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A Generalization of Fractional Calculus and Its Applications to Euler-Darboux Equation

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1. Introduction.

The fractional calculus has been investigated by many mathematicians. (see [14]) In their works the Riemann-Liouville operator (R-L) was most central, while Erdélyi and Kober defined their operator (E-K) in connection with the Hankel transform. (see [8]) Thereafter various generalizations have been made. (see [13]) Here we shall define a certain integral operator involving the Gauss hypergeometric function. (cf. Definition 1) Such an integral was first treated by Love [9] as an integral equation. However, if we regard the integral as an operator with a slight change, it contains both R-L and E-K as its special cases owing to reduction formulas for the Gauss function by restricting the parameters. The more interesting fact is that for our operator two kinds of product rules may be made up by virtue of Erdélyi's

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formulas (see [3]), which were first proved by using the method of fractional integration by parts in the R-L sense. From the rules, of course, the ones for R-L and E-K are deduced. Moreover, our operator is representable by products of R-L's, from which it is possible to obtain the integrability and estimations of Hardy-Littlewood type [6]. We shall also define, in parallel, another integral operator on the interval (x, ∞), which is an extension of operators of Weyl and another Erdélyi-Kober. (cf. Definition 2) Then the formula of integration by parts with respect to our integral operators is obtained. Section 3 is devoted to define the fractional derivative corresponding to the above fractional integrals and to investigate several formulas.

The fractional calculus has been applied to various problems in analysis. (see [13]) For instance, it is known that the calculus is valuable in the theory of equations of mixed type or hyperbolic type with degeneration. Nahušev [11] posed a problem for the degenerate hyperbolic equation

$$y^m \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, \qquad (m > 0), \tag{1.1}$$

in $y > 0$ imposing boundary conditions, one of which is a linear combination of fractional derivatives of the solution on both characteristics. Then many results for various similar problems have been published. (see [12], [2], [22], [23] etc.) In their works, however, the calculus is of the sense of R-L and the orders of the derivatives are fixed to some numbers. The rest of this report, Sections $4 \sim 8$, is devoted to discuss boundary value problems for the Euler-Darboux equation

$$(1.2) \qquad \frac{\partial^2 u}{\partial x \partial y} - \frac{\beta}{x-y}\frac{\partial u}{\partial x} + \frac{\alpha}{x-y}\frac{\partial u}{\partial y} = 0, \quad (\alpha > 0,\ \beta > 0,\ \alpha + \beta < 1),$$

in the domain $\{0 < x < y < 1\}$ as applications of our fractional calculus. Note that the equation (1.2) implies the equation (1.1) ($\alpha = \beta = m/(2m+4)$) or some other degenerate hyperbolic equations being expressed by the characteristic coordinates. The values of the solution of (1.2) on its characteristics $x = 0$ and $y = 1$ are represented by our fractional integrals. Then we set Problem I for (1.2) with such boundary conditions that the generalized fractional integrals or derivatives of the solution on $x = 0$ and $y = 1$ are equal to some given functions. Orders of the fractional integrals or derivatives in these conditions are chosen between some positive and negative numbers depending on α and β. This problem is regarded as a generalization of the Goursat problem, i.e. the characteristic initial value problem. Problem I is reduced to a dominant singular integral equation with the Cauchy kernel, and the expression of the solution is given by virtue of the theory of singular integral equations [5]. In that place, the validity of the Hölder continuity of the fractional integrals and derivatives of Hölder continuous functions is necessary.

Problems II and III are Nahušev's type problems stated above. The boundary conditions are to assume the value of the solution on the non-characteristic boundary $x = y$ and the value of a linear combination of the generalized fractional integrals or derivatives of the solution on both characteristics. Problem II differs from Problem III in coefficients of their second boundary conditions. That is, while that of Problem II are constants, Problem III involves some powers of x and 1-x. Then in calculations and for the solvability different methods are required. In Problem II, the reduced integral equation is the dominant type similar to the case of Problem I, and then the solution can be represented explicitly. But Problem III is reduced to a singular integral equation containing the logarithmic kernel together with the Cauchy kernel. The solvability is discussed on weighted L_p spaces by refering to the theory by Hvedelidze [7], where we need to ask for the aid of Lipschitz spaces in L_p in order to ensure the integrability of our fractional integrals and derivatives, which are discussed in Section 8.

For detailed results and calculations of this report, see [15], [16], [17], [18], [19], [20] and [21].

2. Definitions of Fractional Integrals and Their Properties.

Definition 1. Let $\alpha > 0$, β and η be real numbers. The integral operator $I_{0x}^{\alpha,\beta,\eta}$, which acts on certain functions f(x) on the interval $(0,\infty)$, is defined by

$$(2.1)\qquad I_{0x}^{\alpha,\beta,\eta} f = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} F(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x})\, f(t)dt,$$

where Γ is the gamma function, F means the Gauss hypergeometric series

(2.2) $$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \qquad |z| < 1,$$

and $(a)_n = \Gamma(a+n)/\Gamma(a)$.

