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Normal forms of non-degenerate quasihomogeneous functions with inner modality $\leq$ 4.

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0. Introduction

In this note, we shall study the local classifications of quasihomogeneous functions with isolated singularity and their deformations. The studies of this kind were already dealt with by Arnol'd ([1]), Saito ([2]) and others. In 1, we shall classify quasihomogeneous functions with isolated singularity by Arnol'd's inner modality and have the normal forms for inner modality equal to 2,3 and 4 (Theorem A, in 1). In 2, we shall calculate the deformations of quasihomogeneous functions obtained by our classifications and make sure of our supposition on the boundary of the class of quasihomogeneous functions with inner modality equal to 0,1,2 and 3. This note was written to give a short explanation of the contents of our paper [3].

1. Classification

In [2], K.Saito introduced the following invariant.

__Definition__. Let f be a quasihomogeneous function of type $(1;r_1,\ldots,r_n)$ with isolated singularity. Then the invariant s(f) is defined by

$$s(f):=\max\{i_1r_1+\ldots+i_nr_n \mid x_1^{i_1}\ldots x_n^{i_n} \text{ is a base of the quotient ring } R_f:=\mathcal{O}_{\mathbf{C}^n,0}/(\partial f/\partial x_1,\ldots,\partial f/\partial x_n)\}.$$

He classified quasihomogeneous functions with $s \leqq 1$. The hypersurface singularities with $s = 1$ are called simple elliptic singularities. The duality between $r(f)= r_i$ and $s(f)$, $s(f)+2r(f)=n$, is important. Arnol'd introduced the invariant m_o, which is called <u>inner modality</u>, by developping Saito's $s(f)$.

<u>Definition</u>. Let f be as above. Then the <u>inner modality</u> $m_o(f)$ is defined by

$$m_o(f) := \# \text{ of basis monomials of } R_f \text{ with generalized degree } \geqq 1.$$

He classified quasihomogeneous functions with $m_o = 0$ and 1.

Our results are the classification of quasihomogeneous functions with $m_o = 2,3,4$ and the study of some deformations of them. At first, we shall classify quasihomogeneous functions with $m_o = 2,3$ and 4. The keys are the following two points :

1°. <u>To formulate inner modality directly by weights</u>.

2°. <u>To deal with arithmetical calculations by a computer</u>.

Let's begin with No.1. Let f be a quasihomogeneous function of type $(1;r_1,\dots,r_n)$, where $0 < r_i < \frac{1}{2}$ and $r_i = A_i/N$. Put $D:=Nd=N(N-2\Sigma r_i)$ $=nN-2\Sigma A_i$ and $d:=n-2\Sigma r_i$. Since d is just Saito's invariant s, it is the highest of generalized degrees of the basis monomials of R_f.

<u>Proposition</u> 1. If $m_o(f) \leqq 4$, <u>we have the following assertions</u> :

(1) $\mathrm{corank}(f) \leqq 4$.

(2) $\Sigma r_i \geqq (2n-3)/4$, where $n=\mathrm{corank}(f)$.

(3) $m_o(f) = \#\{k \in N^n | \Sigma k_i r_i \leqq d-1\}$.

<u>Proof</u>. Let's make a randam choice to prove them. According to Arnol'd, the following equation holds :

$$\chi_f(z) = \sum_{j=1}^{D} \mu_j z^j = \prod_{i=1}^{n} \frac{z^{N-A_i} - 1}{z^{A_i} - 1},$$

where μ_j is the number of basis monomials of R_f with generalized degree equal to j/N. If $\Sigma r_i \geqq (2n-3)/4$ and $j \leqq D-N$, for any $k \in N^n$ such that $\Sigma k_i A_i = j$,

the generalized degree$(x_1^{k_1} \ldots x_n^{k_n}) = j/N \leqq n-2\Sigma r_i - 1 \leqq \frac{1}{2}$.

