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Chevalley Groups over $\mathbb{C}((t))$ and Deformations of Simply Elliptic Singularities

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In these notes we are going to relate the deformation theory of the so called simply elliptic singularities to the corresponding Chevalley groups over the formal power series field $\mathbb{C}((t))$. Because of lack of space we can give here only a survey of the main results and the basic concepts involved. Complete details will be found in a forthcoming work on adjoint quotients for certain groups attached to arbitrary Kac Moody Lie algebras. These general results pertain to a much wider class of singularities which in addition includes at least the cusp singularities of degree ≤ 5 whose deformation theory has recently been studied by Looijenga (cf. [13] and his talk at the Kyoto conference). The aim of this article is to explain the special situation given by the simply elliptic singularities where it is possible to avoid the technical machinery needed for the general case.

I. Simple Singularities and Simple Lie Groups

In this part we quickly recall the relation between the simple singularities (equivalently: Kleinian singularities or rational double points) and certain simple Lie groups. For complete details we refer to [21].

1. Simple singularities are normal surface singularities with a very special minimal resolution. The dual graph of the exceptional divisor of such a resolution is a Dynkin diagram of type A_r, $r \geq 1$, D_r, $r \geq 4$, E_6, E_7, E_8. Up to analytic isomorphism these diagrams classify the corresponding

singularities.

2. Let G be a semisimple, simply connected, complex algebraic group and $T \subset G$ a maximal torus with corresponding Weyl group $W = N_G(T)/T$. We have $r = \text{rank}\, G = \dim T$. Denote by $X^*(T)$ the group $\text{Hom}(T,G_m) \cong \mathbb{Z}^r$ of algebraic characters of T and by $X_*(T)$ the dual group $\text{Hom}\,(G_m,T) \cong \mathbb{Z}^r$ of multiplicative one parameter subgroups. Let $\Sigma \subset X^*(T)$ be the system of roots of T in G . For each $\alpha \in \Sigma$ we fix an isomorphism

$$u_\alpha \; : \; G_a \xrightarrow{\sim} U_\alpha \subset G$$

from the additive group G_a onto the root subgroup U_α . For all $s \in T$ we have

$$s\, u_\alpha(c)s^{-1} \; = \; u_\alpha(\alpha(s)c) \quad , \quad c \in \mathbb{C} \, .$$

Let $\Delta = \{\alpha_1,\ldots,\alpha_r\}$ be a system of simple roots of Σ corresponding to the choice of a Borel subgroup $B \supset T$, and let $\Delta^\vee = \{\alpha_1^\vee,\ldots,\alpha_r^\vee\}$ be the simple coroots in $X_*(T)$. Since G is simply connected $X^*(T)$ is spanned freely (over $\mathbb{Z}$) by the fundamental dominant weights $\omega_1,\ldots,\omega_r$ which are determined by the condition $\langle\omega_i,\alpha_j^\vee\rangle = \delta_{ij}$. To each ω_i there corresponds a fundamental irreducible representation

$$\rho_i \; : \; G \longrightarrow GL(V_i)$$

of G on a finite dimensional vector space V_i . Let

$$\chi_i \; : \; G \longrightarrow \mathbb{C}$$

$$\chi_i(g) \; = \; \text{trace}\; \rho_i(g)$$

be the corresponding fundamental character. Then the adjoint quotient of G is given by the morphism

$$\chi : G \longrightarrow \mathbb{C}^r$$

$$\chi(g) = (\chi_1(g), \ldots, \chi_r(g)) .$$

The morphism χ is the algebraic quotient of the adjoint action of G on itself. Any fibre of χ is the union of finitely many conjugacy classes and its dimension is $\dim G - r$. The restriction of χ to T coincides with the natural quotient $T \longrightarrow T/W$ and T/W can be identified with $\mathbb{C}^r$.

3. Now we will look at the fibres of χ more closely. Any fibre of χ can be written in the form $\chi^{-1}(\chi(s))$ for a suitable $s \in T$. Let us first look at $s = 1$. The corresponding fibre consists of the unipotent elements in G, i.e. those which are represented by unipotent matrices in all rational representations of G. It is called the unipotent variety $\mathrm{Uni}(G)$ of G. For arbitrary $s \in T$ there is a reduction to the centralizer $Z(s)$ of s in G which is a reductive subgroup. It is generated by T and the root subgroups U_α for which $\alpha(s) = 1$. The unipotent variety $\mathrm{Uni}(s)$ of $Z(s)$ (i.e. that of its semisimple part) is the product of the unipotent varieties of its simple (almost)-factors. The fibre $\chi^{-1}(\chi(s))$ is G-isomorphic to the homogeneous bundle $G \times^{Z(s)} \mathrm{Uni}(s)$ associated to the principal fibration $G \to G/Z(s)$ and the adjoint action of $Z(s)$ on $\mathrm{Uni}(s)$.

An element $x \in G$ is called regular (resp. subregular) exactly when $\dim Z_G(x) = r$ (resp. $r + 2$) which is the same as the condition $\dim$ (conjugacy class of x) $= \dim G - r$ (resp. $\dim G - r - 2$). There is exactly one regular orbit in the unipotent variety and hence in any fibre of χ. If G is simple there is exactly one subregular unipotent orbit, and this is the orbit of greatest dimension among the nonregular orbits in Uni(G) . If G is semisimple there are as many subregular unipotent orbits as there are simple factors.

4. Now let G be simple of type $\Delta = A_r , D_r , E_r$ and choose a sufficiently small normal slice $S \subset G$ to the subregular unipotent orbit of G . We may assume that S is transversal to all orbits and that it meets the subregular unipotent orbit exactly once.