Definition 2. Under the same assumptions in Definition 1, the integral operator $J_{x\infty}^{\alpha,\beta,\eta}$ is defined by

(2.3) $$J_{x\infty}^{\alpha,\beta,\eta} f = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} t^{-\alpha-\beta} F(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}) f(t)\,dt.$$

Remark 1. When $\alpha+\beta=0$ or $\beta=0$, I and J are reduced to the following integral operators:

(2.4) $$I_{0x}^{\alpha,-\alpha,\eta} f = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,dt = R_{0x}^{\alpha} f,$$

(Riemann-Liouville)

(2.5) $$I_{0x}^{\alpha,0,\eta} f = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\,dt = E_{0x}^{\alpha,\eta} f,$$

(Erdélyi-Kober)

(2.6) $$J_{x\infty}^{\alpha,-\alpha,\eta} f = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)\,dt = W_{x\infty}^{\alpha} f,$$ *(Weyl)*

(2.7) $$J_{x\infty}^{\alpha,0,\eta} f = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} t^{-\alpha-\eta} f(t)\,dt = K_{x\infty}^{\alpha,\eta} f.$$

(Erdélyi-Kober)

Theorem 1. *The following formulas are valid:*

(2.8) $$I_{0x}^{\alpha,\beta,\eta} f = x^{-\alpha-\beta-\eta} I_{0x}^{\alpha,-\alpha-\eta,-\alpha-\beta} f,$$

(2.9) $$I_{0x}^{\alpha,\beta,\eta} x^{\beta-\eta} f = I_{0x}^{\alpha,\eta,\beta} f,$$

(2.10) $$J_{x\infty}^{\alpha,\beta,\eta}\, x^{\alpha+\beta+\eta} f = J_{x\infty}^{\alpha,-\alpha-\eta,-\alpha-\beta} f,$$

(2.11) $$J_{x\infty}^{\alpha,\beta,\eta} f = x^{\eta-\beta} J_{x\infty}^{\alpha,\eta,\beta} f.$$

L_p denotes the usual class of p-th power integrable functions on the interval $(0, \infty)$ with the norm $\|\cdot\|_p$ for $1 \le p < \infty$, and L_∞ essentially bounded functions with $\|\cdot\|_\infty$. If we combine results of Hardy and Littlewood [6], Kober [8] and Flett [4], we obtain

Lemma 1. *Let* $1 \le p \le q \le \infty$, $a < 1-1/p$ *and* $b > \alpha-1/p$. *If functions* $f(x)$ *and* $g(x)$ *satisfy* $x^a f \in L_p$ *and* $x^b g \in L_p$, *and* $\alpha > 1/p - 1/q$, *where* α *may be equal to* $1/p - 1/q$ *except the cases* $1 = p < q < \infty$ *and* $1 < p < q = \infty$, *then* $x^{1/p-1/q-\alpha+a} R_{0x}^{\alpha} f$ *and* $x^{1/p-1/q-\alpha+b} W_{x\infty}^{\alpha} g$ *belong to* L_q *and there hold the estimations*

(2.12) $$\|x^{1/p-1/q-\alpha+a} R_{0x}^{\alpha} f\|_q \le C\|x^a f\|_p ,$$

(2.13) $$\|x^{1/p-1/q-\alpha+b} W_{x\infty}^{\alpha} g\|_q \le C\|x^b g\|_p .$$

Definition 3. Let $1 \le p \le q \le \infty$. The condition $A_1(\alpha,\beta,\eta;a;p,q)$ means that the members satisfy (1°) $a < \min(0,-\beta+\eta) - 1/p + 1$, (2°) (i) $\alpha \ge -\beta + 1/p - 1/r \ge 1/p - 1/q$ for $p \le r \le q$, or (ii) $\alpha \ge -\eta + 1/p - 1/r \ge 1/p - 1/q$ for $p \le r \le q$. (If $1 = p < q < \infty$ or $1 < p < q = \infty$, one of the equal signs in (i) and (ii) should be excluded.) If, instead of (1°), we assume (1°)′ $b > -\min(\beta,\eta) - 1/p$, then that is called the condition $A_2(\alpha,\beta,\eta;b;p,q)$.

Theorem 2. *Let* $1 \le p \le q \le \infty$. *Assume the conditions* $A_1(\alpha,\beta,\eta;a;p,q)$ *and* $A_2(\alpha,\beta,\eta;b;p,q)$, *then* $x^{1/p-1/q+\beta+a} I_{0x}^{\alpha,\beta,\eta} f$ *and* $x^{1/p-1/q+\beta+b} J_{x\infty}^{\alpha,\beta,\eta} g$ *belong to* L_q *for any functions* $f(x)$ *and* $g(x)$ *with* $x^a f \in L_p$ *and* $x^b g \in L_p$, *and there hold the estimations*

$$\|x^{1/p-1/q+\beta+a} I_{0x}^{\alpha,\beta,\eta} f\|_q \le C\|x^a f\|_p, \tag{2.14}$$

$$\|x^{1/p-1/q+\beta+b} J_{x\infty}^{\alpha,\beta,\eta} g\|_q \le C\|x^b g\|_p. \tag{2.15}$$

Furthermore the following decompositions are valid:

Case (i);

$$I_{0x}^{\alpha,\beta,\eta} f = x^{-\alpha-\beta-\eta} R_{0x}^{\alpha+\beta} x^{\eta} R_{0x}^{-\beta} f = R_{0x}^{-\beta} x^{-\alpha-\eta} R_{0x}^{\alpha+\beta} x^{\eta-\beta} f, \tag{2.16}$$

$$J_{x\infty}^{\alpha,\beta,\eta} g = x^{\eta-\beta} W_{x\infty}^{\alpha+\beta} x^{-\alpha-\eta} W_{x\infty}^{-\beta} g = W_{x\infty}^{-\beta} x^{\eta} W_{x\infty}^{\alpha+\beta} x^{-\alpha-\beta-\eta} g, \tag{2.17}$$