Since the generalized degree of any non-zero element of $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ is higher than $\frac{1}{2}$, the monomials $x_1^{k_1} \ldots x_n^{k_n}$ such that $\Sigma k_i A_i =$ j are linearly independent in R_f. Therfore, if $\Sigma r_i \geqq (2n-3)/4$ and $j \leqq D-N$, we have

$$\mu_j = \#\{k \in N^n | \Sigma k_i A_i = j\}.$$

On the other hand, the coefficients of χ_f are symmetric and $\mu_j = \mu_{D-j}$. So $m_o(f) = \sum_{j \geq N} \mu_j = \sum_{j \leq D-N} \mu_j$. Therefore, if $\Sigma r_i \geqq (2n-3)/4$,

$$m_o(f) = \sum_{j \leq D-N} \#\{k \in N^n | \Sigma k_i A_i = j\} = \#\{k \in N^n | \Sigma k_i r_i \leqq d-1\}.$$

Next, put n:= corank(f). Each generalized degree of x_i is less htan $\frac{1}{2}$. Note that the generalized degree of any non-zero element of $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ is higher than $\frac{1}{2}$ and x_i's and 1 are menber of basis monomials of R_f. Note that $m_o(f) = \sum_{j \leq D-N} \mu_j$ and if $n \leqq 5$, from the hypothesis $m_o(f) \leqq 4$, there is an index i such that the generalized degree of x_i is not less than d-1, i.e.

$$r_i \geqq d-1 = n-2\Sigma r_i - 1 \geqq n/3-1.$$

The last inequality results from the inequality $\Sigma r_i \leqq n/3$ by Saito. So $r_i \geqq n/3-1 \geqq 5/3-1 = 2/3$. It's a contradiction to $r_i < \frac{1}{2}$. Therefore if $m_o(f) \leqq 4$,

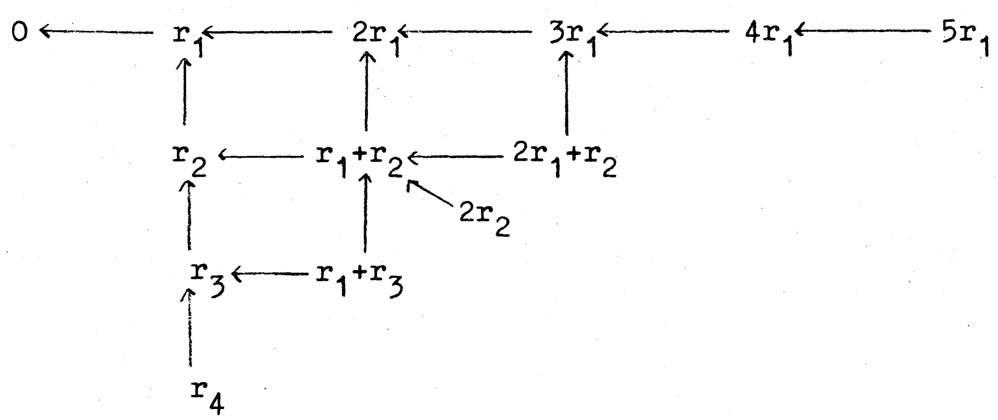
$$n = \text{corank}(f) \leqq 4.$$

Now if we can prove the assertion (2), we can prove the assertion (3) from (2) and the above arguments. We shall show that the assertion (2) holds in each case of corank(f) = 2,3 and 4. Now we are going to prove only the case of corank(f) = 2 because the csaea of corank(f) = 3, 4 can be proved in the similar way. Suppose that $r_1+r_2 < (2n-3)/4 = 1/4$. Put $M:=\{1,x_1,x_2,x_1^2,x_2^2\}$. Then the generalized degree of any element of M is lower than $\frac{1}{2}$. So M is the subet of basis monomials of R_f. If $m_o(f) \leqq 4$, From the equation $m_o(f) = \sum_{j \leqq D-N} \mu_j$, the highest of the generalized degrees of all elements of M is not lower than d-1. There-fore the generalized degree of x_1^2 or x_2^2 is higher than d-1. If the generalized degree of x_1^2 is higher than d-1, we have

