quad \nu:\ \text{real}). \tag{1}$$

But all these treatments are discussed with use of infinite series. The author will treat the solutions of the fractional differintegral equation (1) with use of complex integral [34].

(I) Firstly, we will solve the fractional order's differintegral equation of

$$w_{\nu}(z)+\lambda w(z)=0\ , \tag{2}$$

where $w_{\nu}(z)$ denotes Nishimoto's fractional order's differintegral of function $w(z)$.

That is,

$$w_{\nu}(z)={}_Cw_{\nu}(z)=\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{w(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta, \tag{3}$$

where $C=\{\underline{C},\ \underset{+}{C}\}$. Consequently

(1) and (2) are differential equation for $\nu>0$,

and, (1) and (2) are integral equation for $\nu<0$.

Now, we assume the solution of contour integral form

$$w(z)=\int_L e^{zt}v(t)dt, \tag{4}$$

where $v(t)$ is undetermined function and L is an undetermined inte-integral curve. We have then

$$w(z)=M\int_L \frac{}{t^{\nu}+\lambda}dt=M\ 2\pi i\sum_{k=0}^{m} A_k\ e^{zt_K} \tag{5}$$

$$=\sum_{k=0}^{m} B_k\ e^{z\{\lambda e^{i\pi(1+2K)}\}^{1/\nu}} \qquad (B_k=M\ 2\pi i A_k=\text{const.}) \tag{6}$$

as the general solution of (2) by the residue theorem, where m is determined according as ν, that is

$m=$finite for $\nu=$rational number,

and $m=$infinite for $\nu=$irrational number.

Formula (6) gives the general solution of fractional order's differential equation for $\nu>0$, and gives the solution of fractional order's integral equation for $\nu<0$.

Table 1.

ν	Differintegral equation	Solution
1/2	$w_{(1/2)}(z)+\lambda w(z)=0$	$w(z)=B_0\exp\{z(\lambda e^{i\pi})^2\}$
1/3	$w_{(1/3)}(z)+\lambda w(z)=0$	$w(z)=B_0\exp\{z(\lambda e^{i\pi})^3\}$
2/3	$w_{(2/3)}(z)+\lambda w(z)=0$	$w(z)=B_0\exp\{z(\lambda e^{i\pi})^{3/2}\}+B_1\exp\{z(\lambda e^{i3\pi})^{3/2}\}$
−2/3	$w_{(-2/3)}(z)+\lambda w(z)=0$	$w(z)=B_0\exp\{z(\lambda e^{i\pi})^{-3/2}\}+B_1\exp\{z(\lambda e^{i3\pi})^{-3/2}\}$
1	$w'(y)+\lambda w(z)=0$	$w(z)=B_0\exp\{z(\lambda e^{i\pi})\}=B_0e^{-\lambda z}$
2	$w''(z)+\lambda w(z)=0$	$w(z)=B_0\exp\{z(\lambda e^{i\pi})^{1/2}\}+B_1\exp\{z(\lambda e^{i3\pi})^{1/2}\}$ $=B_0\exp\{i\sqrt{\lambda}z\}+B_1\exp\{-i\sqrt{\lambda}z\}$
−2	$w_{(-2)}(z)+\lambda w(z)=0$	$w(z)=B_0\exp\left(\frac{1}{i\sqrt{\lambda}}z\right)+B_1\exp\left(\frac{-1}{i\sqrt{\lambda}}z\right)$ $=B_0\exp\left(-\frac{i}{\sqrt{\lambda}}z\right)+B_1\exp\left(\frac{i}{\sqrt{\lambda}}z\right)$
5/2	$w_{(5/2)}(z)+\lambda w(z)=0$	$w(z)=B_0\exp\{z(\lambda e^{i\pi})^{2/5}\}+B_1\exp\{z(\lambda e^{i3\pi})^{2/5}\}+\cdots+B_4\exp\{z(\lambda e^{i9\pi})^{2/5}\}$
−5/2	$w_{(-5/2)}(z)+\lambda w(z)=0$	$w(z)=B_0\exp\{z(\lambda e^{i\pi})^{-2/5}\}+B_1\exp\{z(\lambda e^{i3\pi})^{-2/5}\}$ $+\cdots+B_4\exp\{z(\lambda e^{i9\pi})^{-2/5}\}$

Note. In case $\nu=$irrational number, $f(t)=e^{zt}/(t^{\nu}+\lambda)$ has singularities everywhere on the circle $|t|=|\lambda|^{1/\nu}$ (radius of the circle), so these singularities are not isolated. However, every singularities never overlap each other, consequently above result (6) is correct formally, since we choose a closed Jordan curve L such that it encloses all singularities of $f(t)$.

