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Markov Partitions and Densely Periodic of Group Automorphisms

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§ 1 INTRODUCTION.

Specification for solenoidal automorphisms is studied in [4]. However it is unkown yet what kind of zero-dimensional abelian automorphisms admit the specification property. The purpose of this paper is to solve a problem related to [1] ; i.e. in the class of abelian automorphisms, specification for zero-dimensional automorphisms is strictly stronger than ergodicity (Theorems 1 and 2). This shows that the conjecture posed in [8] is false, and that there exist ergodic zero-dimensional automorphisms which are not densely periodic. In fact it will be proved that every zero-dimensional abelian automorphism is ergodic iff it satisfies weak specification (Theorem 4). We remark here that for solenoidal automorphisms, weak specification is strictly stronger than ergodicity (see Theorem 2 of [4]). From our result (Theorem 2) and a result on solenoidal automorphisms with specification proved in [4], it will be discussed that every expansive automorphism with specification has a Markov partition of an arbitrary small diameter (Theorem 3). And for the case of such automorphisms we shall obtain a result which is proved by K. SIGMUND [10] ; i.e. if an axiom A diffeomorphism f is topologically mixing on an infinite basic set X, then for every μ in a dense set of the space $\mathfrak{M}$ of f-invariant probability measures, (X,f,μ) is Bernoullian. Finally we shall prove that every zero-dimensional automorphism holds the pseudo-orbit tracing property (Theorem 5).

Our approach in obtaining the above results is based on the topological dynamics introduced in M. DENKER, C. GRILLENBERG and K. SIGMUND [7] .

In the remainder of this section, we shall give some definitions which are used in the proof of the theorem. Let X be a compact metric abelian group with metric d and σ be an automorphism of X. Then σ preserves the normalized Haar measure of X. Hence we can consider ergodic theoretical properties of (X,σ). The Kolmogorov entropy of σ will be denoted by $h(\sigma)$. It is well hnown that if σ is ergodic under the normalized Haar measure then it is measure-theoretically isomorphic to a Bernoulli shift, so that (X,σ) is topologically mixing (i.e. for any two open sets U and V there is an $M>0$ such that $U \cap \sigma^n V \neq \phi$ for all $n \geq M$). We call that (X,σ) is <u>expansive</u> if there is an open neighborhood U of the identity 0 in X such that $\bigcap_{-\infty}^{\infty} \sigma^n U = \{0\}$. We know that every expansive automorphism has finite entropy. The system (X,σ) is said to satisfy <u>specification</u> if for every $\varepsilon > 0$ there is an $M(\varepsilon) > 0$ such that for every $k \geq 1$ and k points $x_1, \ldots, x_k \in X$ and for every set of integers $a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_k \leq b_k$ with $a_i - b_{i-1} \geq M(\varepsilon)$ $(2 \leq i \leq k)$ and for every integer p with $p \geq b_k - a_1 + M(\varepsilon)$, there is a point $x \in X$ such that $d(\sigma^n x, \sigma^n x_i) < \varepsilon$ for $a_i \leq n \leq b_i$ $(1 \leq i \leq k)$ and $\sigma^p x = x$. The system (X,σ) is said to satisfy <u>weak specification</u> if it has the condition of specification except for the periodic condition $\sigma^p x = x$. It is checked easily that every automorphism with weak specification is ergodic under the Haar measure. A sequence $\{x_i : a < i < b\}$ ($a = -\infty$ or $b = \infty$ is permitted) of points in X is an ε-<u>pseudo-orbit</u> if $d(\sigma x_i, x_{i+1}) < \varepsilon$ $(a < i < b-1)$. A point $x \in X$ ε-traces $\{x_i : a < i < b\}$ if $d(\sigma^i x, x_i) < \varepsilon$ $(a < i < b)$. We say that (X,σ) has the <u>pseudo-orbit tracing property</u> if for every $\varepsilon > 0$ there is an $\delta = \delta(\varepsilon) > 0$ so that every δ-pseudo-orbit $\{x_i : a < i < b\}$ in X is ε-traced by a point $x \in X$. For every k we write $P_k(\sigma) = \{x \in X : \sigma^k x = x\}$ and $P(\sigma) = \bigcup_1^\infty P_k(\sigma)$. Then $P(\sigma)$ is an algebraic subgroup of X. The system (X,σ) is said to be <u>densely periodic</u> if $P(\sigma)$ is dense in X. It is clear from definition that if (X,σ) satisfies specification then it is densely periodic. Let X split into a direct sum $X = \oplus_{-\infty}^{\infty} H_i$ of the copys of a subgroup H. The shift automorphism σ of X defined by $\sigma\{h_n\} = \{h_{n+1}\}$ will be called a <u>Bernoulli automorphism</u>. Every Bernoulli automorphism satisfies specification (see p. 193 of [7]). A Bernoulli automorphism having a group of states which is different from the identity and having no proper non-trivial subgroups will be called a <u>simple Bernoulli automorphism.</u> It is known (Proposition 3.6 of [6]) that every simple Bernoulli automorphism holds the pseudo-orbit tracing property.

