

ON PLURIGENERA OF NORMAL ISOLATED SINGULARITIES, I

by

KIMIO WATANABE

*Institute of Mathematics,*

*University of Tsukuba*

*Ibaraki 305, Japan*

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## 50. Introduction

In this work we show the analogous invariants of plurigenera of compact complex manifolds can also be defined for normal isolated singularities.

Our presentation goes as follows. In Sect. 1, we give the definition of plurigenera  $\delta_m$ ,  $m$  a positive integer, of an  $n$ -dimensional normal isolated singularity and calculate them in the typical three cases. In Sect. 2, we study the normal surface singularities and prove some theorems about  $\delta_m$ . In the last section of this paper we classify the surface singularities such that  $0 \leq \delta_m \leq 1$  for  $m \geq 1$ .

Let  $x$  be a normal singularity of two-dimensional analytic space  $X$ . In [2], Artin introduced a definition for  $x$  to be rational. A point  $x$  is rational if  $R^1 \pi_* \mathcal{O}_{\tilde{X}} = 0$  where  $\pi: \tilde{X} \rightarrow X$  is a resolution of the singularity. Laufer [13] derived a necessary and sufficient criterion for  $x$  to be rational that does not involve a priori knowledge of what a resolution of  $x$  looks like. Yau [27] generalized Laufer's result to higher dimensions. Let  $(X, x)$  be a normal  $n$ -dimensional isolated singularity. It follows from Hironaka's work [7] that a resolution  $\pi: \tilde{X} \rightarrow X$  always exists. The geometric genus of the singularity is defined as

$$p_g(X, x) = \dim (R^{n-1} \pi_* \mathcal{O}_{\tilde{X}})_x .$$

Assume that  $V$  is a Stein neighborhood of  $x$  in  $X$ . Let  $K$  be the canonical line bundle of  $V - \{x\}$ . Then

$$p_g(X, x) = \dim \Gamma(V - \{x\}, \mathcal{O}(K)) / L^2(V - \{x\}) .$$

(Here  $L^2(V - \{x\})$  denotes the set of all square integrable holomorphic  $n$ -forms on  $V - \{x\}$ , see p.601 of [13].) Following the  $m$ -genus of a complex manifold

[22], the plurigenera of an  $n$ -dimensional isolated singularity is defined as

$$\delta_m(X, x) = \dim \Gamma(V - \{x\}, \mathcal{O}(mK)) / L^{2/m}(V - \{x\})$$

where  $L^{2/m}(V - \{x\})$  denotes the set of all  $L^{2/m}$ -integrable  $m$ -ple holomorphic  $n$ -forms on  $V - \{x\}$ .  $\delta_m(X, x)$  can be described in terms of cohomologies of the resolution. These integers are determined independently to the choice of the Stein neighborhoods. Hence  $\delta_m$  can be an invariant attached to the singularity. We consider the asymptotic behavior of  $\delta_m$  when  $m \rightarrow +\infty$ , and calculate the value  $\delta = \limsup_{m \rightarrow \infty} \delta_m / m^n$  in some cases.

Let  $(X, x)$  be defined by a quasihomogeneous polynomial  $f(z_0, z_1, \dots, z_n)$  with weights  $r_0, r_1, \dots, r_n$ :

$$X = \{ (z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \mid f(z_0, z_1, \dots, z_n) = 0 \}.$$

Let  $r(f) = r_0 + r_1 + \dots + r_n$ . Then (Example 1.15),

$$(1) \quad r(f) > 1 \implies \delta_m(X, x) = 0 \quad \text{for } m \geq 1,$$

$$(2) \quad r(f) = 1 \implies \delta_m(X, x) = 1 \quad \text{for } m \geq 1$$

and

$$(3) \quad r(f) < 1 \implies \limsup_{m \rightarrow \infty} \delta_m(X, x) / m^n = \frac{1}{n!} \{1 - r(f)\}^n \frac{1}{r_0 r_1 \dots r_n}.$$

In case (1), (Theorem 1.11)  $(X, x)$  is rational by Burns [4], p.239. If  $(X, x)$  is a quotient singularity then  $\delta_m(X, x) = 0$  for  $m \geq 1$ . Suppose that  $(x, x)$  is a cusp singularity. Then (Theorem 1.16)  $\delta_m(X, x) = 1$  for  $m \geq 1$ .

When  $(X, x)$  is two-dimensional, we prove two fundamental theorems. One is  $\limsup_{m \rightarrow \infty} \delta_m / m^2 < \infty$ . The other is as follows. Let  $\pi: \tilde{X} \rightarrow X$  be the minimal resolution of  $X$ . Let  $A = \pi^{-1}(x)$  be the exceptional set. Suppose  $A'$  is a

connected proper subvariety of  $A$ .  $A'$  is also an exceptional set. Let  $(X', x')$  be the singularity obtained by blowing down  $A'$ . Then (Theorem 2.8)  $\delta_m(X, x) \geq \delta_m(X', x')$ . Let  $p_a(X, x)$  be the arithmetic genus introduced by Wagreich [24]. Then the latter fundamental theorem allows us to introduce the notion of minimality of a singularity.  $(X, x)$  is minimal if  $p_a(X, x) \geq 1$  and  $p_g(X, x) > p_g(X', x')$  for every connected proper subvariety  $A'$  of  $A$ . For instance (Corollary 2.9) Gorenstein singularities have the minimality if  $p_a \geq 1$ . Moreover (Theorem 2.13) a minimal singularity with  $p_a = 1$  is Gorenstein. When the dual graph of a minimal good resolution is star-shaped, (Theorem 2.21) it becomes possible to get an estimate, in terms of the associated graph including the genera of the irreducible components and certain data.

In Sect. 3 we study the classification of singularities such that  $0 \leq \delta_m \leq 1$ .  $\delta_m$  characterize the quotient singularities (Theorem 3.9) :  $(X, x)$  is a quotient singularity  $\iff \delta_m(X, x) = 0$  for  $m \geq 1$ . Knöller [11] proved the analogous theorem :  $(X, x)$  is a rational double point  $\iff \gamma_m(X, x) = 0$  for  $m \geq 1$ . We completely classify all rational singularities with  $0 \leq \delta_m \leq 1$ . This result has the striking resemblance to Wagreich's work [23]. As for the singularity such that  $\delta_m = 1$  for all  $m \geq 1$ , a few exceptions are left in the classification. In particular (Theorem 3.20), if  $(X, x)$  is a Gorenstein singularity such that  $\delta_m = 1$  for all  $m \geq 1$  then  $(X, x)$  is a simple elliptic singularity or a cusp singularity.

The referee has informed the author that Theorem 2.1 is generalized to the case of arbitrary dimensions  $n \geq 2$ . The author wishes to thank the referee for his valuable comments and suggestions.

## §1. Plurigenera of Isolated Singularities

Let  $(X, x)$  be a normal isolated singularity in the  $n$ -dimensional analytic space  $X$ . It follows from Hironaka's work [7] that a resolution  $\pi : \tilde{X} \rightarrow X$  always exists.

**Definition 1.1.** The geometric genus of a normal isolated singularity  $(X, x)$  is  $p_g(X, x) = \dim (R^{n-1} \pi_* \mathcal{O}_{\tilde{X}})_x$ .