Case (ii);

$$I_{0x}^{\alpha,\beta,\eta} f = x^{-\alpha-\beta-\eta} R_{0x}^{\alpha+\eta} x^{\beta} R_{0x}^{-\eta} x^{\eta-\beta} f = R_{0x}^{-\eta} x^{-\alpha-\beta} R_{0x}^{\alpha+\eta} f, \tag{2.18}$$

$$J_{x\infty}^{\alpha,\beta,\eta} g = W_{x\infty}^{\alpha+\eta} x^{-\alpha-\beta} W_{x\infty}^{-\eta} g = x^{\eta-\beta} W_{x\infty}^{-\eta} x^{\beta} W_{x\infty}^{\alpha+\eta} x^{-\alpha-\beta-\eta} g. \tag{2.19}$$

Theorem 3. *Let* $1 \le p \le \infty$, $1 \le q \le \infty$ *and* $1/p + 1/q \ge 1$. *Suppose that constants* a, b, α, β *and* η *satisfy the conditions:* $a < \min(0,-\beta+\eta) - 1/p + 1$, $a + b = 1 - 1/p - 1/q + \beta$ *and* (i) $\alpha \ge -\beta + 1/p - 1/r \ge 1/p + 1/q - 1$ *for* $p \le r \le q/(q-1)$, *or* (ii) $\alpha \ge -\eta + 1/p - 1/r \ge 1/p + 1/q - 1$ *for* $p \le r \le q/(q-1)$. (*If* $1 = p < q < \infty$ *or* $1 = q < p < \infty$, *one of the equal signs in* (i) *and* (ii) *should be omitted.*) *If* $x^a f \in L_p$ *and* $x^b g \in L_q$, *then there holds the equality*

(2.20) $$\int_0^\infty g(x)\, I_{0x}^{\alpha,\beta,\eta} f\, dx = \int_0^\infty f(x)\, J_{x\infty}^{\alpha,\beta,\eta} g\, dx.$$

(The generalized fractional integration by parts)

Theorem 4. *Let $\alpha > \gamma > 0$. Under the same assumptions in Theorem 2, there hold the following decompositions:*

(2.21) $$I_{0x}^{\alpha,\beta,\eta} f = I_{0x}^{\gamma,\delta,\eta}\, I_{0x}^{\alpha-\gamma,\beta-\delta,\gamma+\eta} f$$

$$= I_{0x}^{\alpha-\gamma,\beta-\delta,\eta+\gamma+\delta}\, I_{0x}^{\gamma,\delta,\eta-\beta+\delta} f,$$

(2.22) $$J_{x\infty}^{\alpha,\beta,\eta} g = J_{x\infty}^{\alpha-\gamma,\beta-\delta,\gamma+\eta}\, J_{x\infty}^{\gamma,\delta,\eta} g$$

$$= J_{x\infty}^{\gamma,\delta,\eta-\beta+\delta}\, J_{x\infty}^{\alpha-\gamma,\beta-\delta,\eta+\gamma+\delta} g.$$

Further properties of I and J are found in [15].

3. Generalized Fractional Integral and Derivative.

Definition 4. Let $\alpha > 0$, β, η and a be real numbers. We define the integral

(3.1) $$I_{ax}^{\alpha,\beta,\eta} f = \frac{(x-a)^{-\alpha-\beta}}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} F(\alpha+\beta,-\eta;\alpha;\frac{x-t}{x-a})\, f(t)\, dt$$

for a real valued and continuous function $f(x)$ defined on (a,∞) having order $O((x-a)^k)$ near $x=a$ with $k > \max(0,\beta-\eta) - 1$. When $\alpha \le 0$, by letting n be a positive integer such as $0 < \alpha + n \le 1$, we define

(3.2) $$I_{ax}^{\alpha,\beta,\eta} f = \frac{d^n}{dx^n}\, I_{ax}^{\alpha+n,\beta-n,\eta-n} f,$$

if the right hand has a definite meaning.

The expressions (3.1) and (3.2) are understood as generalizations of the fractional calculus of Riemann-Liouville, and (3.1) also contains the Erdélyi-Kober fractional integral, which are noted in Remark 1.

Another fractional integral (and derivative) are as follows:

Definition 5. Let $\alpha > 0$, β, η and b be real numbers. We define

$$(3.3) \quad J_{xb}^{\alpha,\beta,\eta} f = \frac{(b-x)^{-\alpha-\beta}}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} F(\alpha+\beta,-\eta;\alpha;\frac{t-x}{b-x}) f(t)\, dt,$$

where $f(x)$ is a real valued and continuous function defined on $(-\infty,b)$ having order $O((b-x)^k)$ near $x = b$ with $k > \max(0,\beta-\eta) - 1$. When $0 < \alpha + n \le 1$ for positive integer n, we define

$$(3.4) \quad J_{xb}^{\alpha,\beta,\eta} f = (-1)^n \frac{d^n}{dx^n} J_{xb}^{\alpha+n,\beta-n,\eta-n} f ,$$

if the right hand side has a definite meaning. If $\beta = -\alpha$, we write

$$(3.5) \quad J_{xb}^{\alpha,-\alpha,\eta} f = L_{xb}^{\alpha} f .$$

By changing the parameters in (2.21) we have the following product rules:

$$(3.6) \quad I_{ax}^{\alpha,\beta,\eta} I_{ax}^{\gamma,\delta,\alpha+\eta} = I_{ax}^{\alpha+\gamma,\beta+\delta,\eta}, \quad (\alpha,\gamma > 0),$$

$$(3.7) \quad I_{ax}^{\alpha,\beta,\eta} I_{ax}^{\gamma,\delta,\eta-\beta-\gamma-\delta} = I_{ax}^{\alpha+\gamma,\beta+\delta,\eta-\gamma-\delta}, \quad (\alpha,\gamma > 0).$$

These are still valid for negative α. It is easily seen that

$I_{ax}^{0,0,\eta}$ is the identity operator for any η, then the inverse operator of $I_{ax}^{\alpha,\beta,\eta}$ is given by

$$(3.8) \qquad (I_{ax}^{\alpha,\beta,\eta})^{-1} = I_{ax}^{-\alpha,-\beta,\alpha+\eta},$$

which follows from (3.6) or (3.7). The same formulas as (3.6) ~ (3.8) hold true also for the operator J defined by (3.3) and (3.4).

