数理解析研究所講究録
第 413 巻 1981 年 1-9



# Saddle connection curves of analytic dynamical systems

Shigehiro USHIKI
Department of Mathematics
Faculty of Sciences
Kyoto University

## §1. Introduction

In this note, we shall study the dynamics of an analytic dynamical system $f : R^2 \to R^2$ defined by

(1) $$f(x,y) = ( -y+2x-x^3 , x ).$$

Dynamical system $f$ is obtained by discretizing an ordinary differential equation for nonlinear oscillation of the form :

(2) $$\frac{d^2x}{dt^2} + x^3 = 0.$$

Evidently, this equation is of Hamiltonian type and can be integrated analytically. However, in executing the integration, if we employ numerical methods, some difficulty arises. Can the numerical integration be performed for a long range of time ? Numerical solutions may diverge ? How can we get an error estimate for long time range of solutions ?

We employ, as the first step, the usual finite difference scheme :

(3) $$\frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t^2} + x_n^3 = 0.$$

By putting $\Delta t x_n = u_n$ and $v_n = u_{n-1}$, we obtain the scheme :

(4) $$\begin{cases} u_{n+1} = -v_n + 2u_n - u_n^3 \\ v_{n+1} = u_n \end{cases}$$

so that the dynamics is given by the mapping (1).

Note that the mapping (1) preserves the volume in $R^2$, i.e. $\det(df) = 1$ and that f is invertible. The inverse map $f^{-1} : R^2 \to R^2$ is given by the following :

(5) $f^{-1}(x,y) = ( y, -x + 2y - y^3 )$.

Both f and $f^{-1}$ are defined by polynomials, hence are analytic.

## §2. Saddle connection curves

Let $f : R^2 \to R^2$ be an analytic diffeomorphism of the plane. We call a point x in $R^2$ a saddle point of f if x is a fixed point of f and that one of the absolute values of the eigenvalues of the Jacobian matrix df at the point is greater than one and the other is smaller than one.

For a saddle point x in $R^2$, we denote by $W^s_x$ ( resp. by $W^u_x$ ) the stable manifold ( resp. the unstable manifold ) associated to x.

Let $E^s_x$ ( resp. $E^u_x$ ) denote the eigenspace of linear map $df_x : T_xR^2 \to T_xR^2$ corresponding to the eigenvalue whose absolute value is smaller than one ( resp. greater than one ). Stable manifold $W^s_x$ (resp. unstable manifold $W^u_x$ ) is an injectively immersed one-dimensional manifold and is tangent to $E^s_x$ ( resp. $E^u_x$ ) at the saddle point.

Let $h : I \to R^2$ be an embedding of the unit interval into $R^2$. We call h a saddle connection if the following conditions i), ii), and iii) are satisfied.

i) the image of two boundary points $p = h(0)$ and $q = h(1)$ are saddle points of f.

ii) the image $h(I-\{0,1\})$ contains no fixed point of f.

iii) the image $h(I)$ is invariant under f.

We have the following theorem.

THEOREM If an analytic diffeomorphism $f : R^2 \to R^2$ can be extended to an autonorphism $F : C^2 \to C^2$ of the two dimensional conplex vector space considered as complex manifold, then there does not exist any saddle connection.

See [1] or [2] for the proof.

## §3. Elliptic periodic point

We call a point $p \in R^2$ an elliptic periodic point of $f$ if there is a positive integer $k$ such that $f^k(p) = p$ and that the eigenvalues of the Jacobian matrix $d(f^k)$ at the point are imaginary. It is known that almost all elliptic periodic points are stable if mapping $f$ is an area preserving analytic diffeomorphism. More precisely, let $\lambda$ denote the eigenvalue at the fixed point $p$ of $f^k$ and suppose that $\lambda, \lambda^2, \ldots, \lambda^{2m+2} \neq 1$ for some positive integer $m$, then one may assume the mapping $f^k$ has the form :

$$x_1 = x \cos w - y \sin w + O_{2m+2}$$

$$y_1 = x \sin w + y \cos w + O_{2m+2}$$

$$w = \sum_{k=0}^{m} \gamma_k (x^2 + y^2)^k$$

where $(x,y)$ is a local coordinate around $p$ obtained by power series convergent near the origin and $O_{2m+2}$ denotes a convergent power series in $x,y$ with terms of order greater than or equal to $2m+2$ only. If at least one of the coefficients $\gamma_1, \ldots, \gamma_m$ is not zero, then the fixed point $p$ is a stable fixed point. In fact, in any neighborhood of $p$ there exist an invariant circle surrounding the fixed point $p$. This result was obtained by A.N.Kolmogorov, V.I.Arnold, L.C.Siegel and J.Moser. See [3] for the detail.

## §4. Periodic points of the dynamical system

In this section we examine periodic points of dynamical system (1) .