The geometric genus is in fact independent of the choice of the resolution. Yau [27] derived an intrinsic definition of  $p_g$  that does not involve a priori knowledge of what a resolution of  $x$  looks like, which is a generalization of Laufer's theorem [13] in the 2-dimensional case.

**Theorem 1.2 (Yau [27]).** Let  $x$  be a normal  $n$ -dimensional isolated singularity of  $X$ . Suppose that  $V$  is a (sufficiently small) Stein neighborhood of  $x$  and  $K$  is the canonical line bundle of  $V - \{x\}$ . Then

$$p_g(X, x) = \dim \Gamma(V - \{x\}, \mathcal{O}(K)) / L^2(V - \{x\}),$$

where  $L^2(V - \{x\})$  denotes the set of all square integrable holomorphic  $n$ -forms on  $V - \{x\}$ .

Let  $U = \pi^{-1}(V)$  and  $A = \pi^{-1}(x)$ , then  $\Gamma(U, \mathcal{O}(K)) = L^2(U - A)$  by [13, Theorem 3.1, p.601]. Therefore we obtain  $p_g(X, x) = \dim \Gamma(U - A, \mathcal{O}(K)) / \Gamma(U, \mathcal{O}(K))$ .

For convenience, we denote the line bundle  $K^{\otimes m}$  by  $mK$ . An element of  $\Gamma(V - \{x\}, \mathcal{O}(mK))$  is considered as an  $m$ -ple holomorphic  $n$ -form. Let  $\omega$  be a holomorphic  $m$ -ple  $n$ -form on  $U - A$ . We write  $\omega$  as

$$\omega = \phi(z)(dz_1 \wedge dz_2 \wedge \dots \wedge dz_n)^m,$$

using local coordinates  $(z_1, z_2, \dots, z_n)$ . We associate with  $\omega$  the continuous  $(n, n)$ -form  $(\omega \wedge \bar{\omega})^{1/m}$ , given locally by

$$|\phi(z)|^{2/m} \left(\frac{\sqrt{-1}}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge \dots \wedge dz_n \wedge d\bar{z}_n.$$

Definition 1.3.  $\omega$  is called integrable ( $L^{2/m}$ -integrable) if  $\int_{W-A} (\omega \wedge \bar{\omega})^{1/m} < \infty$  for  $W$ , any sufficiently small relatively compact neighborhood of  $A$  in  $U$ .

Let  $L^{2/m}(U-A)$  be the set of all integrable holomorphic  $m$ -ple  $n$ -forms on  $U-A$ , which is a subspace of  $\Gamma(U-A, \mathcal{O}(mK))$ .  $L^{2/m}(U-A)$  becomes a vector space  $\Gamma(U, \mathcal{O}(mK+(m-1)[A]))$  in the case that  $A$  is a divisor which has at most normal crossings by Sakai [19, Theorem 2.1, p.243]. As for  $L^{2/m}(V-\{x\})$  we replace  $U$  and  $A$  with  $V$  and  $\{x\}$  respectively in the definition of  $L^{2/m}(U-A)$ .

Following Laufer [13], we consider the sheaf cohomology with support at infinity. The following sequence is exact:

$$0 \rightarrow \Gamma(U, \mathcal{O}(mK)) \rightarrow \Gamma_{\infty}(U, \mathcal{O}(mK)) \rightarrow H_{*}^1(U, \mathcal{O}(mK)) \rightarrow \dots$$

By Siu [20], p.374, any section of  $mK$  defined near the boundary of  $U$  has an analytic continuation to  $U-A$ . Therefore there is a natural isomorphism

$$\Gamma_{\infty}(U, \mathcal{O}(mK)) \cong \Gamma(U-A, \mathcal{O}(mK)).$$

By Serre duality,

$$H_*^1(U, \mathcal{O}(mK)) \cong H^{n-1}(U, \mathcal{O}(K-mK)) .$$

Since  $U$  is strongly pseudoconvex,  $H^{n-1}(U, \mathcal{O}(K-mK))$  is finite dimensional.

Hence by the inequality

$$\begin{aligned} \dim \Gamma(U-A, \mathcal{O}(mK)) / L^{2/m}(U-A) &\leq \dim \Gamma(U-A, \mathcal{O}(mK)) / \Gamma(U, \mathcal{O}(mK)) \\ &\leq \dim H_*^1(U, \mathcal{O}(mK)) = \dim H^{n-1}(U, \mathcal{O}(K-mK)), \end{aligned}$$

we have  $\dim \Gamma(U-A, \mathcal{O}(mK)) / L^{2/m}(U-A) < +\infty$ . If  $V \supset V'$ , with  $V'$  another Stein neighborhood of  $x$ , then we have the following commutative diagram of exact sequences.

$$\begin{array}{ccccccccccc} 0 & \rightarrow & \Gamma(U, \mathcal{O}(mK)) & \rightarrow & \Gamma(U-A, \mathcal{O}(mK)) & \rightarrow & H_*^1(U, \mathcal{O}(mK)) & \rightarrow & H^1(U, \mathcal{O}(mK)) & \rightarrow & \dots \\ & & \downarrow \beta_0 & & \downarrow \gamma_0 & & \uparrow \alpha_1 & & \downarrow \beta_1 & & \\ 0 & \rightarrow & \Gamma(U', \mathcal{O}(mK)) & \rightarrow & \Gamma(U'-A, \mathcal{O}(mK)) & \rightarrow & H_*^1(U', \mathcal{O}(mK)) & \rightarrow & H^1(U', \mathcal{O}(mK)) & \rightarrow & \dots \end{array}$$

where  $\beta_0$ ,  $\gamma_0$  and  $\beta_1$  are the restriction maps which are induced by the inclusion map  $j : U' \rightarrow U$  and  $\alpha_1$  is the zero extension map of the cohomology. The restriction map  $\beta_1$  is an isomorphism by Lemma 3.1 of [13]. It follows from an easy diagram chase that

$$\Gamma(U-A, \mathcal{O}(mK)) / \Gamma(U, \mathcal{O}(mK)) \rightarrow \Gamma(U'-A, \mathcal{O}(mK)) / \Gamma(U', \mathcal{O}(mK))$$

is an isomorphism. Thus

$$\Gamma(U-A, \mathcal{O}(mK)) / L^{2/m}(U-A) \cong \Gamma(U'-A, \mathcal{O}(mK)) / L^{2/m}(U'-A).$$

**Definition 1.4.** The plurigenus ( $m$ -genus),  $m$  a positive integer, of a normal isolated singularity  $(X, x)$  is



$$\delta_m(X, x) = \dim \Gamma(V - \{x\}, \mathcal{O}(mK)) / L^{2/m}(V - \{x\}) .$$

Theorem 1.5. Let  $(X, x)$  be a quotient singularity. Then  $\delta_m(X, x) = 0$  for  $m \geq 1$ .