§2. Solutions of fractional order's differintegral equations of type $w_{\nu}(z)+\lambda w(z)=F(z)$ (ν: real)

(I) Case of $F(z)=e^{az}$

A particular solution of fractional differintegral equation

$$w_{\nu}(z)+\lambda w(z)=e^{az} \tag{1}$$

is given by

$$w(z)=\frac{1}{a^{\nu}+\lambda}e^{az}, \tag{2}$$

for $a^{\nu}+\lambda\neq 0$.

Consequently general solution of (1) is given as follows (with use of (2) and (6) in §1.

$$w(z)=\sum_{k=0}^{m} B_k \exp\left[z\left\{\lambda e^{i\pi(1+2k)}\right\}^{1/\nu}\right] + \frac{1}{a^\nu+\lambda}e^{az} \qquad (a^\nu + \lambda \neq 0). \quad (3)$$

(II) Case of $F(z)=\cos az$

The general solution of fractional order's differintegral equation

$$w_\nu(z) + \lambda w(z) = \cos az. \quad (4)$$

is given as follows.

$$w(z)=\sum_{k=0}^{m} B_k \exp\left[z\left\{\lambda e^{i\pi(1+2k)}\right\}^{1/\nu}\right]$$

$$+\frac{1}{a^{2\nu}+\lambda^2+2\lambda a^\nu\cos\frac{\pi}{2}\nu}\left\{(a^\nu\cos\frac{\pi}{2}\nu+\lambda)\cos az+(a^\nu\sin\frac{\pi}{2}\nu)\sin az\right\} \quad (5)$$

for $$a^{2\nu} + 2\lambda a^\nu \cos\frac{\pi}{2}\nu + \lambda^2 \neq 0. \quad (6)$$

§3. Solutions of fractional order's differintegral equation of type $w_{2\mu}(z)+bw_\mu(z)+\lambda w(z)=F(z)$ (μ: real)

In this section, the solution of differintegral equation of type

$$w_{2\mu}(z) + bw_\mu(z) + \lambda w(z) = F(z), \quad (1)$$

where b and λ are constants, and μ is real, is discussed. And $w_\mu(z)$ denotes Nishimoto's fractional order's differintegral of the function $w(z)$ again [12].

(I) $$w_{2\mu}(z) + bw_\mu(z) + \lambda w(z) = 0 \quad (2)$$

is a differential equation for $\mu>0$,

and is an integral equation for $\mu<0$.

Using matrix, we have following general solution for (2).

$$w = \sum_{k=0}^{m}\left\{C_{1,k}\, e^{G_k z} + C_{2,k}\, e^{H_k z}\right\} \qquad (\text{for } b^2 - 4\lambda \neq 0), \quad (3)$$

where

$$\delta^\mu=\frac{-b\pm\sqrt{b^2-4\lambda}}{2}=\begin{cases}\frac{1}{2}(-b+\sqrt{b^2-4\lambda}) \equiv \rho_1 e^{i\phi_1} = \rho_1 e^{i\phi_1}\cdot e^{i2\pi k} & (4)\\ \frac{1}{2}(-b-\sqrt{b^2-4\lambda}) \equiv \rho_2 e^{i\phi_2} = \rho_2 e^{i\phi_2}\cdot e^{i2\pi k} & (5)\end{cases}$$

for $b^2 - 4\lambda \neq 0$, and

$$\delta = \left\{\rho_1 e^{i(\phi_1+2\pi k)}\right\}^{1/\mu} = \rho_1^{1/\mu}\cdot e^{i(\phi_1/\mu)}\cdot e^{i(2\pi/\mu)k}$$

$$= g\sigma^k \equiv G_k \qquad (g=\rho_1^{1/\mu}\cdot e^{i(\phi_1/\mu)},\ \sigma=e^{i(2\pi/\mu)},\ k=0,1,2,\cdots,m) \quad (6)$$

$$\delta = \left\{\rho_2 e^{i(\phi_2+2\pi k)}\right\}^{1/\mu} = \rho_2^{1/\mu}\cdot e^{i(\phi_2/\mu)}\cdot e^{i(2\pi/\mu)k}$$

$$= h\sigma^k \equiv H_k \qquad (h=\rho_2^{1/\mu}\cdot e^{i(\phi_2/\mu)},\ \sigma=e^{i(2\pi/\mu)},\ k=0,1,2,\cdots,m) \quad (7)$$

respectively. And m is determined according as μ, that is

m = finite for μ = rational number

and m = infinite for μ = irrational number.