Let $H^k[a,b]$ be the class of functions which are Hölder continuous on the interval [a,b] with the Hölder index k, where $0 < k < 1$.

Theorem 5. *Let* $\varphi(x) \in H^k[a,b]$, $\alpha \neq 0$ *and* $\eta > \beta-1$. *If* $0 < k+\alpha < 1$, *then* $(x-a)^\beta I_{ax}^{\alpha,\beta,\eta} \varphi \in H^{k+\min(0,\alpha)}[a,b]$. *If* $0 < k+\min(0,-\beta) < 1$ *and* $\varphi(a) = 0$, *then* $I_{ax}^{\alpha,\beta,\eta} \varphi \in H^{k+\min(0,\alpha,-\beta)}[a,b]$.

Remark 2. Completely similar results to Theorem 5 remain valid by replacing $I_{ax}^{\alpha,\beta,\eta}$ by $J_{xb}^{\alpha,\beta,\eta}$, $(x-a)^\beta$ by $(b-x)^\beta$ and $\varphi(a) = 0$ by $\varphi(b) = 0$.

4. Boundary Value Problems for the Euler-Darboux Equation.

Let Ω be the triangle OAB, where $O = (0,0)$, $A = (0,1)$ and $B = (1,1)$. We investigate the Euler-Darboux equation (1.2) in Ω. It is well known that a solution of (1.2) having conditions

$$u(x,x) = \tau(x), \qquad \lim_{y\to x} (y-x)^{\alpha+\beta} \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\right) = \nu(x)$$

is given by the form

$$u(x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \tau[x+(y-x)t]\, t^{\beta-1}(1-t)^{\alpha-1}\, dt$$

$$+ \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)\Gamma(1-\beta)} (y-x)^{1-\alpha-\beta} \int_0^1 \nu[x+(y-x)t]t^{-\alpha}(1-t)^{-\beta}dt$$

and the values of the solution on two characteristic segments OA and AB are written as

$$(4.1)\begin{cases} u^{(1)}(y) \equiv u(0,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} I_{0y}^{\alpha,0,\beta-1}\tau \\ \qquad + \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)} I_{0y}^{1-\beta,\alpha+\beta-1,\beta-1}\nu, \quad 0<y<1, \text{ and} \\ u^{(2)}(x) \equiv u(x,1) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} J_{x1}^{\beta,0,\alpha-1}\tau \\ \qquad + \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\beta)} J_{x1}^{1-\alpha,\alpha+\beta-1,\alpha-1}\nu, \quad 0<x<1, \end{cases}$$

respectively, by making use of the generalized fractional integrals (3.1) and (3.3).

Now we shall set three problems for the equation (1.2).

Problem I. Find a solution $u(x,y)$ of (1.2) in Ω satisfying the boundary conditions

$$(I.1) \qquad I_{0y}^{a,b,-a+\beta-1}u^{(1)} = \varphi_1(y), \quad 0<y<1, \text{ and}$$

$$(I.2) \qquad J_{x1}^{c,d,-c+\alpha-1}u^{(2)} = \varphi_2(x), \quad 0<x<1,$$

where a, b, c and d are constants such that $-\alpha < a < \beta$, $-\alpha < a+b < \beta$, $-\beta < c < \alpha$ and $-\beta < c+d < \alpha$, and $\varphi_1 \in H^{k_1}[0,1]$ $(a-\beta+1 < k_1 < 1,\ \varphi_1(0) = 0)$ and $\varphi_2 \in H^{k_2}[0,1]$ $(c-\alpha+1 < k_2 < 1,\ \varphi_2(1) = 0)$ are given functions.

Problem II. Find a solution $u(x,y)$ of (1.2) in Ω satisfying the boundary conditions

(II.1) $u(x,x) = \varphi_3(x), \quad 0 < x < 1,$ and

(II.2) $AI_{0x}^{a,b,-a+\beta-1} u^{(1)} + BJ_{x1}^{a+\alpha-\beta,c,-a+\beta-1} u^{(2)} = \varphi_4(x), \quad 0 < x < 1,$

where A and B are non-zero constants, constants a, b and c fulfil the inequalities $-\alpha < a < \beta$, $-\alpha < a+b < \beta$ and $-\alpha < a+c < \beta$, and $\varphi_3 \in H^{k_3}[0,1]$ $(\max(a-\beta+1, a+c-\beta+1) < k_3 < 1$, $\varphi_3(0) = \varphi_3(1) = 0)$ and $\varphi_4 \in H^{k_4}[0,1]$ $(a-\beta+1 < k_4 < 1)$ are given functions.