Proof.  $(X, x)$  is a quotient singularity. Hence we can assume that there exist a ball  $B \subset \mathbb{C}^n$  of radius  $\varepsilon$ , sufficiently small, and a finite group  $G$  of unitary linear transformations, no element of which fixes, pointwise, a hyperplane in  $\mathbb{C}^n$ , so that  $(X, x) \cong (B/G, p(0))$  where  $p$  is the quotient map  $B \rightarrow B/G$ . If  $\theta$  is an  $m$ -ple  $n$ -form on  $X - \{x\}$ ,

$$f = p^*\theta / (dz_1 \wedge dz_2 \wedge \dots \wedge dz_n)^m$$

is a holomorphic function on  $B - \{0\}$  and hence extends to be holomorphic also at 0, the origin in  $\mathbb{C}^n$ . Then

$$\int_{X - \{x\}} (\theta \wedge \bar{\theta})^{1/m} = \frac{1}{g} \int_{B - \{0\}} (p^*\theta \wedge \overline{p^*\theta})^{1/m} ,$$

where  $g = \text{ord}(G)$ . Since  $p^*\theta = f(z) (dz_1 \wedge dz_2 \wedge \dots \wedge dz_n)^m$  is holomorphic, the integral in question is finite and so  $\theta \in L^{2/m}(X - \{x\})$ . Thus  $\delta_m(X, x) = 0$ .  $\square$

Theorem 1.6. Let  $\omega$  be a holomorphic  $n$ -form defined on a deleted neighborhood of  $x \in X$ , which is nowhere vanishing on this neighborhood. If  $\omega$  is square integrable in a neighborhood of  $x$ , then  $\delta_m(X, x) = 0$  for all  $m \geq 1$ .

Proof. Let  $V$  be a sufficiently small Stein neighborhood of  $x$ . If  $\theta$  is any holomorphic  $m$ -ple  $n$ -form on  $V - \{x\}$ ,  $f = \theta/\omega^m$  is a holomorphic function on  $V - \{x\}$  and hence extends to be holomorphic also at  $x$ . Thus  $\theta = f\omega^m$  is  $L^{2/m}$ -integrable.  $\square$

Let  $M$  be a compact complex  $(n-1)$ -dimensional manifold, and let  $F$  be a complex analytic line bundle over  $M$ . Assume that  $F$  is positive in the sense of [16]. We denote the total space of the dual line bundle  $F^*$  by  $\tilde{X}$ . The zero section of  $\tilde{X}$  is contractible. Then we get an  $n$ -dimensional normal isolated singularity  $(X, x)$  by blowing down  $\pi : \tilde{X} \rightarrow X$ .

The Leray spectral sequence of  $\pi$  shows

$$H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^0(X, R^i \pi_* \mathcal{O}_{\tilde{X}}) = (R^i \pi_* \mathcal{O}_{\tilde{X}})_x,$$

and the Leray spectral sequence for  $p : \tilde{X} \rightarrow M$ ,  $p$  the projection of  $F^*$ , shows

$$H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^i(M, R^0 p_* \mathcal{O}_{\tilde{X}}) = \bigoplus_{k \geq 0} H^i(M, \mathcal{O}(kF)).$$

Let  $K_M$  be the canonical line bundle of  $M$ . Then all these groups vanish if  $K_M$  is negative. In the case that  $K_M$  is a trivial line bundle,  $H^{n-1}(M, \mathcal{O}_M)$  is one-dimensional. In particular

$$p_g(X, x) = \sum_{k \geq 0} \dim H^{n-1}(M, \mathcal{O}(kF)) = \sum_{k \geq 0} \dim H^0(M, \mathcal{O}(K_M - kF)).$$

**Theorem 1.7.** Let  $(X, x)$  be an  $(n+1)$ -dimensional normal isolated singularity. Assume there exists a resolution  $\pi : \tilde{X} \rightarrow X$  such that  $A = \pi^{-1}(x)$  is an  $n$ -dimensional compact complex manifold. Then

$$\delta_m(X, x) \leq \sum_{k \geq 0} \dim \Gamma(A, \mathcal{O}(mK_A + kN)),$$

where  $N$  is the normal bundle of  $A$  in  $\tilde{X}$ .

**Proof.** Since  $A$  is compact, we can cover  $A$  by finite number of coordinate neighborhoods  $\{U_\alpha\}$  with holomorphic coordinates  $(z_\alpha^1, \dots, z_\alpha^n, t_\alpha)$  with  $A \cap U_\alpha = \{p \in U_\alpha \mid t_\alpha(p) = 0\}$ . Choose a Stein neighborhood  $V$  of  $x$  so that  $\bigcup_\alpha U_\alpha \supset U = \pi^{-1}(V)$ . Then we have an injective homomorphism

$$\Gamma(U-A, \mathcal{O}(mK)) / L^{2/m}(U-A) \rightarrow \sum_{k \geq 0} \Gamma(A, \mathcal{O}(mK_A + kN)) .$$

Let  $\phi \in \Gamma(U-A, \mathcal{O}(mK))$ .  $H_x^1(U, \mathcal{O}(mK))$  is a  $\Gamma(U, \mathcal{O})$ -module of finite dimension over  $\mathbb{C}$ . So  $\phi$  is meromorphic on  $U$  with possible poles on  $A$ .  $\phi$  is given by meromorphic function  $\phi_\alpha$  in  $U_\alpha$  such that

$$\phi_\alpha \{ dz_\alpha^1 \wedge \dots \wedge dz_\alpha^n (dt_\alpha/t_\alpha) \}^m .$$

Expand  $\phi_\alpha$  in a Laurent series

$$\phi_\alpha = \sum_{-\infty < k < \infty} \phi_\alpha^{(k)}(z) t_\alpha^{-k} .$$

In the case where  $k \geq 0$ ,

$$\left( \frac{1}{2\pi\sqrt{-1}} \right)^m \int \dots \int_{|t_\alpha| = \varepsilon_\alpha} (t_\alpha)^k \phi_\alpha \{ dz_\alpha^1 \wedge \dots \wedge dz_\alpha^n (dt_\alpha/t_\alpha) \}^m = \phi_\alpha^{(k)} (dz_\alpha^1 \wedge \dots \wedge dz_\alpha^n)^m ,$$

for  $\varepsilon_\alpha$  sufficiently small. Let  $\{f_{\alpha\beta}\}$  be transition functions of the line bundle  $[A]$ . On  $U_\alpha \cap U_\beta$ ,  $t_\alpha = f_{\alpha\beta} t_\beta$ , then

$$\phi_\alpha^{(k)} (dz_\alpha^1 \wedge \dots \wedge dz_\alpha^n)^m = (f_{\alpha\beta}|_A)^k \phi_\beta^{(k)} (dz_\beta^1 \wedge \dots \wedge dz_\beta^n)^m .$$

Clearly  $[A]|_A$  is the (complex analytic) normal bundle  $N$ . It follows that  $\{\phi_\alpha^{(k)}\} \in \Gamma(A, \mathcal{O}(mK_A + kN))$  and the homomorphism has been constructed. By the definition this homomorphism is injective.  $\square$

Remark. In the case where  $(X, x)$  admits a  $\mathbb{C}^*$ -action, the above homomorphism is surjective. Given  $\{\phi_\alpha^{(k)}\} \in \Gamma(A, \mathcal{O}(mK_A + kN))$ , we can form the global section of  $\Gamma(U-A, \mathcal{O}(mK))$  as follows. Since  $t_\alpha = f_{\alpha\beta}(z) t_\beta$ ,

$$dz_\alpha^1 \wedge \dots \wedge dz_\alpha^n (dt_\alpha/t_\alpha) = dz_\alpha^1 \wedge \dots \wedge dz_\alpha^n (dt_\beta/t_\beta)$$

on  $U_\alpha \cap U_\beta$ . Then

$$\{\phi_\alpha^{(k)} / (t_\alpha)^k\} \{dz_\alpha^1 \wedge \dots \wedge dz_\alpha^{n_\alpha} (dt_\alpha / t_\alpha)\}^m = \{\phi_\beta^{(k)} / (t_\beta)^k\} \{dz_\beta^1 \wedge \dots \wedge dz_\beta^{n_\beta} (dt_\beta / t_\beta)\}^m$$

and hence the homomorphism is surjective.