(II) Case of $F(z) = e^{az}$

The general solution of fractional differintegral equation

$$w_{2\mu}(z) + bw_{\mu}(z) + \lambda w(z) = e^{az} \qquad (\mu\text{: real}) \tag{8}$$

is shown as follows.

$$w(z) = \sum_{k=0}^{m} \left\{ C_{1,k}\, e^{G_k z} + C_{2,k}\, e^{H_k z} \right\} + \frac{1}{a^{2\mu} + ba^{\mu} + \lambda}\, e^{az} \tag{9}$$

for $b^2 - 4\lambda \neq 0$ and for $a \neq G_k$, $a \neq H_k (a^{2\mu} + ba^{\mu} + \lambda \neq 0,\ a \neq \delta)$.

(III) Case of $F(z) = \cos az$

The general solution of

$$w_{2\mu}(z) + bw_{\mu}(z) + \lambda w(z) = \cos az \qquad (\mu\text{: real}) \tag{10}$$

is

$$w(z) = \sum_{k=0}^{m} \left\{ C_{1,k}\, e^{G_k z} + C_{2,k}\, e^{H_k z} \right\}$$

$$+ \frac{(a^{2\mu}\cos \pi\mu + ba^{\mu}\cos \frac{\pi}{2}\mu + \lambda)\cos az + (a^{2\mu}\sin \pi\mu + ba^{\mu}\sin \frac{\pi}{2}\mu)\sin az}{(a^{2\mu}\cos \pi\mu + ba^{\mu}\cos \frac{\pi}{2}\mu + \lambda)^2 + (a^{2\mu}\sin \pi\mu + ba^{\mu}\sin \frac{\pi}{2}\mu)^2} \tag{11}$$

for $b^2 - 4\lambda \neq 0$ and $\alpha^2 + \beta^2 \neq 0$, where

$$\alpha = a^{2\mu} \cos \pi\mu + ba^{\mu} \cos \frac{\pi}{2}\mu + \lambda, \tag{12}$$

$$\beta = a^{2\mu} \sin \pi\mu + ba^{\mu} \sin \frac{\pi}{2}\mu. \tag{13}$$

(IV) Miscellaneous and some examples

Some examples.

(i) In case of $b = -5$ and $\lambda = 4$, equation (2) becomes

$$w_{2\mu}(z) - 5w_{\mu}(z) + 4w(z) = 0, \tag{14}$$

and some solutions of which are shown in Table 1.

Table 1.

Erder μ	Differintegral equation	Solution w
1/2	$w_1 - 5w_{1/2} + 4w = 0$	$C_{1,0}e^{16z} + C_{2,0}e^{z}$
1/3	$w_{2/3} - 5w_{1/3} + 4w = 0$	$C_{1,0}e^{64z} + C_{2,0}e^{z}$
1/4	$w_{1/2} - 5w_{1/4} + 4w = 0$	$C_{1,0}e^{256z} + C_{2,0}e^{z}$
2/3	$w_{4/3} - 5w_{2/3} + 4w = 0$	$C_{1,0}e^{8z} + C_{1,1}\exp\{8e^{i3\pi}z\} + C_{2,0}e^{z} + C_{2,1}\exp\{e^{i3\pi}z\}$
3/2	$w_3 - 5w_{3/2} + 4w = 0$	$C_{1,0}\exp\{4^{2/3}z\} + C_{1,1}\exp\{4^{2/3}e^{i(4/3)\pi}z\} + C_{1,2}\exp\{4^{2/3}e^{i(8/3)\pi}z\}$
		$+ C_{2,0}e^{z} + C_{2,1}\exp\{e^{i(4/3)\pi}z\} + C_{2,2}\exp\{e^{i(8/3)\pi}z\}$
1	$w_2 - 5w_1 + 4w = 0$	$C_{1,0}e^{4z} + C_{2,0}e^{z}$
2	$w_4 - 5w_2 + 4w = 0$	$C_{1,0}e^{2z} + C_{1,1}\exp\{2e^{i\pi}z\} + C_{2,0}e^{z} + C_{2,1}\exp\{e^{i\pi}z\}$
−2/3	$w_{-4/3} - 5w_{-2/3} + 4w = 0$	$C_{1,0}\exp\{(1/8)z\} + C_{1,1}\exp\{(1/8)e^{-i3\pi}z\} + C_{2,0}e^{z} + C_{2,1}\exp\{e^{-i3\pi}z\}$
−3/2	$w_{-3} - 5w_{-3/2} + 4w = 0$	$C_{1,0}\exp\{4^{-2/3}z\} + C_{1,1}\exp\{4^{-2/3}e^{-i(4/3)\pi}z\} + C_{1,2}\exp\{4^{-2/3}e^{-i(8/3)\pi}z\}$
		$+ C_{2,0}e^{z} + C_{2,1}\exp\{e^{-i(4/3)\pi}z\} + C_{2,2}\exp\{e^{-i(8/3)\pi}z\}$