Let p be a fixed point of $f^k$ for some positive integer k. Let $\lambda_1, \lambda_2$ denote the eigenvalues of the Jacobian matrix $d(f^k)$ at p. As f is an area and orientation preserving diffeomorphism we see that $\lambda_1\lambda_2 = \det(d(f^k)) = 1$. We classify the periodic points according to the eigenvalues $\lambda_1$ and $\lambda_2$ as follows:

1) $\lambda_1$ and $\lambda_2$ are real and distinct.
2) $\lambda_1$ and $\lambda_2$ are distinct and imaginary.
3) $\lambda_1 = \lambda_2 = \pm 1$.

In the case 1) the periodic point is a periodic saddle point and is said to be hyperbolic. In the case 2) we have $\lambda_1 = \overline{\lambda}_2$ and the periodic point is called elliptic. In the case 3), we call the periodic point parabolic if the eigenspace is one-dimensional.

There is only one fixed point of f, the origin, which is parabolic. In fact, we have $x = y = 0$ directly from $f(x,y) = (x,y)$. The Jacobian matrix of f is given by

$$df = \begin{pmatrix} 2-3x^2 & -1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues at the origin are given by the equation $\lambda^2 - 2\lambda + 1 = 0$, which has double root $\lambda = 1$. The eigenspace for the eigenvalue is the diagonal line $\{x=y\}$.

There is one periodic orbit of period two, $\{(2,-2),(-2,2)\}$, which is hyperbolic. There are two orbits of period three : $\{(0,\sqrt{3}),(-\sqrt{3},0),(\sqrt{3},-\sqrt{3})\}$ and $\{(0,-\sqrt{3}),(\sqrt{3},0),(-\sqrt{3},\sqrt{3})\}$. They are hyperbolic. There are two orbits of period four : $\{(\sqrt{2},0),(0,\sqrt{2}),(-\sqrt{2},0),(0,-\sqrt{2})\}$ and $\{(\sqrt{2},\sqrt{2}),(-\sqrt{2},\sqrt{2}),(-\sqrt{2},-\sqrt{2}),(\sqrt{2},-\sqrt{2})\}$, which are hyperbolic.

There are at least two orbits of period six. The orbit $\{(1,0),(1,1),(0,1),(-1,0),(-1,-1),(0,-1)\}$ is elliptic. Hyperbolic periodic orbit of period six is computed as follows.

Let $D$ denote the diagonal line $\{ x = y \}$. Let $S$ denote the set of points $(x,y)$ which are mapped by $f$ to the points symmetric with respect to $D$, i.e., $S = \{(x,y) | f(x,y)=(x,y)\}$. More explicitely, $S$ is the curve defined by the equation $y = x - x^3/2$. Let $Q$ denote the coordinate transformation $(x,y) \to (y,x)$, reflection with respect to $D$. Then we see that $f^{-1} = Q \circ f \circ Q$. Let $T$ denote the transformation $(x,y) \to (-x,-y)$. Then we have $f = T \circ f \circ T$. Let $X$ denote the x-axis $\{y=0\}$ and $Y$ the y-axis $\{x=0\}$. Let $L$ be the anti-diagonal line $\{y=-x\}$. Any point $(0,y)$ in $Y$ is mapped by $f$ to a point $(-y,0)$ in $X$. Let $p = (x,0)$ be a point in $X$ and suppose that $p$ is mapped by $f^m$ into $D$ for some positive integer $m$. Then $p$ is a periodic point of period $4m+2$. In fact, $f^{2m}(p) = f^m(f^m(p)) = Q \circ f^{-m} \circ Q \circ f^m(p) = Q \circ f^{-m} \circ f^m(p) = Q(p)$ and that $Q(p)$ belongs to $Y$ hence $f^{2m+1}(p) = f \circ Q(p) = T(p)$, therefore $f^{4m+2}(p) = f^{2m+1} \circ T(p) = T \circ f^{2m+1} \circ T \circ T(p) = T \circ f^{2m+1}(p) = T \circ T(p) = p$.

Similarly, if $f^m(p)$ belongs to $S$ then $p$ is a periodic point of period $4m+4$. Note that if all the points $f^i(p)$ $(i = 1,\ldots,m)$ are in the first quadrant $\{ x \geq 0, y \geq 0 \}$ then the orbit of $p$ does not intersect with the antidiagonal line $L$ (with the exception $p=(0,0)$ ).

Now consider the orbit starting from a point in $L$. Let $q = (x,-x)$ be a point in $L$. If $f^m(q)$ belongs to $D$, then $q$ is a periodic point of period $4m$. If $f^m(q)$ belongs to $S$ then $q$ is a periodic point of period $4m+2$.