Problem III Find a solution $u(x,y)$ of (1.2) in Ω satisfying the boundary conditions

(III.1) $u(x,x) = \varphi_5(x), \quad 0 < x < 1,$ and

(III.2) $$Ax^{b+\alpha+\beta-1} I_{0x}^{a,b,-a+\beta-1} u^{(1)}$$

$$+ B(1-x)^{c+\alpha+\beta-1} J_{x1}^{a+\alpha-\beta,c,-a+\beta-1} u^{(2)} = \varphi_6(x), \quad 0 < x < 1,$$

where A and B are non-zero constants, constants a, b and c fulfil the inequalities $-\alpha < a < \beta$, $-\alpha < a+b < \beta$ and $-\alpha < a+c < \beta$, and $\varphi_5(x)$ and $\varphi_6(x)$ are given functions on the interval (0,1).

In Section 8, further restrictions on constants a, b and c, and on functions $\varphi_5(x)$ and $\varphi_6(x)$ in Problem III will be imposed.

5. Solution of Problem I.

If we note (4.1) and (3.6) and replace y by x, the conditions (I.1) and (I.2) may be read as

$$(5.1)\quad \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} I_{0x}^{a+\alpha,b,-a+\beta-1}\tau + \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\alpha)} I_{0x}^{a-\beta+1,b+\alpha+\beta-1,-a+\beta-1}\nu = \varphi_1(x) \quad \text{and}$$

$$(5.2)\quad \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} J_{x1}^{c+\beta,d,-c+\alpha-1}\tau + \frac{\Gamma(1-\alpha-\beta)}{2\Gamma(1-\beta)} J_{x1}^{c-\alpha+1,d+\alpha+\beta-1,-c+\alpha-1}\nu = \varphi_2(x), \quad 0<x<1.$$

Operating $(I_{0x}^{a+\alpha,b,-a+\beta-1})^{-1} = I_{0x}^{-a-\alpha,-b,\alpha+\beta-1}$ and $(J_{x1}^{c+\beta,d,-c+\alpha-1})^{-1} = J_{x1}^{-c-\beta,-d,\alpha+\beta-1}$ (see (3.8)) on both sides of (5.1) and (5.2), respectively, and subtracting the obtained relations, we have

$$(5.3)\qquad \nu(x) - \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\beta)\Gamma(1-\beta)} R_{0x}^{\alpha+\beta-1} L_{x1}^{1-\alpha-\beta}\nu = \Phi_0(x),$$

where

$$(5.4)\quad \Phi_0(x) = \frac{2\Gamma(1-\alpha)}{\Gamma(1-\alpha-\beta)} R_{0x}^{\alpha+\beta-1} I_{0x}^{-a-\alpha,-b,\alpha+\beta-1}\varphi_1 - \frac{2\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(\beta)\Gamma(1-\alpha-\beta)} R_{0x}^{\alpha+\beta-1} J_{x1}^{-c-\beta,-d,\alpha+\beta-1}\varphi_2 .$$

It is well known (cf. e.g. [23]) the relation

$$(5.5)\quad R_{0x}^{-\alpha} L_{x1}^{\alpha}\varphi = \cos\pi\alpha\,\varphi(x) + \frac{\sin\pi\alpha}{\pi}\int_0^1 \left(\frac{u}{x}\right)^{\alpha}\frac{1}{u-x}\varphi(u)\,du$$

holds valid for $0<\alpha<1$, where the integral is taken in the sense of the Cauchy principal value. Then, after changing the unknown by $\mu(x)=\nu(x)x^{1-\alpha-\beta}$, (5.3) may be written in the form

$$(5.6)\qquad \mu(x) - \frac{\tan\pi\beta}{\pi}\int_0^1 \frac{\mu(u)}{u-x}\,du = \Phi(x), \quad 0<x<1,$$

where

$$(5.7) \qquad \Phi(x) = \frac{\sin \pi\alpha}{\cos \pi\beta \sin \pi(\alpha+\beta)} x^{1-\alpha-\beta} \Phi_0(x).$$

To solve the singular integral equation (5.6) we apply the theory in [5]. Then we find that the solution of (5.6) is written in the form

$$(5.8) \qquad \mu(x) = \frac{1}{1+\tan^2\pi\beta}[\Phi(x) + \frac{\tan \pi\beta}{\pi} (\frac{x}{1-x})^{1-\beta} \int_0^1 (\frac{1-u}{u})^{1-\beta} \frac{1}{u-x} \Phi(u)du],$$

which is sought to be Hölder continuous on the interval (0,1), bounded at $x = 0$ and unbounded but having integrable singularity at $x = 1$. Here we have taken into consideration that $\Phi(x)$ is Hölder continuous on the interval [0,1], whose fact is implied by Theorem 5.

6. Solution of Problem II.

Substituting (4.1) into (II.2), using the product rule (3.6) and operating $(I_{0x}^{a-\beta+1,b+\alpha+\beta-1,-a+\beta-1})^{-1} = I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0}$ (see (3.8)), we find the conditions (II.1) and (II.2) can be unified in the form

$$(6.1) \qquad \frac{A\Gamma(1-\beta)}{B\Gamma(1-\alpha)} \nu(x) + I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0} J_{x1}^{a-\beta+1,c+\alpha+\beta-1,-a+\beta-1} \nu = Q_0(x), \quad 0 < x < 1,$$

where

$$(6.2) \qquad Q_0(x) = -\frac{2A\Gamma(1-\beta)\Gamma(\alpha+\beta)}{B\Gamma(\beta)\Gamma(1-\alpha-\beta)} R_{0x}^{\alpha+\beta-1} \varphi_3$$

$$- \frac{2\Gamma(1-\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(1-\alpha-\beta)} I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0} J_{x1}^{a+\alpha,c,-a+\beta-1} \varphi_3$$

$$+ \frac{2\Gamma(1-\beta)}{B\Gamma(1-\alpha-\beta)} I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0} \varphi_4 .$$