(ii) In case of $b = 1$, $\lambda = -2$ and $\mu = 1/2$, equation (2) becomes

$$w_1(z) + w_{1/2}(z) - 2w(z) = 0. \tag{15}$$

And, (4) and (5) are reduced to

$$\delta^{\frac{1}{2}} = \frac{1}{2}(-1 \pm \sqrt{1+8}) = \begin{cases} 1 = e^{i2\pi k} & (16) \\ -2 = 2e^{i\pi} \cdot e^{i2\pi k} & (17) \end{cases}$$

We have then $\delta = G_k = e^{i4\pi k}$ $\therefore G_o = 1$ (for $k=0$)
from (16), and $\delta = H_k = 2^2 e^{i2\pi} \cdot e^{i4\pi k}$ $\therefore H_o = 4e^{i2\pi} = 4$ (for $k=0$)
from (17). Therefore we obtain

$$w(z) = C_{1,o}\, e^{z} + C_{2,o}\, e^{4z} \tag{18}$$

as the general solution of (15). The same differintegral equation with (15) is treated in the volume of K.B.Oldham and J.Spanier [24], but the author's above solution is different from their solution.

(iii) In case of $b=0$ and $2\mu=\nu$, equation (2), (8) and (10) are reduced to

$$w_\nu(z) + \lambda w(z) = 0 \tag{2}$$

$$w_\nu(z) + \lambda w(z) = e^{az} \tag{8}$$

and

$$w_\nu(z) + \lambda w(z) = \cos az, \tag{10}$$

respectively. The solutions of these equations are obtained by (3), (9) and (11) respectively, of ourse, in which $b=0$ and $2\mu=\nu$. And these solutions are coincide with those of previous section.

(iv) In the same way with (I), we can obtain the solution of linear differintegral equation of type

$$w_{n\mu} + a_{n-1} w_{(n-1)\mu} + \cdots + a_1 w_\mu + a_o w = 0 \quad (\mu = \text{fractional},\ n = \text{integer} > 0), \tag{19}$$

where $a_{n-1}, \cdots, a_1$ and a_o are constants, in general, if we can get the characteristic value of following matrix A, we can obtain the solution of above equation (19).

$$A = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdot & \cdots & -a_1 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & 0 & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & 0 & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

For matrix A, we have following characteristic equation

$$|A - \delta^\mu E_n| = 0 \qquad (E_n = \text{unit matrix of n-th degree}), \tag{20}$$

for characteristic value δ^μ. And (20) is a n-th degree's equation for δ^μ, consequently, in case of $n \geqslant 5$, we can not solve (19) in general. But in special case we can obtain characteristic value for $n \geqslant 5$. Therefore in that case we can solve (19) for $n \geqslant 5$.

Chapter 7. Orthogonality, norm and some properties of some fractional differintegrated functions

§1. Orthogonality and norm of function $(\cos az)_\nu$ [35]

(I) Orthogonality of function $(\cos az)_\nu$

Theorem 1. If $\nu = \mu + 2m + 1$ (m = integer), we obtain

$$\int_{-\pi/2a}^{\pi/2a} w_\mu \, w_\nu \, dz = 0 \qquad (1)$$

for $a \neq 0$, where $w_\nu = (\cos az)_\nu$ and ν is real.

(II) Norm of function $(\cos az)_\nu$

Theorem 2. We have $$\| w_\nu \| = \sqrt{(w_\nu,\ w_\nu)} = a^\nu \sqrt{\frac{\pi}{2a}} \qquad (2)$$

as the norm of the function $w_\nu = (\cos az)_\nu$, for real $a > 0$ and for real ν.