Let  $M_1, M_2$  be compact complex manifolds of dimensions  $n_1, n_2$ , respectively. There exist natural projections  $p_i$  from  $M = M_1 \times M_2$  to  $M_i$ ,  $i = 1, 2$ . Obviously  $p_1^*K_1 + p_2^*K_2$  is the canonical line bundle of  $M$ , which we denote by  $K$ . Suppose  $F_1, F_2$  be positive line bundles on  $M_1, M_2$ , respectively.  $F = p_1^*F_1 + p_2^*F_2$  is also positive. The zero section, which is identified with  $M$ , of the total space of  $F^*$  is contractible. By blowing down  $M$ , we get an  $(n_1 + n_2 + 1)$ -dimensional normal isolated singularity. Using Theorem 1.7, we get

$$\begin{aligned} \delta_m &= \sum_{k \geq 0} \dim \Gamma(M, O(mK - kF)) \\ &= \sum_{k \geq 0} \dim \Gamma(M_1 \times M_2, O(m(p_1^*K_1 + p_2^*K_2) - k(p_1^*F_1 + p_2^*F_2))) \\ &= \sum_{k \geq 0} \{ \dim \Gamma(M_1, O(mK_1 - kF_1)) \times \dim \Gamma(M_2, O(mK_2 - kF_2)) \} \end{aligned}$$

( by Künneth formula ) .

Proposition 1.8. If  $K_2$  is trivial and  $K_1$  is positive, then  $\delta_m \sim m^{n_1}$ .  
In fact  $\limsup_{m \rightarrow \infty} \delta_m / m^{n_1} = (1/n_1!) \{c_1(K_1)\}^{n_1}$ , where  $c_1(K_1)$  is the first

Chern class of  $K_1$ .

Example 1.9. Let  $(X, x)$  be the  $n$ -dimensional normal isolated singularity obtained by blowing down the zero section, denoted by  $M$ , of a negative line bundle. Then

$$(1) K_M : \text{negative} \implies \delta_m = 0 \quad (m \geq 1)$$

$$(2) K_M : \text{trivial} \implies \delta_m = 1 \quad (m \geq 1)$$

$$(3) K_M : \text{positive} \implies \limsup_{m \rightarrow \infty} \delta_m / m^n > 0.$$

Next we consider a normal isolated singularity defined by a quasi-homogeneous polynomial.

Definition 1.10. Suppose that  $(r_0, r_1, \dots, r_n)$  are fixed positive rational numbers. A polynomial  $f(z_0, z_1, \dots, z_n)$  is said to be quasi-homogeneous of type  $(r_0, r_1, \dots, r_n)$  if it can be expressed as a linear combination of monomials  $z_0^{i_0} z_1^{i_1} \dots z_n^{i_n}$  for which  $i_0 r_0 + i_1 r_1 + \dots + i_n r_n = 1$ .

Let  $d$  denote the smallest positive integer so that  $r_0 d = q_0$ ,  $r_1 d = q_1$ ,  $\dots$ ,  $r_n d = q_n$  are integers. Then  $f(t^{q_0} z_0, \dots, t^{q_n} z_n) = t^d f(z_0, \dots, z_n)$ .

Theorem 1.11. Let  $(X, x)$  be an  $n$ -dimensional normal isolated singularity defined by a quasihomogeneous polynomial  $f$  of type  $(r_0, r_1, \dots, r_n)$ . Then  $(X, x)$  is rational in the sense of Burns [4] if and only if  $r(f) = r_0 + r_1 + \dots + r_n > 1$ .

Proof. By virtue of [4, Proposition (3.2), p.239] it suffices to show that  $\omega = dz_1 \wedge \dots \wedge dz_n / \frac{\partial f}{\partial z_0}$  is square integrable in a neighborhood of  $x$ . Let  $d$  denote the smallest positive integer such that there exists, for each  $i$ , an integer  $q_i$  so that  $r_i d = q_i$ . Let  $\phi : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{n+1}$  be defined by  $\phi(w_0, \dots, w_n) = (w_0^{q_0}, \dots, w_n^{q_n})$  and let  $X' = \phi^{-1}(X)$ . Then  $X'$  is defined by a homogeneous polynomial  $\phi^* f$  of degree  $d$  with the singular locus  $S(X')$ . Let

$\underline{A}$  be an irreducible hypersurface defined by  $\phi^*f$  in  $\mathbb{P}_n$ ;  $X'$  is a "cone over  $\underline{A}$ ", and let  $\rho : A \rightarrow \underline{A}$  be a resolution of the singularity. Suppose that  $H_A^* \rightarrow A$  is the line bundle induced on  $A$  by the "tautological" line bundle on  $\mathbb{P}_n$  (dual to the hyperplane bundle). Then the "tautological" map  $\pi : H_A^* \rightarrow X'$  is a resolution of the singularity with  $\pi^{-1}(0) = A$ . Since  $\phi^*\omega$  is locally square integrable at any point  $p \in S(X') - \{0\}$ ,  $\pi^*\phi^*\omega$  is holomorphic off  $A$ . By easy calculation,  $\pi^*\phi^*\omega$  has zero of order  $q_0 + q_1 + \dots + q_n - d - 1$  at  $A$ . From [13],  $\omega$  is square integrable if and only if  $q_0 + q_1 + \dots + q_n - d - 1 \geq 0$ , i.e.,  $r_0 + r_1 + \dots + r_n > 1$ .  $\square$

Corollary 1.12 (Burns [4]). Arnold's singularities [1] are rational.

When  $(X, x)$  is defined by a quasihomogeneous polynomial, then  $\delta_m$  is completely determined by its weights  $\{r_0, r_1, \dots, r_n\}$ .

Theorem 1.13. Let  $(X, x)$  be an  $n$ -dimensional normal isolated singularity defined by a quasihomogeneous polynomial  $f$  of type  $(r_0, r_1, \dots, r_n)$ . Let  $d$  denote the smallest positive integer such that there exists, for each  $i$ , an integer  $q_i$  so that  $r_i d = q_i$ . Then

$$\delta_m(X, x) = \#\{(\lambda_0, \dots, \lambda_n) \in \mathbb{N}^{n+1} \mid m[d - (q_0 + \dots + q_n)] \geq \lambda_0 q_0 + \dots + \lambda_n q_n\} \\ - \#\{(\lambda_0, \dots, \lambda_n) \in \mathbb{N}^{n+1} \mid m[d - (q_0 + \dots + q_n)] - d \geq \lambda_0 q_0 + \dots + \lambda_n q_n\}.$$