The second term on the left hand side of (6.1) may be written as

(6.3) $$I_{0x}^{-a+\beta-1,-b-\alpha-\beta+1,0} J_{x1}^{a-\beta+1,c+\alpha+\beta-1,-a+\beta-1} \nu$$

$$= - \frac{\sin \pi(a-\beta)}{\pi} x^{b+\alpha+\beta-1} \int_0^1 u^{a-\beta+1} (1-u)^{-a-c-\alpha} (u-x)^{-1} \nu(u)\, du$$

$$- \cos \pi(a-\beta)\, x^{a+b+\alpha} (1-x)^{-a-c-\alpha} \nu(x) ,$$

which is derived by using various formulas for the Gauss function. (see also [21])

If we set $\mu(x) = x^{a-\beta+1}(1-x)^{-a-c-\alpha} \nu(x)$ and use the relation (6.3), we find that (6.1) is reduced to the singular integral equation

(6.4) $$P(x)\mu(x) + \int_0^1 \frac{1}{u-x} \mu(u)\, du = Q(x) , \quad 0 < x < 1,$$

where

$$P(x) = \pi \cot \pi(a-\beta) - \frac{\pi}{\sin \pi(a-\beta)} \frac{A\Gamma(1-\beta)}{B\Gamma(1-\alpha)} x^{-a-b-\alpha} (1-x)^{a+c+\alpha} ,$$

$$Q(x) = - \frac{\pi}{\sin \pi(a-\beta)} x^{-b-\alpha-\beta+1} Q_0(x) .$$

The solution of the equation (6.4) is represented by

(6.5) $$\mu(x) = \frac{P(x)Q(x)}{P^2(x)+\pi^2} - \frac{Z(x)}{\sqrt{P^2(x)+\pi^2}} \int_0^1 \frac{Q(u)}{Z(u)\sqrt{P^2(u)+\pi^2}} \frac{1}{u-x} du ,$$

where

$$(6.6)\qquad Z(x) = \exp\left\{ \frac{1}{2\pi i} \int_0^1 \frac{1}{u-x} \log \frac{P(u)-\pi i}{P(u)+\pi i} du \right\}.$$

Here the branch of the logarithm in (6.6) should be selected such that the value at $u=1$ is equal to $2\pi(-a+\beta-1)i$. The solution (6.5) is obtained by applying the theory in [5] and it is Hölder continuous on the interval $(0,1)$, bounded at $x=0$ and unbounded but having integrable singularity at $x=1$. To obtain the solution (6.5), the Hölder continuity of the coefficient $P(x)$ and the free term $Q(x)$ have to be guaranteed, and the fact can be deduced by multiplying the both sides of (6.4) by $x^{a+b+\alpha}$ and by using Theorem 5.

7. Solution of Problem III.

By similar calculations to Problem II we have from (III.1) and (III.2) the equation

$$(7.1)\qquad \frac{A\Gamma(1-\beta)}{B\Gamma(1-\alpha)} \nu(x) + I_{0x}^{-a+\beta-1,0,-b-\alpha-\beta+1} J_{x1}^{a-\beta+1,0,-a-c-\alpha} \nu = R_0(x),$$

where

$$(7.2)\qquad R_0(x) = -\frac{2A\Gamma(1-\beta)\Gamma(\alpha+\beta)}{B\Gamma(\beta)\Gamma(1-\alpha-\beta)} R_{0x}^{\alpha+\beta-1}\varphi_5$$

$$-\frac{2\Gamma(1-\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(1-\alpha-\beta)} I_{0x}^{-a+\beta-1,0,-b-\alpha-\beta+1} J_{x1}^{a+\alpha,-\alpha-\beta+1,-a-c-\alpha}\varphi_5$$

$$+\frac{2\Gamma(1-\beta)}{B\Gamma(1-\alpha-\beta)} I_{0x}^{-a+\beta-1,0,-b-\alpha-\beta+1}\varphi_6 .$$

The second term on the left hand side of (7.1) is calculated as

(7.3) $$I_{0x}^{-a+\beta-1,0,-b-\alpha-\beta+1} J_{x1}^{a-\beta+1,0,-a-c-\alpha} \nu$$

$$= \int_0^1 K_0(x,u)\nu(u)\,du + \left(\frac{x}{1-x}\right)^{a-\beta+1} \cos\pi(a-\beta+1)\,\nu(x),$$

where

(7.4) $K_0(x,u)$

$$= \frac{\Gamma(-b-\alpha-\beta+2)}{\Gamma(a-b-\alpha-2\beta+3)\Gamma(-a+\beta-1)} u^{a-b-\alpha-2\beta+2} x^{b+\alpha+\beta-2} (1-u)^{-a-c-\alpha}$$

$$\times F_1(-b-\alpha-\beta+2; a-\beta+2, -c-\alpha-\beta+1; a-b-\alpha-2\beta+3; \frac{u}{x}, u), \quad (0 < u < x),$$

$$= \frac{\Gamma(-b-\alpha-\beta+2)}{\Gamma(-a-b-\alpha+1)\Gamma(a-\beta+1)} u^{a-\beta} (1-u)^{-a-c-\alpha}$$

$$\times F_1(-b-\alpha-\beta+2; -a+\beta, -c-\alpha-\beta+1; -a-b-\alpha+1; \frac{x}{u}, x), \quad (x < u < 1).$$

To obtain the relation (7.3), we have used several formulas for the Gauss function and the Appell hypergeometric function F_1 by referring to [1], [10] and [20]. (see also [21]) Note that in case of $b = c$, $K_0(x,u)$ can be represented by means of the Gauss function.