§2. Orthogonality and norm of function $(\sin az)_\nu$

(I) Orthogonality of function $(\sin az)_\nu$

Theorem 1. If $\nu = \mu + 2m + 1$ (m = integer), we obtain

$$\int_{-\pi/2a}^{\pi/2a} w_\mu \, w_\nu \, dz = 0 \qquad (1)$$

for $a \neq 0$, where $w_\nu = (\sin az)_\nu$ and ν is real.

(II) Norm of function $(\sin az)_\nu$

Theorem 2. We have $$\| w_\nu \| = \sqrt{(w_\nu,\ w_\nu)} = a^\nu \sqrt{\frac{\pi}{2a}}\ , \qquad (8)$$

as the norm of the function $w_\nu = (\sin az)_\nu$, for real $a > 0$ and for real ν.

§3. Relationships of $(\cos az)_\nu$ and $(\sin az)_\nu$ [35]

We have following theorems through the results of §1 and §2.

Theorem 1. We have

$$\|(\cos az)_\nu\| = \|(\sin az)_\nu\| = a^\nu \sqrt{\frac{\pi}{2a}}\ , \qquad (1)$$

for real $a > 0$ and real ν.

Theorem 2. We have

$$\int_{-\pi/2a}^{\pi/2a} (\cos az)_\mu (\cos az)_\nu dz = \int_{-\pi/2a}^{\pi/2a} (\sin az)_\mu (\sin az)_\nu dz \qquad (2)$$

$$= a^{\mu+\nu-1} \frac{\pi}{2} \cos \frac{\pi}{2} (\nu - \mu) \qquad (\text{for } a \neq 0). \qquad (3)$$

§4. Differintegration (with respect to order ν) of the fractional differintegrated trigonometric functions $(\cos az)_\nu$ and $(\sin az)_\nu$ [35]

We have following functional relationships for $(\cos az)_\nu$ and $(\sin az)_\nu$, since they are regular functions for the order ν. (for $a \neq 0$).

Theorem 1.

(i) $\frac{d}{d\nu}(\cos az)_\nu = (\cos az)_\nu \log a - \frac{\pi}{2} (\sin az)_\nu$ (1)

(ii) $\frac{d}{d\nu}(\cos z)_\nu = - \frac{\pi}{2} (\sin z)_\nu$ (2)

(iii) $\frac{d}{d\nu}(\sin az)_\nu = (\sin az)_\nu \log a + \frac{\pi}{2} (\cos az)_\nu$ (3)

(iv) $\frac{d}{d\nu}(\sin z)_\nu = \frac{\pi}{2} (\cos z)_\nu$ (4)

Theorem 2.

(i) $$\frac{d}{d\nu}\left\{(\sin az)_\nu(\cos az)_\nu\right\} = 2(\cos az)_\nu(\sin az)_\nu \log a + \frac{\pi}{2}\left[\left\{(\cos az)_\nu\right\}^2 - \left\{(\sin az)_\nu\right\}^2\right] \tag{5}$$

(ii) $$\frac{d}{d\nu}\left\{(\sin z)_\nu (\cos z)_\nu\right\} = \frac{\pi}{2}\left[\left\{(\cos z)_\nu\right\}^2 - \left\{(\sin z)_\nu\right\}^2\right] \tag{6}$$

(iii) $$\frac{d}{d\nu}\left\{\frac{(\sin az)_\nu}{(\cos az)_\nu}\right\} = \frac{\pi}{2}\left[1 + \left\{\frac{(\sin az)_\nu}{(\cos az)_\nu}\right\}^2\right] = \frac{\pi}{2} a^{2\nu}\left\{(\cos az)_\nu\right\}^{-2} \tag{7}$$

(iv) $$\frac{d}{d\nu}\left\{\frac{(\sin z)_\nu}{(\cos z)_\nu}\right\} = \frac{\pi}{2}\left[1 + \left\{\frac{(\sin z)_\nu}{(\cos z)_\nu}\right\}^2\right] = \frac{\pi}{2}\left\{(\cos z)_\nu\right\}^{-2} \tag{8}$$

Theorem 3. Omitting the constants of integration, we have then

(i) $$\int (\cos az)_\nu\, d\nu = \frac{1}{1 + (\frac{2}{\pi}\log a)^2}\left\{\frac{2}{\pi}(\sin az)_\nu + (\frac{2}{\pi})^2(\cos az)_\nu \log a\right\} \tag{9}$$