Let G denote the dual group of X. We define the dual automorphism γ of G by $(\gamma g)(x) = g(\sigma x)$, $g \in G$ and $x \in X$. It follows from Pontrjagin's duality theorem that (X,σ) is densely periodic iff $\bigcap_n (\gamma^n - I)G = \{0\}$ where I denotes the idenity map of G. We call that (G,γ) is *finitely generated under* γ if there is a finite set F in G such that $G = gp \bigcup_{-\infty}^{\infty} \gamma^j F$ (the notation gp E means the subgroup generated by a set E). It is proved in p. 258 of [15] that X is connected iff G is torsion free (i.e. every $0 \neq g \in G$ has no finite order) and X is zero-dimensional iff G is a torsion group (i.e. for every $g \in G$ there is an $n > 0$ such that $ng = 0$).

The discrete abelian groups in which the orders of all elements are powers of a fixed prime p are called *p-primary groups*. In particular let G be a countable discrete abelian group annihilated by multiplication by a prime p and γ be an automorphism of G. If $\mathbf{Z}/p\mathbf{Z}[x,x^{-1}]$ is the ring of polynomials in x and x^{-1} with coefficients in the field $\mathbf{Z}/p\mathbf{Z}$ (the notation $\mathbf{Z}$ means the ring consisting of all integers), then the ring acts on G by $q(x,x^{-1})g = q(\gamma,\gamma^{-1})g$, $g \in G$. So we can consider G to be a $\mathbf{Z}/p\mathbf{Z}[x,x^{-1}]$-module. We write $\mathbf{R}_k(p) = \mathbf{Z}/p\mathbf{Z}[x^k,x^{-k}]$ $(k \geqslant 1)$. It is clear that $\mathbf{R}_k(p)$ is a subring of $\mathbf{R}_1(p)$. We say that G is *$\mathbf{R}_1(p)$-torsion free* if every $0 \neq g \in G$ has $ag \neq 0$ for all $0 \neq a \in \mathbf{R}_1(p)$. When every $g \in G$ has $ag = 0$ for some $0 \neq a \in \mathbf{R}_1(p)$, G is said to be a *$\mathbf{R}_1(p)$-torsion group.* It is easily checked that (G,γ) is aperiodic except the identity iff G is $\mathbf{R}_1(p)$-torsion free. We say that the *$\mathbf{R}_1(p)$-rank* of G is $r > 0$ when there are r elements $g_1, \dots, g_r \in G$ such that $a_1 g_1 + \dots + a_r g_r = 0$ $(a_1, \dots, a_r \in \mathbf{R}_1(p))$ implies $a_1 = \dots = a_r = 0$, and every