Theorem (Brieskorn, [2]):

i) $S \cap \mathrm{Uni}(G)$ is a simple singularity of type Δ .

ii) The restriction $\chi_{|S} : S \to T/W$ of the adjoint quotient realizes a semiuniversal deformation of the simple singularity $S \cap \mathrm{Uni}(G)$.

From this theorem we can derive many useful informations concerning simple singularities and their deformations. To determine the singularities in the fibres of a semiuniversal deformation we have to look at the singularities in $S \cap \chi^{-1}(\chi(s))$ for $s \in T$ sufficiently close to 1 . In this case a basis $\Delta(s)$ of the root system $\Sigma(s) = \{\alpha \in \Sigma \mid \alpha(s) = 1\}$ of Z(s) may be embedded into a base Δ of Σ , and for each connected component of $\Delta(s)$ there exists a simple singularity in $S \cap \chi^{-1}(\chi(s))$ of the corresponding type. This description also shows that the discriminant (i.e. the critical set) of $\chi_{|S}$ coincides with the discriminant of the ramified covering $T \longrightarrow T/W$ (near $\chi(1)$).

5. All deformations of a simple singularity admit a simultaneous resolution. This fact can be derived from the following construction. Let $B \supset T$ be a Borel subgroup of G . Then B can be written as a semidirect product $B = T \ltimes U$ where U is the unipotent radical of B . Let $G \times^{B} (B)$ be the

bundle associated to the principal fibration $G \to G/B$ and the adjoint action of B on itself (B). We obtain a commutative diagram

$$\begin{tikzcd}
G \times {}^{B}(B) \arrow[r, "\phi"] \arrow[d, "\theta"] & G \arrow[d, "\chi"] \\
T \arrow[r, "\psi"] & T/W
\end{tikzcd}$$

where $\phi(g * b) = gbg^{-1}$, $\theta(g * b) = \theta(g * tu) = t$ and ψ is the natural quotient map (we denote the class of (g,b) in $G \times {}^{B}(B)$ by $g * b$).

Theorem (Grothendieck, Springer):

The diagram above is a simultaneous resolution of χ, i.e. θ is smooth, ϕ is proper and for all $s \in T$ the restriction $\phi_s : \theta^{-1}(s) \to \chi^{-1}(\psi(s))$ is a resolution of singularities.

II. Simply Elliptic Singularities

We now review some properties of simply elliptic singularities and their semiuniversal deformations. Details can be found in the references [9], [10], [11], [12], [13], [14], [18], [19], [20].

6. A normal surface singularity (X_o,x) with isolated singular point x is called simply elliptic exactly when the exceptional divisor $E = \pi^{-1}(x)$ in the minimal resolution $\pi : Y \to X_o$ consists of a single elliptic curve. The selfintersection number of E is necessarily negative, $E \circ E = -d$ for some integer $d \geq 1$. We call d the degree of the singularity. Up to analytic isomorphism (X_o,x) is determined by its degree d and the analytic structure of E. Hence any simply elliptic singularity can be obtained as

the contraction of the zero section in some negative line bundle over a suitable elliptic curve E .

The embedding dimension of (X_o,x) is $\max(3,d)$. For $d = 1,2,3$ we obtain the "parabolic" hypersurfaces in the sense of Arnol'd [1] which were studied by Saito [20].

$$X^6 + Y^3 + Z^2 + \tau XYZ = 0 \qquad d = 1$$

$$X^4 + Y^4 + Z^2 + \tau XYZ = 0 \qquad d = 2$$

$$X^3 + Y^3 + Z^3 + \tau XYZ = 0 \qquad d = 3$$

(Here the parameter τ is related to the j-invariant of the elliptic curve E , cf. [20]). For $d = 4$ we obtain the complete intersection of two quadrics in $\mathbb{C}^4$.

To the first six simply elliptic singularities there is associated an affine Dynkin diagram $\tilde{\Delta}$:

d	1	2	3	4	5	6
$\tilde{\Delta}$	$\tilde{E}_8$	$\tilde{E}_7$	$\tilde{E}_6$	$\tilde{D}_5$	$\tilde{A}_4$	$A_2 \widetilde{\times} A_1$

The deformation theory of these singularities can be described completely in terms of the corresponding diagrams. This was suggested already in the work of Saito [20] and established precisely in the works of Knörrer [9], Looijenga [10], [11], [12], Pinkham [19] and Merindol [14]. The basic tool in Pinkham's approach is the theory of the corresponding Del Pezzo surfaces.

7. We first give a rough picture of the semiuniversal deformation $\phi : X \to V$ of a simply elliptic singularity X_o , cf. [14], [18]. We may

choose ϕ to be equivariant with respect to natural G_m-actions on X and V and we may assume that there is a projection $p : V \to \Omega^+ = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\}$ as well as a section $s : \Omega^+ \to V$ of p mapping Ω^+ onto the fixed points of G_m in V with the following properties. Decompose $V = V_e \cup V_f$, where $V_e = s(\Omega^+)$, $V_f = V-V_e$. Then a fibre $\phi^{-1}(s(\lambda))$ has a simply elliptic singularity of the same degree d as X_o and the exceptional elliptic curve E of its minimal resolution is isomorphic to $\mathbb{C}^*/<\lambda^i|i \in \mathbb{Z}>$. A fibre over V_f is either smooth or has at most simple singularities.