(ii) $$\int (\cos z)_\nu\, d\nu = \frac{2}{\pi}(\sin z)_\nu, \tag{10}$$

(iii) $$\int (\sin az)_\nu\, d\nu = \frac{1}{1+(\frac{2}{\pi}\log a)^2}\left\{-\frac{2}{\pi}(\cos az)_\nu + (\frac{2}{\pi})^2(\sin az)_\nu \log a\right\} \tag{11}$$

(iv) $$\int (\sin z)_\nu\, d\nu = -\frac{2}{\pi}(\cos z)_\nu . \tag{12}$$

Theorem 4. Omitting the constants of integration, we have

(i) $$\log a\cdot \int (\cos az)_\nu\, d\nu - \frac{\pi}{2}\int (\sin az)_\nu\, d\nu = (\cos az)_\nu \tag{13}$$

(ii) $$\log a\cdot \int (\sin az)_\nu\, d\nu + \frac{\pi}{2}\int (\cos az)_\nu\, d\nu = (\sin az)_\nu \tag{14}$$

(iii) $$2\log a\cdot \int (\cos az)_\nu (\sin az)_\nu\, d\nu + \frac{\pi}{2}\int\left\{(\cos az)_\nu\right\}^2 d\nu - \frac{\pi}{2}\int\left\{(\sin az)_\nu\right\}^2 d\nu = (\sin az)_\nu(\cos az)_\nu \tag{15}$$

(iv) $$\int\left\{(\cos z)_\nu\right\}^2 d\nu - \int\left\{(\sin z)_\nu\right\}^2 d\nu = \frac{2}{\pi}(\cos z)_\nu(\sin z)_\nu \tag{16}$$

(v) $$\int \frac{1}{\left\{(\cos az)_\nu\right\}^2}\, d\nu = \frac{2}{\pi a^{2\nu}}\cdot\frac{(\sin az)_\nu}{(\cos az)_\nu} \tag{17}$$

(vi) $$\int \frac{1}{\left\{(\cos z)_\nu\right\}^2}\, d\nu = \frac{2}{\pi}\cdot\frac{(\sin z)_\nu}{(\cos z)_\nu} \tag{18}$$

Theorem 5. $(\cos az)_\nu$ and $(\sin az)_\nu$ satisfy following differential equations (19), (20) and (21), (22) respectively for $a \neq 0$.

(i) $$\frac{d^2w}{d\nu^2} - \log a\cdot\frac{dw}{d\nu} + (\frac{\pi}{2})^2 w = -\frac{\pi}{2}\log a\cdot(\sin az)_\nu \tag{19}$$

(ii) $$\frac{d^2w}{d\nu^2} + (\frac{\pi}{2})^2 w = 0 \qquad (\text{for } a=1) \tag{20}$$

(iii) $$\frac{d^2w}{d\nu^2} - \log a\cdot\frac{dw}{d\nu} + (\frac{\pi}{2})^2 w = \frac{\pi}{2}\log a\cdot(\cos az)_\nu \tag{21}$$

(iv) $$\frac{d^2w}{d\nu^2} + (\frac{\pi}{2})^2 w = 0, \qquad (\text{for } a=1) \tag{22}$$

where $w = (\cos az)_\nu$ for (i) and (ii), and $w = (\sin az)_\nu$ for (iii) and (iv).

§5. Some fractional differintegral equations satisfied with

$(\sin az)_\nu$ and $(\cos az)_\nu$

Theorem 1. $w_\nu = (\sin az)_\nu$ satisfies following differintegral equations.

(i) $$w_{\nu+2} + a^2 w_\nu = 0, \tag{1}$$

(ii) $$w_{\nu+2} + aw_{\nu+1} + a^2 w_\nu = a^2 (\cos az)_\nu \tag{2}$$

where $a \neq 0$, and ν is arbitrary.

Theorem 2. $w_\nu = (\cos az)_\nu$ satisfies following differintegral equations.

(i) $$w_{\nu+2} - a^2 w_\nu = 0 \tag{3}$$

(ii) $$w_{\nu+2} - aw_{\nu+1} + a^2 w_\nu = a^2 (\sin az)_\nu \tag{4}$$

where $a \neq 0$, and ν is arbitrary.

Note 1. In case of ν = integer, equations (1), (2), (3) and (4) become integer order's differintegral equations.