Proof. If  $\theta$  is any holomorphic  $m$ -ple  $n$ -form on  $X - \{x\}$ ,  $g = \theta/\omega^m$  is a holomorphic function on  $X - \{x\}$  and hence extends to be holomorphic also at  $x$ . Expand  $g$  in a power series:  $g = \sum a_{\lambda_0 \dots \lambda_n} z_0^{\lambda_0} \dots z_n^{\lambda_n}$ . Notation being as in Theorem 1.11,  $\pi^*\phi^*z^\lambda \omega^m$  has zeros of order

$$\lambda_0 q_0 + \lambda_1 q_1 + \dots + \lambda_n q_n + m(q_0 + q_1 + \dots + q_n - d - 1)$$

at A. From [19, Theorem 2.1, p.243],  $z^\lambda \omega^m \in L^{2/m}(X - \{x\})$  if and only if

$$0 \leq (m-1) + \{ \lambda_0 q_0 + \lambda_1 q_1 + \dots + \lambda_n q_n + m(q_0 + q_1 + \dots + q_n - d - 1) \}$$

Hence

$$\theta \sim \tilde{g} \omega^m \pmod{L^{2/m}(X - \{x\})}$$

where  $\tilde{g} = \sum a_{\lambda_0 \lambda_1 \dots \lambda_n} z_0^{\lambda_0} z_1^{\lambda_1} \dots z_n^{\lambda_n}$  with

$$m\{d - (q_0 + q_1 + \dots + q_n)\} \geq \lambda_0 q_0 + \lambda_1 q_1 + \dots + \lambda_n q_n .$$

Assume moreover that  $\tilde{g} \equiv 0$  on  $X$ . Then there exists a polynomial  $p(z)$  such

that  $\tilde{g}(z) = f(z)p(z)$ , where  $p(z) = \sum b_{\lambda_0 \lambda_1 \dots \lambda_n} z_0^{\lambda_0} z_1^{\lambda_1} \dots z_n^{\lambda_n}$  with

$$m\{d - (q_0 + q_1 + \dots + q_n)\} - d \geq \lambda_0 q_0 + \lambda_1 q_1 + \dots + \lambda_n q_n .$$

Thus we get the desired result. □

Corollary 1.14.

$$p_g(X, x) = \#\{ (\lambda_0, \dots, \lambda_n) \in \mathbb{N}^{n+1} \mid d - (q_0 + \dots + q_n) \geq \lambda_0 q_0 + \dots + \lambda_n q_n \} .$$

One can easily check that Theorem 1.13 gives the following example.

Example 1.15. If we are as above, then

$$r(f) > 1 \iff \delta_m = 0 , \text{ for } m \geq 1,$$

$$r(f) = 1 \iff \delta_m = 1 , \text{ for } m \geq 1,$$

$$r(f) < 1 \iff \limsup_{m \rightarrow \infty} \delta_m / m^n = (1/n!)(1-r(f))^n (1/r_0 r_1 \dots r_n) .$$

Let  $k$  be a totally real field of degree  $n$  over the rationals and  $M$  an additive subgroup of  $k$  which is a free abelian group of rank  $n$ . Let  $U_M^+$  be the group of those units  $\epsilon$  of  $k$  which are totally positive and satisfy  $\epsilon M = M$ . For a given pair  $(M, E)$  with  $E \subset U_M^+$  (where  $E$  has rank  $n-1$ ) one defines

$$G(M, E) = \left\{ \begin{pmatrix} \epsilon & \mu \\ 0 & 1 \end{pmatrix} \mid \epsilon \in E, \mu \in M \right\} .$$

Let  $\mathbb{H}$  be the upper half plane. The group  $G(M, E)$  operates freely and properly discontinuously on  $\mathbb{H}^n$  by  $z_j \mapsto \epsilon^{(j)} z_j + \mu^{(j)}$ , where  $x \mapsto x^{(j)}$ ,  $1 \leq j \leq n$ , denote the  $n$  different embeddings of  $k$  into the reals. Then  $\mathbb{H}^n / G(M, E)$  defined a complex manifold which acquires a normal singularity when an additional point  $\infty$  is added with neighborhoods

$$|\operatorname{Im}(z_1) \operatorname{Im}(z_2) \dots \operatorname{Im}(z_n)| > c ,$$

where  $c$  is a constant. The singularity at  $\infty$  will be called a cusp singularity of type  $(M, E)$ .

Theorem 1.16. Let  $(X, x)$  be a cusp singularity. Then  $\delta_m(X, x) = 1$  for any  $m \geq 1$ .

The proof will be found in [26]. In the 2-dimensional case the proof was given in [6].



## §2. Normal Surface Singularities

We begin by recalling some theorems and definitions in Section 1. Let  $(X, x)$  be a (germ of) 2-dimensional normal singularity and  $\pi : \tilde{X} \rightarrow X$  be a resolution of the singularity. We assume that  $V$  is a Stein neighborhood of  $x$  in  $X$ . Then  $U = \pi^{-1}(V)$  is a strongly pseudoconvex neighborhood of  $A = \pi^{-1}(x)$ . Let  $K$  be the canonical line bundle of  $U$ . The following integer is defined by Küller [11] :

$$\gamma_m(X, x) = \dim \Gamma(U-A, \mathcal{O}(mK)) / \Gamma(U, \mathcal{O}(mK)) \quad (m \geq 1).$$

This integer is determined independently to the choice of the Stein neighborhoods. Hence  $\gamma_m$  is an invariant attached to the singularity. Küller considered the asymptotic behavior of  $\gamma_m$  when  $m \rightarrow +\infty$ , and showed

Theorem 2.1 (Küller [11]). There is a positive constant  $c$  such that  $\gamma_m \leq cm^2$  for a 2-dimensional normal singularity.

By the definition  $\Gamma(U, \mathcal{O}(mK)) \subset L^{2/m}(U-A)$ , we have  $\delta_m \leq \gamma_m$ . Therefore we obtain the following.

Theorem 2.2. For any 2-dimensional normal singularity, we have

$$\delta = \limsup_{m \rightarrow \infty} \delta_m / m^2 < \infty.$$

We call this theorem the first fundamental theorem of  $\{\delta_m\}$  for 2-dimensional normal singularities.

Let  $\pi^{-1}(x) = A = \cup A_i$ ,  $1 \leq i \leq n$ , be the decomposition of the exceptional set  $A$  into irreducible components. We associate a weighted

graph  $\Gamma$  to  $\pi$  in the following way.

Definition 2.3. We associate to a resolution  $\pi$  a weighted graph  $\Gamma_\pi$  with weighted vertices  $\gamma_i(b_i, g_i)$ ,  $i = 1, \dots, n$  where  $b_i = A_i \cdot A_i$  and  $g_i = \text{genus}(A_i)$ ;  $\gamma_i$  and  $\gamma_j$  are joined by 1-simplex if  $A_i \cap A_j \neq \emptyset$ . A vertex of weight  $(b, g)$  is denoted by  $\textcircled{b}$  and  $\textcircled{b}$  will denote  $\textcircled{b}$  .  
[g] [0]

Definition 2.4. A vertex  $\gamma_i$  of  $\Gamma$  is said to be center if either  $g_i > 0$  or  $g_i = 0$  and  $\gamma_i$  is joined to at least three other vertices. We say  $\Gamma$  is star-shaped if there is at most one center.

Definition 2.5. If  $\gamma$  is a vertex of  $\Gamma$  we defined  $\Gamma - \{\gamma\}$  to be the weighted graph obtained from  $\Gamma$  by removing  $\gamma$  and all edges joined to  $\gamma$ . If  $\gamma$  is the center of a star-shaped graph then the components of  $\Gamma - \{\gamma\}$  are called the branches of  $\Gamma$ .