By this procedure, we can find infinitely many periodic orbits of even periods. For a hyperbolic periodic point of period six, we have the equation in $\alpha$ :

$$\alpha^2\{(\alpha^2-3)^3+2\}-4 = (\alpha^2-4)(\alpha^6-5\alpha^4+7\alpha^2-1) = 0.$$

The real root $\alpha = 0.4008905\ldots$ gives a hyperbolic periodic

point $(\alpha,-\alpha)$.
Let $\beta= 0.2774894...$ be a real root of equation

$$\beta^2\{(\beta^2-3)^3-1\}+2 = 0.$$

Then the point $(\beta,-\beta)$ is an elliptic periodic point of period eight. The eigenvalues of $d(f^8)$ at $(\beta,-\beta)$ are $\lambda = \exp(\pm\pi\vartheta i)$, where $\vartheta= 0.5819047...$

Numerical observation shows that we may apply Moser's theorem on the existence of invariant circles around elliptic fixed points.

There are also hyperbolic periodic points of period eight. Let $\gamma= 0.53228493...$ be a real root of equation

$$\gamma^2\{(\gamma^2-2)^3-2\}+2 = 0.$$

Then the point $(\gamma,0)$ is a hyperbolic periodic point of period eight. Eigenvalues of $d(f^8)$ at $(\gamma,0)$ are real and distinct :
$\lambda_1 = 5.5152932...$ and $\lambda_2 = 0.18131301...$

Numerical observation shows that the stable manifold and the unstable manifold of this hyperbolic periodic point intersect transversally, which implies the existence of homoclinic points. These invariant manifolds surround the elliptic periodic points of period eight mentioned above. Inside the domain surrounded by these invariant manifolds there are many invariant circles around the elliptic periodic points.

We observed numerically that some orbits starting near the hyperbolic periodic points give quite a chaotic plot of points of the orbit but they diverge after hundreds of thousands of iterations. We observed also that there are invariant circles inside the domain contaning the origin and limited by stable and unstable manifolds of the hyperbolic periodic points. There are elliptic periodic points and hyperbolic periodic points in the domain limited by the invariant circle observed numerically. The existence of invariant circle means that if we start the iteration of the dynamical system with a initial point in the domain inside the circle, the orbit never diverge however long a period one may continue the iteration.

Let $\delta = 0.36502000\ldots$ be a real root of equation

$$\delta^2\{(\delta^2-2)^3-1\}+1 = 0.$$

Then the point $(\delta,0)$ is an elliptic periodic point of period ten. The eigenvalues of $d(f^{10})$ at $(\delta,0)$ are

$\lambda = \exp(\pm\pi\vartheta i)$, where $\vartheta = 0.21696420\ldots$

Figure 1 is the plot of several orbits near this point. Invariant circles of period ten are observed numerically. Let $\varepsilon = 0.18099892\ldots$ be a real root of equation in $\varepsilon$ :

$$R = Q - \frac{1}{2}Q^2, \quad R = \varepsilon(3-\varepsilon^2), \; Q = -\varepsilon+2R-R^3.$$

Then the point $(\varepsilon,-\varepsilon)$ is a hyperbolic periodic point of period ten. There seems to exist an invariant annulus containing these two orbits of period ten, one of which is hyperbolic and the other is elliptic (see figure 2). Figure 3 illustrates several orbits near this hyperbolic periodic point. Near this point are observed chaotic orbits. By the theorem obtained by the author, there exist no invariant curves connecting hyperbolic periodic points. If a stable manifold and an unstable manifold of hyperbolic periodic points intersect, the set of intersection points cannot contain any portion of regular curves. This fact suggests strongly the existence of homoclinic points which are observed numerically.

Finally we remark that the orbit structure of this dynamical system reminds us of the picture of the orbit structure concerning the three body problem considered by H.Poincaré in his book [4]. He dared not illustrate the configuration. A figure of this dynamical system was given by Arnold and Avez [5]. Similar phenomena observed numerically in Hamiltonian dynamical systems and in dynamics of a Cremona transformation are reported by Hénon and Heiles [6].

PARAMETER
A= 0.0000
B= 0.0000
C= 0.0000
D= 0.0000
WINDOW
X : 0.3650190000000000
0.36502100000000U0
Y : -0.0000010000000000
0.0000010000000000
ORDINARY MODE;
ORBITMODE
X0= 0.365019400000
Y0= 0.000000000000