$$2r_1 \geqq d-1 = 1-2(r_1+r_2),\ 4(r_1+r_2) \geqq 4r_1+2r_2 \geqq 1 \text{ and } r_1+r_2 \geqq 1/4,$$

but it's a contradiction to $r_1+r_2 < 1/4$. And also the same contradiction occurs in the case of x_2^2. Therefore, if corank = 2 and $m_o(f) \leqq 4$, we have $r_1+r_2 \geqq 1/4$. This completes the proof of Proposition 1.

Next we proceed to No.2. Let $r_i \in R$ and $0 < r_1 \leqq r_2 \ \ldots \leqq r_n$. Put $W:=\{\Sigma k_i r_i \mid k_i \in N\}$. Here we shall arrange elements of W in order. For any elements a and b of W, we describe $a \leqq b$ by $a \longleftarrow b$. Then we have the following diagram :



For any subset $S \subset W$, we denote the second least element of S by $Min^{(2)}S$. Let's put :

$D_3 := \text{Min } 2r_1,\ r_2\ ,\ E_2 := \max\ 2r_1,\ r_2\ ,$

$$D_4 := \begin{cases} \text{Min}\{E_2,\ 3r_1\} & \text{if } n = 2, \\ \text{Min}\{E_2,\ 3r_1,\ r_3\} & \text{if } n = 3,\ 4, \end{cases}$$

$$E_3 := \begin{cases} \text{Min}^{(2)}\{E_2,\ 3r_1\} & \text{if } n = 2, \\ \text{Min}^{(2)}\{E_2,\ 3r_1,\ r_3\} & \text{if } n = 3,4, \end{cases}$$

$$D_5 := \begin{cases} \text{Min}\{E_3,\ 4r_1, r_1+r_2\} & \text{if } n = 2,3, \\ \text{Min}\{E_3,\ 4r_1,\ r_1+r_2,\ r_4\} & \text{if } n = 4, \end{cases}$$

Then we have the following inequality from the above diagram :

$$0 < r_1 \leqq D_3 \leqq D_4 \leqq D_5,$$

and the other elements of W are greater than or equal to D_5. Therefore we have the following proposition :

Proposition 2. *Let* f *be a quasihomogeneous function of type* $(1; r_1, \dots, r_n)$, *where* $0 < r_1 \leqq \dots \leqq r_n < \frac{1}{2}$. *Then we have* :

$$m_o(f) = 0 \Longleftrightarrow d-1 < 0 \text{ and } \sum r_i \geqq (2n-3)/4.$$
$$m_o(f) = 1 \Longleftrightarrow 0 \leqq d-1 < r_1 \text{ and the same condition.}$$
$$m_o(f) = 2 \Longleftrightarrow r_1 \leqq d-1 < D_3 \text{ and the same condition.}$$
$$m_o(f) = 3 \Longleftrightarrow D_3 \leqq d-1 < D_4 \text{ and the same condition.}$$
$$m_o(f) = 4 \Longleftrightarrow D_4 \leqq d-1 < D_5 \text{ and the same condition.}$$

If the function germ of f has an isolated singular point, we can see easily that f contains the monomial $x_i^m x_j$ or x_i^m for each i. Therefore we have the following propositions.

Proposition 3. *Every quasihomogeneous function of two variables of corank* 2 *with isolated singularity contains one, at least, of the three systems of monomials in the following table* :

Class	Monomials	r_1	r_2
I	x^a, y^b	1/a	1/b
II	$x^a y$, y^b	(b-1)/ab	1/b
III	$x^a y$, xy^b	(b-1)/(ab-1)	(a-1)/(ab-1)

Proposition 4. For corank = 3, we have the seven systems.