By a cycle, we shall mean an element of the vector space over the rational numbers generated by  $\{A_i\}$ . A cycle  $Z = \sum r_i A_i$  is called effective if all  $r_j$ 's are non-negative. A cycle  $Z = \sum r_i A_i$  is called integral if all  $r_j$ 's are rational integers. A cycle  $Z = \sum r_i A_i$  is positive if  $Z$  is effective and  $r_j > 0$  for some  $j$ . We let  $|Z| = \cup A_i$ ,  $r_i \neq 0$ , denote the support of  $Z$ . We abbreviate a positive integral cycle to a PI-cycle.

In the following, by a curve we shall mean a compact irreducible 1-dimensional analytic subset of  $\tilde{X}$ , i.e., a curve is the one of  $\{A_j\}$ .

The intersection number  $Z_1 \cdot A_2$  of cycles  $Z_1$  and  $Z_2$  can be naturally defined. Note that this is in general a rational number. Let  $F$  be a line bundle on  $\tilde{X}$ . We define the intersection number  $F \cdot C$  of  $F$  with a curve  $C$  by

the degree of the line bundle  $F|_C$  restricted to  $C$ . Denote by  $[Z]$  the line bundle over  $\tilde{X}$  defined by the integral cycle  $Z$ . Then it is easy to see that  $[Z] \cdot C = Z \cdot C$  for any curve  $C$ . Since the intersection matrix  $S = (A_i \cdot A_j)$  is a non-singular matrix, we can uniquely determine the cycle  $Z = \sum r_i A_i$  satisfying

$$Z \cdot C = F \cdot C \quad (*)$$

for all curves  $C$ . In general  $\det(S)$  is not  $\pm 1$ , therefore the coefficients  $r_j$  may be rational numbers. The cycle  $Z$  defined by  $(*)$  is said to be numerically equivalent to  $F$ . We define the intersection number of line bundles  $F_1$  and  $F_2$  by  $F_1 \cdot F_2 = Z_1 \cdot Z_2$ , where  $Z_i$  are numerically equivalent cycles to  $F_i$  ( $i = 1, 2$ ).  $(\det(S))F_1 \cdot F_2$  is an integer. It is easy to check that for integral cycles  $Z_1$  and  $Z_2$  we have  $[Z_1] \cdot [Z_2] = Z_1 \cdot Z_2$ .

We define the virtual genus of PI-cycle  $Z$  to be

$$p(Z) = (1/2)(Z \cdot Z + K \cdot Z) + 1,$$

where  $K$  is the canonical line bundle on  $\tilde{X}$ . Now we define

$$p_a(X, x) = \sup p(Z)$$

where  $Z$  ranges over all PI-cycle on  $\tilde{X}$ .

The fundamental cycle is the unique PI-cycle  $Z_0$  on  $\tilde{X}$  such that

- (1)  $Z_0 \cdot A_i \leq 0$  for every component  $A_i$  of  $\pi^{-1}(x)$ ,
- (2) if  $Z$  is a PI-cycle such that  $Z \cdot A_i \leq 0$  for any  $i$ , then  $Z \geq Z_0$ .

The existence of  $Z_0$  is shown by Artin [2]. Given the intersection matrix  $(A_i \cdot A_j)$ , one can easily determine  $Z_0$ . We define

$$p_f(X, x) = p(Z_0).$$

In fact  $p_a$  and  $p_f$  are independent of the choice of the resolution (for details see [24]).

Definition 2.6. We call  $p_a(X,x)$  the arithmetic genus of  $(X,x)$ .

The invariants defined thus far are not independent. One can easily see that  $p_g \geq p_a \geq p_f$ . Furthermore Artin [2] has proven the following theorem :

$$p_g = 0 \iff p_a = 0 \iff p_f = 0 .$$

Moreover Wagreich [24] has proven that  $p_a = 1 \iff p_f = 1$ .

Definition 2.7. Let  $(X,x)$  be a normal surface singularity. We say  $(X,x)$  is rational (resp. elliptic) if  $p_g(X,x) = 0$  (resp.  $p_g(X,x) = 1$ ).

Remark. For the definition of elliptic singularities some authors work instead with  $p_a$  :  $(X,x)$  is elliptic  $\iff p_a(X,x) = 1$ . In this case they say  $(X,x)$  is strongly elliptic if  $p_g(X,x) = 1$ .

Let  $A'$  be any connected proper subvariety of  $A$ . Then  $A'$  is exceptional in  $U$  by [12], Lemma 5.11, p.89.  $A'$  has a pseudoconvex neighborhood  $U'$ . Blowing down  $A'$ , we get a normal surface singularity, which is denoted by  $(X',x')$ . The singularity which appears in this way will be simpler than the original singularity  $(X,x)$ , provided that  $A$  is the exceptional set of the minimal resolution.

We call the following theorem the second fundamental theorem of  $\{ \delta_m \}$  for normal surface singularities.

Theorem 2.8. In the case of a minimal resolution, for any  $m \geq 1$ , we have

$$\gamma_m(X, x) \geq \gamma_m(X', x') ,$$

$$\delta_m(X, x) \geq \delta_m(X', x') .$$

Proof. Let  $A^{0,q}(F)$  be the sheaf of germs of  $(0,q)$ -forms with coefficients in a complex analytic line bundle  $F$ . Then we have a fine resolution  $\{ A^{0,q}(F) \}$  of  $\mathcal{O}(F)$ .

Following Laufer [13], we have a diagram :

$$\begin{array}{ccccccc} 0 & \rightarrow & \Gamma(U', \mathcal{O}(mK)) & \rightarrow & \Gamma(U' - A', \mathcal{O}(mK)) & \rightarrow & H_*^1(U', \mathcal{O}(mK)) \rightarrow \dots \\ & & & & & & \downarrow \\ 0 & \rightarrow & \Gamma(U, \mathcal{O}(mK)) & \rightarrow & \Gamma(U - A, \mathcal{O}(mK)) & \rightarrow & H_*^1(U, \mathcal{O}(mK)) \rightarrow H^1(U, \mathcal{O}(mK)) . \end{array}$$

Given an element  $\omega \in \Gamma(U' - A', \mathcal{O}(mK))$ , there exists  $\xi \in \Gamma(U', A^{0,0}(mK))$  such that  $\xi = \omega$  near the boundary of  $U'$ .  $\bar{\partial}\xi = \bar{\partial}\omega = 0$  near the boundary of  $U'$ . Hence  $\bar{\partial}\xi$  has compact support, i.e.,  $\bar{\partial}\xi \in \Gamma_*(U', A^{0,1}(mK))$ .  $\bar{\partial}(\bar{\partial}\xi) = 0$ , so  $\bar{\partial}\xi$  is a cocycle in  $H_*^1(U', \mathcal{O}(mK))$ . Let  $\widetilde{\bar{\partial}\xi}$  be the zero extension of  $\bar{\partial}\xi$  from  $U'$  to  $U$ . Then  $\bar{\partial}(\widetilde{\bar{\partial}\xi}) = 0$ , and  $\widetilde{\bar{\partial}\xi}$  is a cocycle in  $H_*^1(U, \mathcal{O}(mK))$ . By [10], Vanishing Theorem, p.246,  $H^1(U, \mathcal{O}(mK)) = 0$ . Therefore  $\widetilde{\bar{\partial}\xi}$  is the  $\bar{\partial}$ -image of some  $\zeta \in \Gamma(U, A^{0,0}(mK))$ :  $\bar{\partial}\zeta = \widetilde{\bar{\partial}\xi}$ . Since  $\widetilde{\bar{\partial}\xi}$  is zero near the boundary of  $U$ ,  $\zeta$  is holomorphic there. By Siu [20], p.374, there exists  $\tilde{\omega} \in \Gamma(U - A, \mathcal{O}(mK))$  such that  $\tilde{\omega} = \zeta$  near the boundary of  $U$ . It is easy to check that the map  $\omega \mapsto \tilde{\omega}$  induces a well-defined homomorphism

$$\Gamma(U' - A', \mathcal{O}(mK)) \rightarrow \Gamma(U - A, \mathcal{O}(mK)) / \Gamma(U, \mathcal{O}(mK)) \rightarrow \Gamma(U - A, \mathcal{O}(mK)) / L^{2/m}(U - A) .$$