Substituting (7.3) into (7.1) and changing the unknown by $\mu(x) = (1-x)^{-a-c-\alpha}\nu(x)$, we have the equation

(7.5) $$\left\{\frac{A\Gamma(1-\beta)}{B\Gamma(1-\alpha)}(1-x)^{a-\beta+1} + \cos\pi(a-\beta+1)\,x^{a-\beta+1}\right\}\mu(x)$$

$$+ \int_0^1 K(x,u)\mu(u)\,du = R(x), \quad 0 < x < 1,$$

where

(7.6) $K(x,u) = (1-x)^{-c-\alpha-\beta+1}(1-u)^{a+c+\alpha} K_0(x,u)$ and

$$(7.7) \qquad R(x) = (1-x)^{-c-\alpha-\beta+1} R_0(x).$$

The kernel $K(x,u)$ for $0 < u < x$ is also written in the form

$$K(x,u) = \frac{\Gamma(-b-\alpha-\beta+2)}{\Gamma(a-b-\alpha-2\beta+3)\Gamma(-a+\beta-1)} u^{a-b-\alpha-2\beta+2}$$

$$\times x^{b+\alpha+\beta-1} (1-x)^{-c-\alpha-\beta+1} (1-u)^{c+\alpha+\beta-1} (x-u)^{-1}$$

$$\times F_1(a-\beta+1;-b+c,-c-\alpha-\beta+1;a-b-\alpha-2\beta+3;\frac{u}{x},\frac{u(1-x)}{x(1-u)}).$$

Then we can obtain the relation

$$(7.8) \qquad \lim_{u\to x-0} (u-x)K(x,u) = \frac{1}{\pi} \sin \pi(a-\beta+1)\, x^{a-\beta+1}.$$

In case of $x < u < 1$, similar discussions imply

$$(7.9) \qquad \lim_{u\to x+0} (u-x)K(x,u) = \frac{1}{\pi} \sin \pi(a-\beta+1)\, x^{a-\beta+1}.$$

Thus if we separate the singular part from the integral term in (7.5) by the use of (7.8) and (7.9), we have the equation

$$(7.10) \qquad P(x)\mu(x) + \frac{Q(x)}{\pi i}\int_0^1 \frac{1}{u-x}\mu(u)du + \int_0^1 k(x,u)\mu(u)du = R(x),$$

where

$$(7.11) \qquad P(x) = \frac{A\Gamma(1-\beta)}{B\Gamma(1-\alpha)} (1-x)^{a-\beta+1} + \cos \pi(a-\beta+1)\, x^{a-\beta+1},$$

$$(7.12) \qquad Q(x) = i \sin \pi(a-\beta+1)\, x^{a-\beta+1},$$

$$(7.13) \qquad k(x,u) = (1-x)^{-c-\alpha-\beta+1} (1-u)^{a+c+\alpha} K_0(x,u) - \frac{\sin \pi(a-\beta+1)}{\pi} \frac{x^{a-\beta+1}}{u-x}.$$

It can be shown that the kernel $k(x,u)$ has the property

$$k(x,u) = O(\log|x-u|), \quad (u \to x). \tag{7.14}$$

In order to state results for the solvability of the equation (7.10), we give some notations. Let $p_0 > 1$ be a number which is determined in Section 8. Classes $L_{p_0}(U;\omega)$ and $L_{q_0}(U;\omega^{1-q_0})$ denote the weighted L_p spaces, where $q_0 = p_0/(p_0-1)$, U stands for the open interval (0,1) and the function $\omega(x)$ is such as

$$\omega(x) = x^{\gamma_1(p_0-1)}(1-x)^{\gamma_2(p_0-1)} \tag{7.15}$$

with $0 < \gamma_1, \gamma_2 < 1$. As will be seen in the next section, we have $R(x) \in L_{p_0}(U;\omega)$. Denoting the left hand side of (7.10) by $M\mu$ and setting

$$M^*\chi = P(x)\chi(x) - \frac{1}{\pi i}\int_0^1 \frac{Q(u)\chi(u)}{u-x}\,du + \int_0^1 k(u,x)\chi(u)\,du,$$

we obtain the following theorem from the theory by Hvedelidze [7]:

Theorem 6. *The homogeneous equations* $M\mu = 0$ *and* $M^*\chi = 0$ *have finite numbers of linearly independent solutions in* $L_{p_0}(U;\omega)$ *and* $L_{q_0}(U;\omega^{1-q_0})$, *respectively. A necessary and sufficient condition for the solvability of the equation* (7.10) *in* $L_{p_0}(U;\omega)$ *is to hold the relation*

$$\int_0^1 R(x)\chi_j(x)\,dx = 0 \qquad (j = 1,\cdots,n),$$

where $\{\chi_j(x)\}$ $(j = 1,\cdots,n)$ *is a complete system of linearly independent solutions of* $M^*\chi = 0$ *in* $L_{q_0}(U;\omega^{1-q_0})$.

8. Integrability of Free Term of (7.10).

We shall investigate that the function R(x) defined in (7.7) belongs to some L_p class under certain restrictions on the given functions $\varphi_5(x)$ and $\varphi_6(x)$, and constants a, b and c in Problem III.

We begin with the definition of the Lipschitz space in L_p. $\overline{f}(x)$ stands for the extended function of f(x) defined on U such that $\overline{f}(x) = f(-x)$, $(-1/2 < x < 0)$; $= f(x)$, $(0 < x < 1)$; $= f(2-x)$, $(1 < x < 3/2)$. $L_p(U)$ denotes the L_p space on the interval U.