Proposition 5. For corank = 4, the nineteen systems.

As for Proposition 4 and 5, see pages 193-195 in Inv. math.55 (1979).

Now, it is necessary to determine exponents a, b(a,b,c or a,b, c,d) according to Proposition 2 so that functions with systems of monomials given in Proposition 3-5 may have inner modality 2,3 and but its calculations are beyond us and we had to use a computer. In using a computer, we have to determine the upper limits of exponents for given inner modality and it can be done easily by use of Saito's inequality. Then we have determined quasihomogeneous functions corresponding to the exponents obtained by a computer. In this way, we succeeded in determining all the quasihomogeneous functions with inner modality $\leqq 4$. As for the lists of quasihomogeneous functions with inner modality = 2, 3 and 4, see pages 187-188 in Inv. math.55 (1979).

Theorem A. (1) We have 20 normal forms with inner modality = 2.

(2) 24 normal forms with inner modality = 3.

(3) 28 normal forms with inner madality = 4.

In the next section, we consider the deformations of quasihomogeneous functions obtained by Theorem A.

2. Deformations

We consider the boundary of each group of quasihomogeneous functions with fixed inner modality. According to Saito, isolated hypersurface singularities except simple singularities A_k, D_k, E_k are deformed into one of simple elliptic singularities $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. This fact shows that simple elliptic singularities are, in a sense, the boundary of simple singularities. In the category of quasihomogeneous functions, we shall develop this concept and define the boundary of quasihomogeneous functions with fixed inner modality.

Definition. A quasihomogeneous function f is a boundary of quasihomogeneous functions with inner modality k if $m_0(f) = k+1$ and # of basis monomials with generalized degree > 1 is less than k+1.

We suppose that this boundary will be an actual boundary in the sense of deformations. In order to make sure of it for boundary of each group of quasihomogeneous functions with inner modality = 0,1,2 and 3, we have determined normal forms of boundaries in the similar way about quasihomogeneous functions with inner modality = 0,1,2,3 and 4. And we have calculated the deformations of quasihomogeneous functions with inner modality = 1,2,3,4 and made sure that our supposition is true in the case of boundaries of quasihomogeneous functions with inner modality = 0,1,2 and 3.

Theorem B. *For k = 0,1,2 and 3, quasihomogeneous functions with inner modality = k+1 are deformed to the boundary of quasihomogeneous functions with m_0 = k.*

3. Modality and Inner modality

In this section, we consider the relationship between modarity and inner modarity. Arnol'd defined modality for general functions with isolated singularity separately from inner modality for quasihomogeneous functions.

Definition. The modality of a germ of a function f with isolated singularity at 0 is the smallest number m such that a sufficiently small neighbourhood of jf(0) in the jet space J(n,1) is covered by a finitely many m-parameter families of orbits.

Arnol'd conjectured that modality will be equal to inner modality for quasihomogeneous functions. He made sure that this conjecture is true for inner modality = 0 and 1. Since the inequality, inner modality $\leqq$ modality, we can prove Arnol'd's conjecture for inner modality = 2,3 and 4 if we can calculate modality of quasihomogeneous functions with inner modality = 2,3 and 4. Arnol'd's calculations can cover quasihomogeneous functions with inner modality $\leqq$ 2. From this, we can prove by our classification that his conjecture is true for inner modality = 2.

Theorem C. *Quasihomogeneous functions with inner modality $\leqq$ 2 have the same modality as their inner modality.*

References

1. Arnol'd, V.I. : Normal forms of functions in the neighbourhoods of degenerate critical points. Russian Math. Surveys 29, 10-50 (1974).

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3. Yoshinaga, E. and Suzuki, M. : Normal forms of non-degenerate quasihomogeneous functions with inner modality $\leqq$ 4. Inv. math. 55 185-206 (1979).