$\zeta$  is holomorphic outside of some compact set in  $U'$ .  $\tilde{\omega}$  has possible poles on  $A'$ . Since  $\bar{\partial}(\zeta - \xi) = 0$  on  $U'$ ,  $\zeta - \xi = \lambda$ ,  $\lambda \in \Gamma(U', \mathcal{O}(mK))$ . Hence  $\tilde{\omega} - \omega = \lambda$  on  $U'$ .

Therefore, if  $\tilde{\omega} \in \Gamma(U, \mathcal{O}(mK))$ , then  $\omega \in \Gamma(U', \mathcal{O}(mK))$ ; besides, if  $\tilde{\omega} \in L^{2/m}(U-A)$ , then  $\omega \in L^{2/m}(U'-A')$ . Thus homomorphisms

$$\phi : \Gamma(U'-A', \mathcal{O}(mK)) / \Gamma(U', \mathcal{O}(mK)) \rightarrow \Gamma(U-A, \mathcal{O}(mK)) / \Gamma(U, \mathcal{O}(mK))$$

$$\psi : \Gamma(U'-A', \mathcal{O}(mK)) / L^{2/m}(U'-A') \rightarrow \Gamma(U-A, \mathcal{O}(mK)) / L^{2/m}(U-A)$$

are defined and injective. □

Remark. Note that  $\Omega \in \Gamma(U-A, \mathcal{O}(mK))$  having poles exactly on  $A$  is not in the image of  $\phi$  and  $\psi$  respectively.

Corollary 2.9. Let  $(X, x)$  be a Gorenstein singularity, i.e., there is some neighborhood  $V$  of  $x$  in  $X$  and a holomorphic 2-form  $\omega$  on  $V - \{x\}$  such that  $\omega$  has no zeros on  $V - \{x\}$ . If  $p_g(X, x) \geq 1$ , then  $p_g(X, x) > p_g(X', x')$ .

Proof. Let  $\pi : \tilde{X} \rightarrow X$  be the minimal resolution of the singularity. The support of  $\pi^*\omega$  is empty or  $A = \pi^{-1}(x)$ . If  $\pi^*\omega \in \Gamma(U, \mathcal{O}(K))$ , then  $(X, x)$  is a rational singularity, and so  $p_g(X, x) = 0$ , a contradiction. Hence the support of  $\pi^*\omega$  is  $A$ . Thus  $\pi^*\omega$  is not in the image of  $\phi_1 = \psi_1$  for any  $(U', A')$  as in the proof of Theorem 2.8, where  $\phi_1$  (resp.  $\psi_1$ ) denotes  $\phi$  (resp.  $\psi$ ) with  $m = 1$ . □

Arnold defined the inner modality for quasihomogeneous isolated singularities. In the 2-dimensional case, the following theorem is proved in [29].

Theorem. Let  $\mu_0$  be the inner modality of a quasihomogeneous isolated singularity of dimension 2. Then  $p_g \leq \mu_0 \leq \delta_2$ . Furthermore,

$$p_g = 1 \implies \mu_0 \leq 4 \quad ; \quad \mu_0 = 1 \text{ or } 2 \implies p_g = 1 .$$

Definition 2.10. A normal surface singularity  $(X, x)$  with  $p_a \geq 1$  is minimal if  $p_g(X, x) > p_g(X', x')$  for any  $(X', x')$ . Moreover  $(X, x)$  is minimally elliptic if  $p_a = 1$  and  $p_g(X', x') = 0$  for any  $(X', x')$  (see Laufer [15, Theorem 3.4 (3)]).

Remark. The definition of "minimally elliptic" is equivalent to that of Laufer [15, p.1263].

Theorem 2.11. Let  $(X, x)$  be a normal surface singularity. If  $(X, x)$  is Gorenstein and  $p_a \geq 1$ , then  $(X, x)$  is minimal.

Proof. Obvious by Corollary 2.9. □

Corollary 2.12 (Laufer [15]). If  $(X, x)$  is Gorenstein and  $p_g(X, x) = 1$ , then  $(X, x)$  is a minimally elliptic singularity.

Theorem 2.13. Let  $(X, x)$  be a normal surface singularity. If  $(X, x)$  is minimal and  $p_a = 1$ , then  $(X, x)$  is Gorenstein.

Proof. Suppose  $p_g = n$ , then  $n = p_g \geq p_a = 1$ . Choose  $\omega_1, \omega_2, \dots, \omega_n \in \Gamma(U-A, \mathcal{O}(K))$  to be a basis for  $\Gamma(U-A, \mathcal{O}(K))/\Gamma(U, \mathcal{O}(K))$ , where  $U$  is a strongly pseudoconvex neighborhood of  $A$ . Let  $C_i$  (resp.  $D_i$ ) be the pole (resp. zero) locus of  $\omega_i$ , then  $K = (\omega_i) = -C_i + D_i$ , and  $C_i$  is a positive cycle. By the definition of  $p_a$ ,

$$1 = p_a \geq p(C_i) = (1/2)(C_i \cdot C_i + C_i \cdot K) + 1 = (1/2)C_i \cdot D_i + 1 .$$

Since  $C_i \cdot D_i \geq 0$ ,  $p(C_i) = 1$ .

Should  $|C_i| \cap |C_j| = \emptyset$  for some  $i$  and  $j$ , then there would exist a positive cycle  $Z$  such that  $p(Z) \geq 0$ ,  $Z \cdot C_i \geq 1$  and  $Z \cdot C_j \geq 1$ , so

$$1 = p_a \geq p(C_i + Z + C_j) = p(C_i) + p(Z) + p(C_j) + C_i \cdot Z + C_j \cdot Z + C_i \cdot C_j - 2 \geq 2,$$

which is a contradiction. Hence  $\cup |C_i|$  is a connected analytic subvariety of  $A$ . Since  $(X, x)$  is minimal,  $A = \cup |C_i|$ . Now consider linear combinations of  $\{\omega_i\}$  with coefficients  $\sum \alpha_i \omega_i$ . In those forms there exists a meromorphic 2-form  $\omega$  such that the support of the pole of  $\omega$  is  $A$ . We write the divisor  $(\omega)$  as  $(\omega) = -C + D$  where  $C$  is the pole locus and  $D$  is the zero locus of  $\omega$  respectively. From the definition of  $p_a$ ,

$$1 = p_a \geq p(C) = (1/2)(C \cdot C + K \cdot C) + 1 = (1/2)C \cdot D + 1.$$

Since  $C \cdot D \geq 0$ ,  $C \cdot D = 0$ . Hence  $\omega$  has no zeros near  $A$ . Thus  $(X, x)$  is Gorenstein. □

Lemma 2.14. With the notation being above let  $f$  be a holomorphic function on  $U$  and non-vanishing off  $A$ . Then  $f(A) \neq 0$ .