Note 2. Equations (1), (2), (3) and (4) are fractional order's differintegral equations with constant coefficients, hence we can solve them with the method which is shown in Chapter 6.

§6. Some fractional differintegral equations satisfied with $(\sin z \cdot z)_\nu$

Theorem 1. $w_\nu = (\sin z \cdot z)_\nu$ satisfies following fractional differintegral equations.

(i) $$w_{\nu+1} - \frac{\nu+1}{z} w_\nu = \frac{z^2 + \nu^2 + \nu}{z} (\cos z)_\nu \tag{1}$$

(ii) $$w_{\nu+2} + w_{\nu+1} - \frac{\nu+1}{z} w_\nu = \left\{ \frac{z^2 + z(\nu+2) + \nu^2 + \nu}{z} \right\} (\cos z)_\nu - (\sin z)_\nu \cdot z \tag{2}$$

(iii) $$w_{\nu+2} - \frac{\nu+2}{z} w_{\nu+1} + \frac{(\nu+1)(\nu+2)}{z^2} w_\nu = -z(\sin z)_\nu - \frac{\nu(\nu+1)(\nu+2)}{z^2} (\cos z)_\nu \tag{3}$$

where ν is arbitrary and $z \neq 0$.

§7. Some fractional differintegral equations satisfied with $(e^{az}\, z)_\nu$

Theorem 1. $w(z) = (e^{az}\, z)(az \neq 0)$ satisfies following fractional differintegral equations.

(i) $$w_\nu(z) - a^\nu + \frac{\nu a^{\nu-1}}{z} w(z) = 0 \tag{1}$$

(ii) $$w_{\nu+2}(z) - aw_{\nu+1}(z) - \frac{a^2}{az+\nu} w_\nu(z) = 0 \tag{2}$$

(iii) $$w_{\nu+2}(z) - \frac{\nu+2}{z} w_{\nu+1}(z) + \frac{(\nu+1)(\nu+2)}{z^2} w_\nu(z)$$
$$= a^{\nu+2} e^{az} z + a^{\nu-1} \nu(\nu+1)(\nu+2) e^{az} \frac{1}{z^2} \tag{3}$$

for arbitrary ν. That is, $w = e^{az}z$ is the solution of fractional differintegral equation of (1) and $w_\nu = (e^{az}z)_\nu$ is the solution of fractional differintegral equation of (2) and (3).

§8. Orthogonality and norm of function $(e^{az}z)_\nu$

(I) Orthogonality of function $(e^{az}z)_\nu$

Theorem 1. If

$$\begin{cases} \mu = \frac{1}{2}(1+\delta) \\ \nu = \frac{1}{2}(1-\delta) \end{cases} \quad \text{or} \quad \begin{cases} \mu = \frac{1}{2}(1-\delta) \\ \nu = \frac{1}{2}(1+\delta) \end{cases} \tag{1}$$

we have then $$\int_{-1}^{1} w_\mu w_\nu dz = 0 \tag{2}$$

for $a \neq 0$, where $w_\nu = (e^{az}z)_\nu$ and $\delta = \sqrt{1+4a^2}$.

That is, functions $(e^{az}z)_\mu$ and $(e^{az}z)_\nu$ are orthogonal on the interval $[-1, 1]$, for μ and ν which are shown in (1) and for $a \neq 0$.

(II) Norm of function $(e^{az}z)_\mu$

Theorem 2. We have

$$\|w_\mu\| = \sqrt{(w_\mu, w_\mu)} = \left[a^{2\mu-1}\left\{\left(1+\frac{1}{2a^2}-\frac{\mu}{a^2}+\frac{\mu^2}{a^2}\right)\sinh 2a + \frac{1}{a}(2\mu-1)\cosh 2a\right\}\right]^{\frac{1}{2}} \tag{3}$$

as the norm of the function $w_\mu = (e^{az}z)_\mu$ for real $a > 0$ and for real μ such that

$$a^{2\mu-1}\left\{\left(1+\frac{1}{2a^2}-\frac{\mu}{a^2}+\frac{\mu^2}{a^2}\right)\sinh 2a + \frac{1}{a}(2\mu-1)\cosh 2a\right\} > 0.$$