The dimension of V is $\max(11-d,1)$ and it is smooth exactly when $d \leq 5$. Then $V \cong \Omega^+ \times \mathbb{C}^{r+1}$, where $r = 9 - d$. For $d = 6$ we obtain $V \cong \Omega^+ \times C(\mathbb{P}^1 \times \mathbb{P}^2)$ where $C(\mathbb{P}^1 \times \mathbb{P}^2)$ is the affine cone over the Segrê embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ into $\mathbb{P}^5$. For $d = 7$ each slice $V_\lambda = p^{-1}(\lambda)$ is a cone over a surface of degree 7 in $\mathbb{P}^6$ (depending on λ). For $d = 8$ there are two components $V = V_1 \cup V_2$ which intersect along V_e. Each slice $V_{1,\lambda}$ is a cone over an embedding of the elliptic curve $\mathbb{C}^*/<\lambda^i|i \in \mathbb{Z}>$ into $\mathbb{P}^7$ and $V_2 \cong \Omega^+ \times \mathbb{C}^2$. For $d = 9$ we have $V_{red} \cong \Omega^+ \times \mathbb{C}$, however V has embedded components along V_e. For $d \geq 10$ we have $V_{red} = V_e \cong \Omega^+$, but again V is not reduced. (Under the isomorphisms given above p and s will always have the canonical form.)

A simply elliptic singularity can be smoothed by deformation if and only if $d \leq 9$. If $d = 9$ there are no singular fibres over V_f, and if $d = 8$ there are none above $V_1 \cap V_f$. In the other cases $(d \leq 8)$ the discriminant of ϕ is of particular interest. It was described in a uniform way by Pinkham and Looijenga $(d \leq 3)$.

8. We now recall this construction. It suffices to consider the discriminant $D_\lambda \subset V_{f,\lambda} = V_f \cap V_\lambda$ of the restriction $\phi_\lambda : \phi^{-1}(V_{f,\lambda}) \longrightarrow V_{f,\lambda}$. The

exceptional elliptic curve E in the resolution of $\phi^{-1}(s(\lambda))$ is then isomorphic to $\mathbb{C}^*/\langle\lambda^i\rangle$.

Let $X_*(T)$ denote the lattice generated by the coroots of some root system Σ and let $T = X_*(T) \otimes \mathbb{C}^*$ be a maximal torus of the corresponding simply connected complex Lie group. By A we denote the abelian variety $X_*(T) \otimes E = T/X_*(T) \otimes \langle\lambda^i\rangle$. The Weyl group W of Σ acts naturally on $X_*(T)$ and A . There is an essentially unique $\mathbb{C}^*$-bundle L over A endowed with a W-action and such that its first Chern class $c_1(L)$ equals the negative normalized Killing form on $X_*(T)$ (value 2 on short coroots). Here we use the Appell-Humbert identification $c_1(\mathrm{Pic}\ A) \cong S^2X_*(T) \subset H^2(A,\mathbb{C})$. The isotropy groups of W on L are generated by reflections. Therefore L/W is a smooth space. Let $D \subset L/W$ denote the discriminant of the ramified covering $L \to L/W$.

<u>Theorem</u> (Looijenga, Pinkham):

<u>Let</u> Σ <u>be a root system of type</u> E_8 , E_7 , E_6 , D_5 , A_4 , $A_2 \times A_1$. <u>Then the pair</u> $(L/W,D)$ <u>is isomorphic to the pair</u> $(V_{f,\lambda},D_\lambda)$ <u>for the corresponding simply elliptic singularity</u>. <u>Let</u> $\bar{s} \in V_{f,\lambda} = L/W$ <u>be the image of a point</u> $s \in L$ <u>and</u> W_s <u>the stabilizer of</u> s <u>in</u> W . <u>Then there is a type preserving bijection between the irreducible factors of</u> W_s <u>and the (simple) singularities in the fibre</u> $\phi^{-1}(\bar{s})$ <u>of the deformation</u> ϕ .

Pinkham actually gives a construction of the total space of ϕ , too. His method also extends to the cases $d = 7$ and 8 . If $d = 8$ the pair $(V_2 \cap V_{f,\lambda},D_\lambda)$ is obtained by putting $\Sigma = A_1$. The case $d = 7$ can be described similarly by using a rank two lattice containing an A_1-system (for precise details cf. [14]).

For later use we note the following. We may pull back L to a trivial

$\mathbb{C}^*$-bundle $T \times \mathbb{C}^*$ over T equipped with an action of the affine Weyl group $\tilde{W} = W \ltimes X_*(T)$. The translations $X_*(T)$ will then operate on T as the subgroup $X_*(T) \otimes \langle\lambda^i\rangle \cong X_*(T)$ of T , and the action on $T \times \mathbb{C}^*$ will be determined by an automorphy factor $e : T \times X_*(T) \to \mathbb{C}^*$. Since $X_*(T)$ acts freely on $T \times \mathbb{C}^*$ we obtain the same isotropy groups for $\tilde{W}$ on $T \times \mathbb{C}^*$ as for the action of W on L . By the same reason the discriminants of the ramified coverings $T \times \mathbb{C}^* \longrightarrow (T \times \mathbb{C}^*)/\tilde{W} = L/W$ and $L \to L/W$ coincide.

III. Chevalley Groups over $\mathbb{C}((t))$

Extended Dynkin diagrams, affine root systems and affine Weyl groups arise in the study of algebraic groups over local fields [3], [5]. It has been natural to ask (cf. for example [19]) whether there would be a similar relation between simply elliptic singularities and the corresponding Chevalley groups over $\mathbb{C}((t))$ as there was between simple singularities and simple complex Lie groups.