Definition 6. Let $p \geq 1$ and $0 < k \leq 1$. A subclass $L_{p,k}$ of $L_p(U)$ is called a Lipschitz space in L_p if its elements satisfy the property

$$\left(\int_0^1 |\overline{f}(x+h) - f(x)|^p \, dx\right)^{1/p} = O(|h|^k), \qquad (h \to 0).$$

Now we shall prepare the following lemmas. The first two lemmas have been given by Hardy and Littlewood [6], where the lower limit of the Riemann-Liouville fractional integral of a function f(x), (2.4), is taken to be $-\infty$, but in our case by making the extension of f(x) such that $f(x) = 0$ for $x < 0$ the results remain valid.

Lemma 2. *If* $p \geq 1$, $0 < \alpha < 1$ *and* $f \in L_p(U)$, *then for the function* $g(x) = R_{0x}^{\alpha} f$ *we have the relation*

$$\left(\int_0^1 |\overline{g}(x+h) - g(x)|^p \, dx\right)^{1/p} = o(|h|^{\alpha}), \qquad (h \to 0).$$

Lemma 3. *Let* $p \geq 1$, $0 < k \leq 1$ *and* $0 < k+\alpha < 1$. *If* $f \in L_{p,k}$, *then* $R^{\alpha}_{0x} f$, $L^{\alpha}_{x1} f \in L_{p,k+\alpha}$.

Lemma 4. *Let* $p > 1$ *and* $0 < k \leq 1$. *If* $f \in L_{p,k}$, *then* $x^t f$, $(1-x)^t f \in L_{r,k}$ *for* $1 \leq r < p$ *and* $t-k > 1/p - 1/r$.

To find the integrability of $R(x)$, we put restrictions on constants a, b and c, besides assumptions in Problem III.

(A.1) $c \leq 0$.

(A.2) For sufficiently large $p_1 > 1$ with $1/p_1 < -a-c+\beta$, there exist q_1 and r such that $p_1 > q_1 > r > 1$, $-a+c+\alpha+2\beta-2 > 1/p_1-1/q_1$ and $-a-b-\alpha > 1/q_1-1/r$.

(A.3) For sufficiently large $p_2 > 1$, there exists q_2 such that $p_2 > q_2 > 1$ and $-a-b-\alpha > 1/p_2-1/q_2$.

The boundary functions $\varphi_5(x)$ and $\varphi_6(x)$ are supposed to satisfy the following:

(A.4) $\varphi_5 \in L_{p_1,k_1}$, where p_1 is assumed in (A.2) and k_1 satisfies $1-\alpha-\beta < k_1 < \min(-a+c+\beta-1-1/p_1+1/q_1,\ -a-b-2\alpha-\beta+1-1/q_1+1/r)$.

(A.5) $\varphi_6 \in L_{p_2,k_2}$, where p_2 is assumed in (A.3) and k_2 satisfies $a-\beta+1 < k_2 < -b-\alpha-\beta+1-1/p_2+1/q_2$.

Under the assumptions (A.1) ~ (A.5), we find that $R(x) \in L_{p_0}(U)$ ($p_0 = \min(r,q_2) > 1$) by using Lemmas 2 ~ 4 and Theorem 2, and then we have $R(x) \in L_{p_0}(U;\omega)$.

9. Product Operations in Fractional Calculus.

More than those of Problems II and III , we can set various boundary conditions. That is, if we assume, instead of (II.2) or (III.2),

$$(9.1)\quad AI_{0x}^{a,b,-a-\alpha}\, x^{\alpha+\beta-1}\, u^{(1)} + BJ_{x1}^{a+\alpha-\beta,c,-a-\alpha}\,(1-x)^{\alpha+\beta-1}\, u^{(2)} = \varphi(x), \quad \text{or}$$

$$(9.2)\quad Ax^{b-\alpha-\beta+1}\, I_{0x}^{a,b,-a-\alpha}\, x^{\alpha+\beta-1}\, u^{(1)} + B(1-x)^{c-\alpha-\beta+1}\, J_{x1}^{a+\alpha-\beta,c,-a-\alpha}\,(1-x)^{\alpha+\beta-1}\, u^{(2)} = \varphi(x),$$

a similar equation to (6.1) or (7.1) can be obtained. But, in these cases, calculations of their products

$$(9.3)\quad I_{0x}^{-a+\beta-1,-b,-\alpha-\beta+1}\, J_{x1}^{a-\beta+1,c,-a-\alpha}\, \nu \quad \text{for (9.1), and}$$

$$(9.4)\quad I_{0x}^{-a+\beta-1,-\alpha-\beta+1,-b}\, J_{x1}^{a-\beta+1,\alpha+\beta-1,-a-c+\beta-1}\, \nu \quad \text{for (9.2)}$$

are needed, which are difficult comparing with (6.3) and (7.3). Because they are reduced to compute the Kampé de Fériet hypergeometric series and the behavior of the series near the boundary points of the domain of convergence cannot be known.

To say more generally, what integral is the product

$$(9.5)\quad I_{0x}^{-p,q,r}\, J_{x1}^{p,s,t}\, \varphi, \quad (0 < p < 1)$$

reduced to ? The product (9.5) can be calculated in case of special combinations of parameters p, q, r, s and t. For example, the formula (5.5) is for the case $q = p$, $s = -p$, (6.3) for $r = 0$, $t = -p$ and (7.3) for $q = 0$, $s = 0$. But (9.3) and (9.4) are contained in the hard case to know. For calculations of other combinations of parameters, see [21].

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