In case of $a = \mu$, (3) is reduced to

$$\|w_\mu\| = \left[\mu^{2\mu-1}\left\{\left(\frac{4\mu^2+1-2\mu}{2\mu^2}\right)\sinh 2\mu + \left(2-\frac{1}{\mu}\right)\cosh 2\mu\right\}\right]^{\frac{1}{2}}, \tag{4}$$

for $$\mu^{2\mu-1}\left\{\left(\frac{4\mu^2+1-2\mu}{2\mu^2}\right)\sinh 2\mu + \left(2-\frac{1}{\mu}\right)\cosh 2\mu\right\} > 0.$$

And in case of $\mu = \frac{1}{2}$, (3) is reduced to

$$\|w_{\frac{1}{2}}\| = \sqrt{1+1/4a^2}\,\sqrt{\sinh 2a} \qquad (\text{for real } a > 0). \tag{5}$$

Chapter 8

§1. Table of fractional differintegrations of elementary functions (by Nishimoto)

Following is a table of fractional differintegrations of elementary functions which is obtained by the author's definition for fractional calculus.

Table 1

Nishimoto's Fractional differintegrations of elementary functions

$f(z)$	$f_\nu(z)$	
1. 1	$0 \qquad (\nu \neq -m,\ m = \text{integer} > 0)$	
2. $e^{az} \quad (a \neq 0)$	$a^\nu e^{az}$	
3. $e^{-az} \quad (a \neq 0)$	$e^{-i\pi\nu} a^\nu e^{-az}$	
4. $\cosh az \quad (a \neq 0)$	$(-ia)^\nu \cosh(az + i\frac{\pi}{2}\nu)$	
5. $\sinh az \quad (a \neq 0)$	$(-ia)^\nu \sinh(az + i\frac{\pi}{2}\nu)$	
6. $\cos az \quad (a \neq 0)$	$a^\nu \cos(az + \frac{\pi}{2}\nu)$	
7. $\sin az \quad (a \neq 0)$	$a^\nu \sin(az + \frac{\pi}{2}\nu)$	
8. z^a	$e^{-i\pi\nu} \frac{\Gamma(\nu-a)}{\Gamma(-a)} z^{a-\nu} \left(\left\lvert \frac{\Gamma(\nu-a)}{\Gamma(-a)} \right\rvert \leqq M,\ M = \text{const.} \right)$	
9. $\log az \quad (a \neq 0)$	$-e^{-i\pi\nu} \Gamma(\nu) z^{-\nu} \quad (\lvert \Gamma(\nu) \rvert \leqq M,\ M = \text{const.})$	
10. $\log z$	$\frac{1}{m!} z^m \log z + z^m \sum_{k=0}^{m-1} \frac{(-1)^{m-k}}{K!\{(m-K)!\}(m-K)}$ $(\nu = -m,\ m = \text{integer} \geqq 1)$	
11. $(z^2-1)^\nu$ $(\arg \sqrt{z^2-1} = \phi)$	$\frac{2^\nu \Gamma(\nu+1)}{2\pi} \int_{-\pi}^{\pi} \{Z + \sqrt{Z^2-1} \cos(\theta-\phi)\}^\nu d\theta$	$Re(z) > 0$, $\nu > 0$
	$2^\nu \Gamma(\nu+1) P_\nu(z) \qquad (\phi = 0)$	
	$\frac{2^\nu}{\Gamma(-\nu)} \int_{-\infty}^{\infty} \{-Z + \sqrt{Z^2-1} \cosh(\varphi - i\phi)\} d\varphi$	$\nu < 0$
	$\frac{2^\nu}{\Gamma(-\nu)} \cdot 2Q_\nu(-Z) \qquad (\phi = 0,\ z(\text{real}) < -1)$	
12. $z^{\nu+1-\gamma} (1-z)^{\gamma-\alpha-1}$	$e^{i\pi\{-\alpha \pm (\gamma-\nu-1)\}} \frac{\Gamma(\alpha)}{\nu\Gamma(\alpha-\nu)} z^{-\alpha}$ $\times\ {}_2F_1(\alpha, \alpha-\gamma+1 : \alpha-\nu : \frac{1}{z})$ $\left(+ \text{ for } \lvert\arg z\rvert < \frac{\pi}{2},\ - \text{ for } \frac{\pi}{2} < \lvert\arg z\rvert < \pi \text{ for double sign } \pm \text{ and } \lvert z\rvert > 1. \right)$	

Note. There are some tables for fractional integral or fractional derivatives, for example, the tables of fractional integral by A.Erdélyi et al. [23], and that of fractional derivative by T.J.Osler et al. [25]. The author hopes that all readers of this paper compare above results in Table 1 with that of another papers.

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