A first attempt is to repeat the construction of the morphism $\chi : G \to T/W$ over the base field $K = \mathbb{C}((t))$. However, this will lead only to forms of simple singularities over K (cf. [21] Appendix 1). In particular one does not end up with finite dimensional objects over $\mathbb{C}$. To remedy these defects one has to modify a Chevalley group over K in a way suggested by the theory of the closely related Euclidean Kac Moody Lie algebras [6], [15], [16].

9. Let G be a semisimple simply connected algebraic group over $\mathbb{C}$. Let $K = \mathbb{C}((t)) = \{ \sum_{i \geq i_o} a_i t^i \mid a_i \in \mathbb{C} , i_o \in \mathbb{Z}\}$ be the field of power series over $\mathbb{C}$ and $G(K)$ the group of points of G over K . The most important

modification of $G(K)$ will be the following semidirect product. By $\Omega \cong \mathbb{C}^*$ we denote the group of $\mathbb{C}$-automorphisms of K given by

$$ {}^{\lambda}p(t) = p(\lambda t) $$

where $\lambda \in \mathbb{C}^*$ and $p(t)$ is a power series in t. This group acts naturally on $G(K)$ and we may form the semidirect product $G(K) \rtimes \Omega$. The projection $p : G(K) \rtimes \Omega \to \Omega$ is invariant under conjugation by $G(K)$. If (g,λ) is an element of the fibre $p^{-1}(\lambda)$ conjugation by an element $x \in G(K)$ will look like

$$\begin{tikzcd} G(K)\rtimes\Omega \arrow[r] & \Omega = \mathbb{C}^* \\ (g(t),\lambda) \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \lambda \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

$$ x(g,\lambda)x^{-1} = (xg\,{}^{\lambda}x^{-1},\lambda) $$

where ${}^{\lambda}x^{-1} = \lambda x^{-1}\lambda^{-1} = x(\lambda t)^{-1}$ if we write x as a power series in t. It will turn out later that for $|\lambda| \neq 1$ the corresponding conjugacy classes will have finite $\mathbb{C}$-codimension in $G(K) \rtimes \Omega$ and that it is possible to define a quotient of (a part of) $G(K) \rtimes \Omega$ with respect to conjugation. However, to obtain a complete picture we need a further modification of the group $G(K)$.

10. The Kac Moody Lie algebra $\hat{\underline{g}}$ corresponding to $G(K)$ is given as the following one-dimensional central extension of the points $\underline{g} \otimes \mathbb{C}[t,t^{-1}]$ of the Lie algebra $\underline{g}$ of G over the Laurent polynomial ring $\mathbb{C}[t,t^{-1}]$. We have $\hat{\underline{g}} = (\underline{g} \otimes \mathbb{C}[t,t^{-1}]) \oplus \mathbb{C}\,.\,c$ as a $\mathbb{C}$-vector space and the Lie bracket for elements $x \otimes u$, $y \otimes v$, $x,y \in \underline{g}$, $u,v \in \mathbb{C}[t,t^{-1}]$ is defined by

$$ [x \otimes u\ ,\ y \otimes v] = [x,y] \otimes uv - \mathrm{Res}(udv)(x,y)\,.\,c $$

where (,) is the Killing form on $\underline{g}$ and Res(udv) means the residue of the differential form udv (cf. [4]). Kac has developed a theory of highest weight representations for $\hat{\underline{g}}$ including an analogue of the Weyl character formula [7]. If G is simple of rank r then there are r + 1 fundamental representations corresponding to fundamental dominant weights similarly as in the classical theory. All these representations are of infinite dimension over $\mathbb{C}$. In [4] Garland has shown how to lift these representations to representations of a central extension $\tilde{G}$ of $G(K) \rtimes \Omega$

$$1 \rightarrow \mathbb{C}^* \rightarrow \tilde{G} \xrightarrow{\varepsilon} G(K) \rtimes \Omega \rightarrow 1 .$$

center of $\tilde{G}$

To describe this extension it is, according to the theory of Moore [17], sufficient to know the restriction of this extension to the maximal torus K^* of some $SL_2(K)$-subgroup of G(K) associated to a long root. Garland shows that the extension ε is defined by the inverse of the tame symbol, i.e. two elements u,v of the torus K^* lifted to $\tilde{G}$ in a special way multiply according to $u,v \in K^* \subset SL_2(K) \subset G(K) \subset G(K) \rtimes \Omega \subset \tilde{G}$

$$u \cdot v = uv\,(-1)^{\nu(u)\nu(v)}\, c(v^{\nu(u)} u^{-\nu(v)})$$

where $u \cdot v$ is the product in $\tilde{G}$, uv is the product in K , $\nu : K \rightarrow \mathbb{Z}$ is the t-valuation on $K = \mathbb{C}((t))$ and $c : K \rightarrow \mathbb{C}$ is the constant term of power series. In the formula above the value of c lies in $\mathbb{C}^*$ and is regarded as an element of the center of $\tilde{G}$.

To define and analyse characters and conjugacy classes of $\tilde{G}$ we have to introduce some important subgroups of $\tilde{G}$.