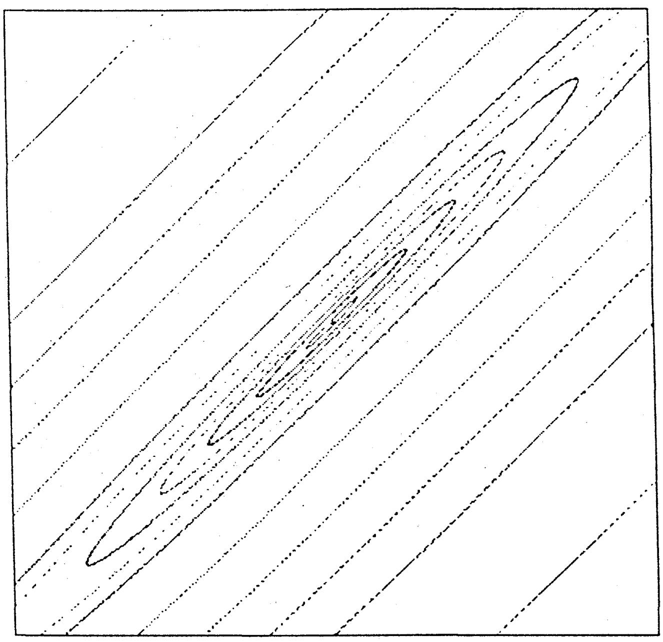

Figure 1

Orbits near an elliptic periodic point

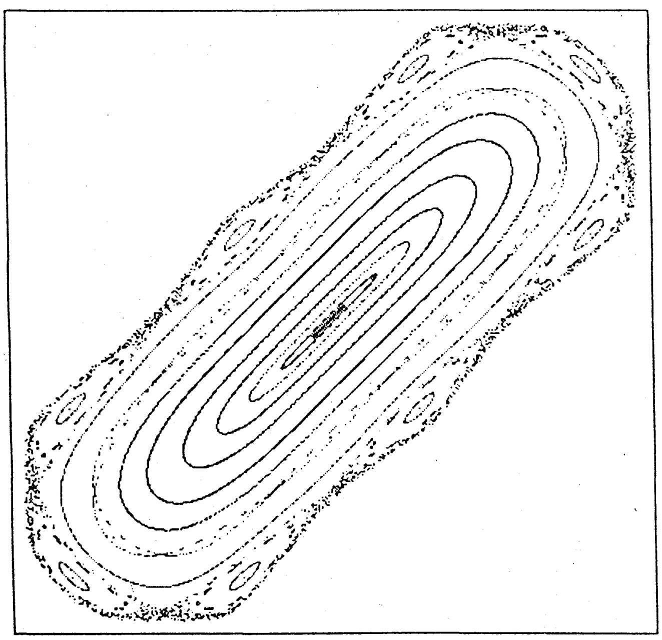

Figure 2

several orbits of dynamical system f islands surrounding elliptic periodic points are observed Observe invariant circles.

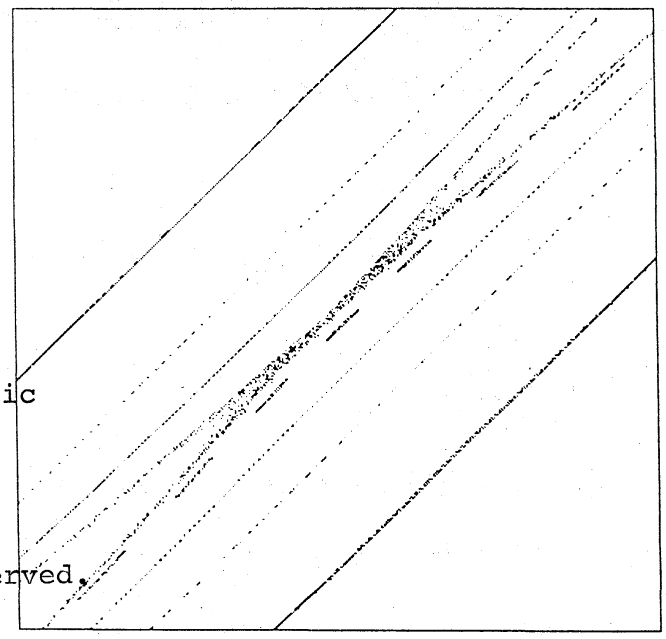


Figure 3

Orbits near a hyperbolic periodic point.

Invariant circles and chaotic orbit are observed.

REFERENCES

[1] S.Ushiki : Sur liaison-cols de système dynamique analytique, (à paraître).

[2] S.Ushiki : On unstable manifolds of analytic diffeomorphisms of the plane, (to appear).

[3] C.L.Siegel & J.K.Moser : Lectures on celestial mechanics, Die Grundlehren der mathematschen Wissenschaften in Einzeldarstellungen Band 187, Springer,1971.

[4] H.Poincaré : Les méthodes nouvelles de la mécanique céleste, tôme III, Gauthier-Villars, Paris, 1967.

[5] V.I.Arnold & A.Avez : Problèmes ergodiques de la mécanique classique, Gauthier-Villars, Paris, 1967.

[6] M.Hénon & C.Heiles : The applicability of the third integral of motion : some numerical experiments, Astronomical Journal 69, N1, p.73-79, 1964.