$0 \neq g \in G$ is expressed as $ag = a_1g_1 + \dots + a_rg_r$ for some $0 \neq a \in \underline{R}_1(p)$ and some $a_1, \dots, a_r \in \underline{R}_1(p)$ with $(a_1, \dots, a_r) \neq (0, \dots, 0)$. Obviously, $\underline{R}_1(p)$-rank$(G) = 1$ implies $\underline{R}_k(p)$-rank$(G) = k$ $(k \geq 1)$. If G is $\underline{R}_1(p)$-torsion free, then for $0 \neq f \in G$, $\underline{R}_1(p)f$ is expressed as a restricted direct sum $\underline{R}_1(p)f = \oplus_{-\infty}^{\infty} \underline{\gamma}^j\langle f\rangle$ of the subgroups where $\langle f\rangle$ denotes a cyclic group of order p.

Throughout this paper, given an automorphism of a group, its restriction on a subgroup and its factor automorphism on a factor group will be denoted by the same symbol as the original automorphism if there is no confusion.

§ 2 RESULTS.

In this paper the followings will be proved.

THEOREM 1. Let $\underline{\sigma}$ be an automorphism of a zero-dimensional compact metric abelian group X and $(G,\underline{\gamma})$ denote the dual of $(X,\underline{\sigma})$ as before. Assume that every $g \in G$ is annihilated by multiplication by a prime p and the $\underline{R}_1(p)$-rank of G is one. If $(X,\underline{\sigma})$ satisfies specification, then $\underline{\sigma}$ is a simple Bernoulli automorphisms.

Applying Theorem 1, we shall obtain the following

THEOREM 2. Let X and $\underline{\sigma}$ be as in Theorem 1. Then the followings are equivalent ;

(A) $(X,\underline{\sigma})$ satisfies specification,

(B) there exists a zero-dimensional compact metric abelian group $\overline{X}$ and a Bernoulli automorphism $\overline{\sigma}$ of $\overline{X}$ such that (X,σ) is an algebraic factor of $(\overline{X},\overline{\sigma})$.

(C) X contains a sequence $X = F_0 \supset F_1 \supset \ldots$ of completely σ-invariant subgroups such that $\bigcap F_n = \{0\}$ and for every $n > 0$, $\sigma_{F_n/F_{n+1}}$ is a Bernoulli automorphism.

It is proved in [1] that there exists an ergodic automorphism σ of a zero-dimensional compact metric abelian group X such that X has no periodic points under σ except the identity. From this together with Theorem 1 , it will be followed that in the class of zero-dimensional automorphisms, specification is strictly stronger than ergodicity. The following is an easy conclusion of Theorem 2.

<u>COROLLARY 1.</u> Let X and σ be as in Theorem 1 , if (X,σ) is expansive and ergodic, then (X,σ) satisfies specification.

<u>THEOREM 3.</u> Let σ be an automorphism of a compact metric abelian group X . If (X,σ) is ergodic and expansive, then (X,σ) has a Markov partition of an arbitrary small diameter (cf. see p. 246 of [7] for definition) .

<u>COROLLARY 2.</u> Under the notations of Theorem 3, if (X,σ) is ergodic and expansive, then the measures μ such that (X,σ,μ) is measure-theoretically isomorphic to a Bernoulli shift form a dense subset of the space $\mathcal{M}$ of σ-invariant measures with weak topology.

PROOF. Since $(X,\underline{\sigma})$ has a Markov partition by Theorem 3, the conclusion follows from Lemma 4 of K. SIGMUND [10].

THEOREM 4. Let X and $\underline{\sigma}$ be as in Theorem 1. Then the followings are equivalent ;

(A) $(X,\underline{\sigma})$ is ergodic,

(B) $(X,\underline{\sigma})$ satisfies weak specification.

THEOREM 5. Every automorphism of a zero-dimensional compact metric abelian group holds the pseudo-orbit tracing property.

The results mentioned above are proved in [18] and so we omit the proofs here.

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