$X_*(T)$ は rank l の lattice

$$\begin{tikzcd} G \arrow[r, phantom, "<"] & G(K) \arrow[r, phantom, "\subset"] & G(K)\rtimes\Omega & \tilde{G} \arrow[l, "\varepsilon"'] \arrow[r, phantom, "\supset"] & \tilde{N} \\ X_*(T)\otimes\mathbb{C}^* = T \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\subset"] & T(K) \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\subset"] & T(K)\rtimes\Omega \arrow[u, phantom, "\cup"] & & \\ & & T\times\Omega \arrow[u, phantom, "\cup"] & \tilde{T} \arrow[l] \arrow[r, phantom, "\subset"] & \tilde{N} \end{tikzcd}$$

11. In the following we will keep the notations of section 2. The composition $\tilde{G} \overset{\varepsilon}{\to} G(K) \rtimes \Omega \to \Omega$ is also denoted by p . By $T = X_*(T) \otimes \mathbb{C}^*$ we denote the complex points of a maximal torus of G . We regard T as a subgroup of $T(K) \subset G(K)$ which are the K-valued points of T and G . Let $\tilde{T} = \varepsilon^{-1}(T \times \Omega)$. Then $\tilde{T}$ is a maximal $\mathbb{C}$-torus in $\tilde{G}$ of dimension $r + 2$ which, using a section of T in $\tilde{G}$, can be written as a product $T \times \mathbb{C}^* \times \Omega$. Using the ordinary Bruhat decomposition of $G(K)$ one proves:

<u>Proposition 1:</u> <u>Let $\tilde{N}$ be the normalizer of $\tilde{T}$ in $\tilde{G}$. Then there is an exact sequence</u>

$$1 \to \tilde{T} \to \tilde{N} \to \tilde{W} \to 1$$

<u>where $\tilde{W}$ is the affine Weyl group</u> $\tilde{W} = W \ltimes X_*(T)$.

$$\begin{tikzcd} \tilde{T}_\lambda \arrow[r, phantom, "\subset"] \arrow[d] & \tilde{T} \arrow[r, phantom, "\subset"] \arrow[d] & \tilde{N} \arrow[r, phantom, "\subset"] \arrow[d] & \tilde{G} \arrow[d, "\varepsilon"] \\ T_\lambda = \varepsilon(\tilde{T}_\lambda) \arrow[r, phantom, "\subset"] & \varepsilon(\tilde{T}) \arrow[r, phantom, "\subset"] & \varepsilon(\tilde{N}) \arrow[r, phantom, "\subset"] & G(K)\rtimes\Omega \\ & & X_*(T) \arrow[u, phantom, "\cup"] \arrow[ull, "\text{act}"] \arrow[uull, bend right, "\text{act}"'] \arrow[r, phantom, "\subset"] & \tilde{W} \end{tikzcd}$$

There is a particularly nice section of $X_*(T)$ into the image of $\tilde{N}$ in $G(K) \rtimes \Omega$ given by the subgroup

$$X_*(T) \otimes \langle t^i \mid i \in \mathbb{Z}\rangle \subset X_*(T) \otimes K^* = T(K)$$

in $T(K)$

Let $\overset{(l+1)/\mathbb{C}}{\tilde{T}_\lambda} = p^{-1}(\lambda) \cap \tilde{T}$. Then $\tilde{T}_\lambda$ is a $\mathbb{C}^*$-bundle over $\overset{l/\mathbb{C}}{T_\lambda} = \varepsilon(\tilde{T}_\lambda) \subset G(K) \rtimes \Omega$ equipped with the natural action of $\tilde{N}/\tilde{T} \cong \tilde{W}$. The subgroup $X_*(T)$ of $\tilde{W}$ operates on T_λ by translations through the subgroup $X_*(T) \otimes \langle\lambda^i | i \in \mathbb{Z}\rangle$ of T . Comparing the automorphy factor for the action of $X_*(T)$ on $\tilde{T}_\lambda$ with the automorphy factor of the construction at the end of section 8 one obtains:

$$\begin{tikzcd}
\widetilde{W} \arrow[r, phantom, "\curvearrowright"] \arrow[d, equal] & \widetilde{T}_\lambda \arrow[d, "\mathbb{C}^*"] & \ell+1/\mathbb{C} \arrow[l] \\
\widetilde{W} \arrow[r, phantom, "\curvearrowright"] & T_\lambda & \ell/\mathbb{C} \arrow[l]
\end{tikzcd}$$

Proposition 2: The $\tilde{W}$-actions on the $\mathbb{C}^*$-bundle $\tilde{T}_\lambda$ and the $\mathbb{C}^*$-bundle $T \times \mathbb{C}^*$ defined in section 8 for the same λ coincide with respect to the natural identification

$$\begin{tikzcd}
 & & \widetilde{W} \arrow[dll, bend right] \arrow[drr, bend left] & & \\
\tilde{T}_\lambda \arrow[d] & = & T \times \mathbb{C}^* \times \{\lambda\} \arrow[d] & \simeq & T \times \mathbb{C}^* \arrow[d, "\text{section 8}"] \\
T_\lambda & = & T \times \{\lambda\} & \simeq & T \\
 & & \widetilde{W} \arrow[ull, bend left] \arrow[urr, bend right] & &
\end{tikzcd}$$

12. Consider the points $G(\mathbb{C}[[t]])$ of G over the formal power series ring and the natural reduction homomorphism "mod t" :

$$\begin{tikzcd}
\widetilde{G} \arrow[r, "\varepsilon"] & G(K) \rtimes \Omega & & & & \\
\cup & \cup & G(\mathbb{C}[[t]]) \arrow[rr, "r"] & & G = G(\mathbb{C}) & t \longmapsto 0 \\
 & & \cup & & \cup & \\
\widetilde{B} \arrow[r] & B' \rtimes \Omega \supset & B' = r^{-1}(B) \arrow[rr] & & B \supset T &
\end{tikzcd}$$

r : $G(\mathbb{C}[[t]]) \xrightarrow{r} G = G(\mathbb{C})$

Let $B \subset G$ be a Borel subgroup containing T and $B' = r^{-1}(B)$ its preimage under r . We consider B' as a subgroup of $G(K)$ and call $\tilde{B} := \varepsilon^{-1}(B' \rtimes \Omega)$ an Iwahori subgroup of $\tilde{G}$. According to Moore's theory [17] we have an isomorphism of groups $\tilde{B} \cong (B' \rtimes \Omega) \times \mathbb{C}^*$ (with $\mathbb{C}^*$ the center of $\tilde{G}$) giving rise to a semidirect product decomposition $\tilde{B} = \tilde{T} \ltimes \tilde{U}$ where $\tilde{U}$ is the kernel of the obvious projection $\tilde{B} \to \tilde{T}$. We then have the affine Bruhat decomposition originally due to Iwahori and Matsumoto [5] and adapted to our context by Garland [4] :

fixed Iwahori subgroup

$$\widehat{V} = \begin{pmatrix} 1 & & a_{ij}(t) \\ & \ddots & \\ t b_{ij}(t) & & 1 \end{pmatrix}, \quad \widehat{B} = \begin{pmatrix} a_1 & & a_{ij}(t) \\ & \ddots & \\ t b_{ij}(t) & & a_n \end{pmatrix}$$

Theorem 1: The group $\tilde{G}$ is the disjoint union of the distinct double cosets $\tilde{U} w \tilde{B}$, $w \in \tilde{N}/\tilde{T} = \widehat{W}$

Let $\Sigma \subset X^*(T)$ be the root system of T in G and $u_\alpha : G_a \xrightarrow{\sim} U_\alpha \subset G$ the fixed additive one parameter subgroup corresponding to an $\alpha \in \Sigma$ (cf: section 2). Through the projection $\tilde{T} \to T$ we consider Σ as a subset of

$\tilde{G} \xrightarrow{p} \Omega = \mathbb{C}^*$

$$\begin{tikzcd}[row sep=small] X^*(\tilde{T}) & X^*(T) \\ \tilde{\Sigma} \arrow[u, phantom, "\cup" rotate=90] & \Sigma \arrow[u, phantom, "\cup" rotate=90] \\ a \arrow[u, phantom, "\in" rotate=90] & \alpha \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

$X^*(\tilde{T})$. Let $\delta \in X^*(\tilde{T})$ be the character defined by the composition $\tilde{T} \hookrightarrow \tilde{G} \xrightarrow{p} \Omega \cong \mathbb{C}^*$. The affine root system of $\tilde{T}$ in $\tilde{G}$ is now defined as $\tilde{\Sigma} = \{\alpha + i\delta \in X^*(\tilde{T}) \mid \alpha \in \Sigma , i \in \mathbb{Z}\}$. For any affine root $a = \alpha + i\delta \in \tilde{\Sigma}$ we obtain a complex one parameter group

$$a = \alpha + i\delta \in \tilde{\Sigma} \Longrightarrow u_a : \mathbb{C} \xrightarrow{\sim} \tilde{U}_a \subset \tilde{G}$$

with the property

$$s\, u_a(c)s^{-1} = u_a(a(s)c)$$

for all $s \in \tilde{T}$, $c \in \mathbb{C}$, by composing

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\sim"] & \mathbb{C}t^i \arrow[r, hook] & K \arrow[r] & U_\alpha(K) \\ c \arrow[r, mapsto] & ct^i \arrow[rr, mapsto] & & u_\alpha(ct^i) \end{tikzcd}$$

with a fixed grouptheoretic section $U_\alpha(K) \longrightarrow \tilde{G}$.

Using either the ordinary Bruhat decomposition for $G(K)$ or the affine one for $\tilde{G}$ one can investigate the structure of the centralizers $Z(s)$ of elements $s \in \tilde{T}$.

$$\begin{tikzcd} s \in \tilde{T} \subset \tilde{G} \arrow[r, "p"] & \Omega \cong \mathbb{C}^* \\ G(K) \rtimes \Omega \arrow[u, phantom, "\supseteq" rotate=90] \arrow[ur] & \end{tikzcd}$$

Theorem: Let $s \in \tilde{T}$ such that $|p(s)| = |\delta(s)| \neq 1$ and $\tilde{\Sigma}(s) = \{a \in \tilde{\Sigma} \mid a(s) = 1\}$. Then $\tilde{\Sigma}(s)$ is a finite subroot system of $\tilde{\Sigma}$ and $Z(s)$ is a finite dimensional complex reductive group with root system $\tilde{\Sigma}(s)$ generated by the subgroups $\tilde{T}$ and $\tilde{U}_a$, $a \in \tilde{\Sigma}(s)$.

13. Let $\tilde{G}^o$ consist of the elements in $\tilde{G}$ which are conjugate into a fixed Iwahori subgroup $\tilde{B} = \tilde{T} \ltimes \tilde{U}$. From a refinement of the ordinary or

$$\begin{tikzcd}[row sep=small] \tilde{G} \supset & \tilde{B} = \tilde{T} \ltimes \tilde{U} \\ \tilde{G}^o \arrow[u, phantom, "\cup" rotate=90] \arrow[ur, bend right] & \end{tikzcd}$$

affine Bruhat decomposition one deduces:

Proposition 1: Let s, s' be elements in $\tilde{T}$ conjugate under $\tilde{G}$. Then s and s' are conjugate by an element in $\tilde{N}$.

$$\begin{tikzcd}
\tilde{T} \arrow[r, phantom, "\subset"] \arrow[d] & \tilde{N} \arrow[r, phantom, "\subset"] \arrow[d] & \tilde{G} \arrow[d] \\
\square \arrow[r, phantom, "\subset"] \arrow[d] & \square \arrow[r, phantom, "\subset"] \arrow[d] & \tilde{G}/~~\mathrm{Ad}~~\,\tilde{N} \arrow[d] \\
\square \arrow[r, phantom, "\subset"] & \square \arrow[r, phantom, "\subset"] & \tilde{G}/\mathrm{Ad}\,\tilde{G}
\end{tikzcd}$$

As a corollary we obtain a set-theoretic map

$$\begin{tikzcd}
\tilde{G}^0 \arrow[r, phantom, "\supset"] & \tilde{B} = \tilde{T} \ltimes \tilde{U} \arrow[dd] & \\
g \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\underset{\mathrm{Ad}\tilde{G}}{\sim}"] & b \arrow[u, phantom, "\cup"] \arrow[d, mapsto] & \\
 & s \in \tilde{T} \arrow[r] & \tilde{T}/\tilde{W}
\end{tikzcd}$$

$$\tau : \tilde{G}^0 \longrightarrow \tilde{T}/\tilde{W}$$

$$\uparrow$$

[illegible] $(\mathbb{C}^*)^{\ell+2}/\mathbb{Z}^{\ell} \times W$

(here $\tilde{T}/\tilde{W}$ is the set-theoretic quotient!) defined uniquely in the following way. An element $g \in \tilde{G}^0$ is conjugate to some $b \in \tilde{B}$. Let s be the projection of b onto $\tilde{T}$. Then $\tau(g)$ is the class of s in $\tilde{T}/\tilde{W}$.

$\tilde{G}^0 \supset \tilde{B}$

To form an analytic quotient $\tilde{T}/\tilde{W}$ we have to delete the points $s \in \tilde{T}$ with $|p(s)| = 1$. Now let G be simple and $\rho_i : \tilde{G} \to GL(V_i)$, $i = 0,\ldots,r$ the fundamental irreducible representations of $\tilde{G}$ introduced in section 10. By χ_i we denote the formal character of $\tilde{T}$ on V_i given by the Kac-Weyl character formula [7]. Let $\tilde{T}_{>1} := \{s \in \tilde{T} \mid |p(s)| > 1\}$.

大きな事実

Proposition 2: The characters χ_i are $\tilde{W}$-invariant holomorphic functions on $\tilde{T}_{>1}$. The map $\tilde{T}_\lambda/\tilde{W} \to \mathbb{C}^{r+1}$ sending the $\tilde{W}$-orbit of an element $s \in \tilde{T}_\lambda$ to $(\chi_0(s),\ldots,\chi_r(s))$ is an analytic isomorphism onto $\mathbb{C}^{r+1} \setminus \{0\}$ for all $\lambda \in \Omega$, $|\lambda| > 1$, with the possible exception of a discrete subset of Ω bounded from above.

$$\begin{tikzcd}
\tilde{G} \arrow[r, "P"] & \Omega \\
\tilde{G}^0 \arrow[u, phantom, "\cup"] &
\end{tikzcd}$$

Let $\tilde{G}^0_{>1} = \{g \in \tilde{G}^0 \mid |p(g)| > 1\}$. The map $\tau : \tilde{G}^0_{>1} \longrightarrow \tilde{T}_{>1}/\tilde{W}$ may be composed with the morphism $\tilde{T}_{>1}/\tilde{W} \to \Omega \times \mathbb{C}^{r+1}$,

$(s \bmod \tilde{W}) \longmapsto (p(s), \chi_0(s), \ldots, \chi_r(s))$, to give the algebraic trace map

$$\chi : \tilde{G}^0_{>1} \longrightarrow \Omega \times \mathbb{C}^{r+1}.$$

$$\begin{tikzcd}
\tilde{G}^0 \arrow[r, "\tau"] & \tilde{T}/\tilde{W} & {[s]} \arrow[l, phantom, "\ni"] \arrow[dd, mapsto] \\
\tilde{G}^0_{>1} \arrow[u, phantom, "\cup"] \arrow[r] & \tilde{T}_{>1}/\tilde{W} \arrow[u, phantom, "\cup"] \arrow[d, phantom, "\wr\|"] & \\
 & \Omega \times \mathbb{C}^{\ell+1} & (P(s), \chi_g(s)
\end{tikzcd}$$

$$\begin{tikzcd}
 & g \in \widetilde{G} \arrow[r, "\varepsilon"] \arrow[ddl, "\rho_i"'] & G(K) \rtimes \Omega & & \\
 & \widetilde{G}^0 \arrow[u, phantom, "\cup"] \arrow[r, "\tau"] & \widetilde{T}/\widetilde{W} & \ni [s] \arrow[r, mapsto] & (p(s), \chi_j(s)) \\
 L(V_i) & \widetilde{G}^0_{>1} \arrow[u, phantom, "\cup"] \arrow[l, dashed] \arrow[r] & \widetilde{T}_{>1}/\widetilde{W} \arrow[u, phantom, "\cup"] \arrow[rr] & & \Omega \times \mathbb{C}^{l+1}
\end{tikzcd}
\Bigg\} \Rightarrow \boxed{\begin{matrix} \chi_j \circ \tau(g) \\ \| \\ \mathrm{trace}\, \rho_j(g) \end{matrix}}$$

If $g \in \tilde{G}^{o}_{>1}$ maps to a convergent power series in $G(\mathbb{C}((t))) \rtimes \Omega$ then $\rho_i(g)$ may be considered as an operator of trace class on V_i with respect to the hermitian product introduced by Garland [4]. Then $\chi_i(g) = \chi_i(\tau(g))$ will coincide with the analytic trace of $\rho_i(g)$.

If $G = G_1 \times \ldots \times G_k$ is semisimple with simple factors G_j the extension $\tilde{G}$ is only a quotient of the product $\tilde{G}_1 \times \ldots \times \tilde{G}_k$ by a $(k-1)$-dimensional central torus. Accordingly fundamental characters of $\tilde{G}$ are given by products of fundamental characters of the $\tilde{G}_i$ satisfying certain conditions. These characters will in general not be algebraically independent as is the case for simple G.

14. Let $s \in \tilde{T}$, $|p(s)| > 1$. We know from section 12 that the centralizer $Z(s)$ of s in $\tilde{G}$ is a finite dimensional reductive group. We denote its unipotent variety by $\mathrm{Uni}(s)$.

$$\begin{tikzcd}
\widetilde{G}^0_{>1} \arrow[d, "\tau"'] & \supset \tau^{-1}(\tau(s)) \arrow[d] \\
\widetilde{T}/\widetilde{W} & \ni \tau(s)
\end{tikzcd}$$

Theorem 1: The fibre $\tau^{-1}(\tau(s))$ is $\tilde{G}$-isomorphic to the associated bundle $\tilde{G} \times^{Z(s)} \mathrm{Uni}(s)$.

Corollary: The fibre $\tau^{-1}(\tau(s))$ contains only finitely many conjugacy classes which are all of finite $\mathbb{C}$-codimension.

Let $\tilde{B}_{>1} = \tilde{B} \cap \tilde{G}^{o}_{>1}$. Then there is a commutative diagram

$$\begin{tikzcd}
\widetilde{G}^0_{>1} \arrow[d] & \widetilde{G} \times^{Z(s)} \mathrm{Uni}(s) \arrow[d] & \subset & \tilde{G} \times^{\tilde{B}} (\tilde{B}_{>1}) \arrow[r, "\phi"] \arrow[d, "\theta"'] & \tilde{G}^{o}_{>1} \arrow[d, "\tau"] & \supset \tau^{-1}\tau(s) \arrow[d] \\
\widetilde{T}_{>1}/\widetilde{W} & \tau(s) & \in & \tilde{T}_{>1} \arrow[r, "\psi"] & \tilde{T}_{>1}/\tilde{W} & \ni \tau(s)
\end{tikzcd}$$

defined in the same way as in section 5.

Theorem 2: The diagram above is a simultaneous resolution for τ .

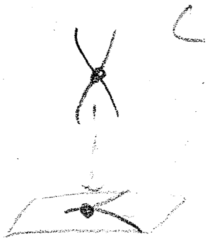
The proof of these theorems is analogous to the proof in the classical case (cf. [21]), the crucial starting point being a theory of Jordan normal form for elements in $\tilde{G}_o$.

IV. Conclusion

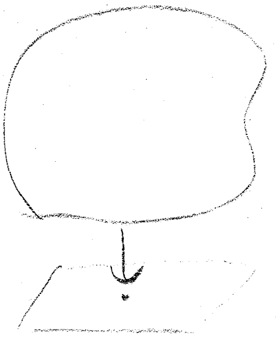
15. Combining the descriptions of the semiuniversal deformation $\phi : X \to V$ of a simply elliptic singularity and the fibres of $\tau : \tilde{G}^o_{>1} \longrightarrow \tilde{T}_{>1}/\tilde{W}$ for the corresponding group $\tilde{G}$ we obtain the following result. Let $X_f := \phi^{-1}(V_f)$. $\boxed{\odot}$

$$\begin{tikzcd} X \arrow[r, phantom, "\supset"] \arrow[d] & X_f \arrow[r, phantom, "\supset"] \arrow[d] & X_f(x) \arrow[r, hook] & \tilde{G}^o_{>1} \\ V \arrow[r, phantom, "\supset"] & V_f \arrow[rr, phantom, "\cong"] & & \tilde{T}_{>1}/\tilde{W} \end{tikzcd}$$

Theorem: For simply elliptic singularities of degree $d \leq 6$ there is an identification $V_f \cong \tilde{T}_{>1}/\tilde{W}$ such that for all $x \in X_f$ there is a neighborhood $X_f(x)$ of x and an inclusion $X_f(x) \hookrightarrow \tilde{G}^o_{>1}$ making the following diagram commute



$$\begin{tikzcd} X \arrow[r, phantom, "\supset"] \arrow[d, "\phi"] & X_f \arrow[r, phantom, "\supset"] \arrow[d] & X_f(x) \arrow[r, hook] & \tilde{G}^o_{>1} \arrow[d, "\tau"] \\ V \arrow[r, phantom, "\supset"] & V_f \arrow[rr, "\sim", equal] & & \tilde{T}_{>1}/\tilde{W} \end{tikzcd}$$



A similar statement is true for the part $\phi_2 : \phi^{-1}(V_2) \to V_2$ of the semiuniversal deformation in case $d = 8$.

The root systems attached to simply elliptic singularities form only a small part of all root systems. Some further root systems can be attached to simply elliptic singularities equipped with a group of symmetries (cf. [8]). An analogue of the theorem above then holds for the deformations conserving symmetries. Here groups over $\mathbb{C}((t))$ come into play which are not of Chevalley type. They will be dealt with in the work on general Kac Moody algebras mentioned in the introduction.

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