Proof. Suppose  $f(A) = 0$ . Then  $(f) = Z$  where  $Z = \sum d_i A_i$ . Now  $(f) \cdot A_i = 0$  for any  $i$ , therefore  $Z \cdot Z = 0$ . It contradicts the fact that the intersection matrix  $(A_i \cdot A_j)$  is negative definite. □

In the case where  $(X, x)$  is a minimally elliptic singularity we get more information about the singularity.

Theorem 2.15 (Laufer [15]). Let  $(X, x)$  be a minimally elliptic singularity. Then  $(X, x)$  is Gorenstein and  $p_g(X, x) = 1$ .

Proof. Let  $U$  be a strongly pseudoconvex neighborhood of the



exceptional set  $A$ . Since  $p_g \geq p_a = 1$ , there exists at least one non-zero element in  $\Gamma(U-A, \mathcal{O}(K))/\Gamma(U, \mathcal{O}(K))$ . Let  $\omega \in \Gamma(U-A, \mathcal{O}(K))$  be the representative of the above element. Denote by  $C$  (resp.  $D$ ) the pole (resp. zero) locus of  $\omega$ , then  $K = (\omega) = -C + D$ , where  $C$  is a positive cycle. So

$$1 = p_a \geq p(C) = (1/2)(C \cdot C + C \cdot K) + 1 = (1/2)C \cdot D + 1$$

Since  $C \cdot D \geq 0$ ,  $C \cdot D = 0$ . Now  $(X, x)$  is minimally elliptic, therefore  $|C| = A$ . Hence we may assume that  $\omega$  does not vanish off  $A$ . This implies  $(X, x)$  is Gorenstein.

Let  $\omega'$  be another non-zero element in  $\Gamma(U-A, \mathcal{O}(K))/\Gamma(U, \mathcal{O}(K))$ . By the similar argument  $\omega'$  does not vanish off  $A$  and the support of its pole locus is  $A$ . Then  $f = \omega'/\omega$  is a nowhere vanishing holomorphic function on  $U-A$  and hence extends to be holomorphic also at  $A$ . We claim that

$$(f-f(A))\omega \in \Gamma(U, \mathcal{O}(K)) .$$

Suppose otherwise, i.e.,  $(f-f(A))\omega \notin \Gamma(U, \mathcal{O}(K))$ . By the same argument  $(f-f(A))\omega$  does not vanish off  $A$ . By Lemma 2.14  $(f-f(A))$  does not vanish on  $A$ , which is a contradiction. Therefore

$$\omega' - f(A)\omega = (f-f(A))\omega \in \Gamma(U, \mathcal{O}(K)) .$$

Thus  $p_g = \dim \Gamma(U-A, \mathcal{O}(K))/\Gamma(U, \mathcal{O}(K)) = 1$ . □

**Theorem 2.16.** Let  $(X, x)$  be a normal surface singularity. If  $(X, x)$  is Gorenstein and  $p_g \geq 2$ , then  $1 \leq p_f < p_g$ .

**Proof.** We assume  $\pi : \tilde{X} \rightarrow X$  to be the minimal resolution of the singularity. Let  $U$  be a strongly pseudoconvex neighborhood of the exceptional set  $\pi^{-1}(x) = A$ . Suppose that  $p_f = p_g$ . Then  $p(Z_0) = p_g \geq 2$ .

Kato's theorem [10], p.246, says

$$\dim \Gamma(U-A, \mathcal{O}(-Z_0)) / \Gamma(U, \mathcal{O}(-Z_0)) + \dim H^1(U, \mathcal{O}(-Z_0)) = (1-p(Z_0)) + p_g(X, x) .$$

By the hypothesis the right hand side is equal to 1. Since  $|Z_0| = A$ , a non-zero constant function is not a zero element in

$$\Gamma(U-A, \mathcal{O}(-Z_0)) / \Gamma(U, \mathcal{O}(-Z_0)) ,$$

so  $\dim \Gamma(U-A, \mathcal{O}(-Z_0)) / \Gamma(U, \mathcal{O}(-Z_0)) \geq 1$ . Hence  $\dim H^1(U, \mathcal{O}(-Z_0)) = 0$ . Now consider the sheaf cohomology with support at infinity. Let  $K$  be the canonical line bundle of  $U$ . Then the following sequence is exact :

$$0 \rightarrow \Gamma(U, \mathcal{O}(K+Z_0)) \xrightarrow{\phi} \Gamma(U-A, \mathcal{O}(K+Z_0)) \rightarrow H_*^1(U, \mathcal{O}(K+Z_0)) \rightarrow \dots .$$

Serre duality gives  $H^1(U, \mathcal{O}(-Z_0))$  as dual to  $H_*^1(U, \mathcal{O}(K+Z_0))$ . Then  $\phi$  is an isomorphism. As  $(X, x)$  is Gorenstein, there is a holomorphic 2-form  $\omega$  on  $U-A$  such that  $\omega$  has no zeros on  $U-A$ . Let  $(\omega) = \sum_{i=1}^n \lambda_i A_i$  denote the divisor of  $\omega$ , where  $A_i$  ( $i = 1, 2, \dots, n$ ) are the irreducible components of  $A$ . Then we obtain  $n$  linear equations

$$K \cdot A_j = (\omega) \cdot A_j = \sum_{i=1}^n \lambda_i A_i \cdot A_j \quad (j = 1, 2, \dots, n)$$

in  $n$  unknowns  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Since  $(X, x)$  is not a rational double point,  $K \cdot A_{j_0} > 0$  for some  $j_0$  [cf. 11]. By Lemma 3.2  $\lambda_i < 0$  for all  $i$ , i.e.,  $\lambda_i \leq -1$  for all  $i$ . Then  $-(\omega)$  is a PI-cycle. Now  $(\omega)$  is a cycle on  $A$  and  $\omega + Z_0 \in \Gamma(U-A, \mathcal{O}(K+Z_0))$ . Hence  $\omega + Z_0 \in \Gamma(U, \mathcal{O}(K+Z_0))$ , so  $Z_0 \geq -(\omega)$ . Since  $\pi$  is the minimal resolution,  $0 \leq K \cdot A_j = (\omega) \cdot A_j$  for any  $j$ , so  $-(\omega) \geq Z_0$  by the minimality of the fundamental cycle  $Z_0$ . Therefore  $K = (\omega) = -Z_0$ . Thus  $p(Z_0) = 1$ , which is a contradiction.  $\square$

Corollary 2.17 (Yoshinaga-Ohtanagi [28]). Let  $(X, x)$  be a normal surface singularity. If  $(X, x)$  is Gorenstein and  $p_g = 2$ , then  $p_a = 1$ .

Proof. By Theorem 2.16  $p_f = 1$ , and which implies  $p_a = 1$ .  $\square$

Thus, by Theorems 2.11 and 2.13 we obtain the following theorems.

Theorem 2.18. Let  $(X, x)$  be a normal surface singularity with  $p_g = 2$ . Then  $(X, x)$  is Gorenstein if and only if  $(X, x)$  is minimal and  $p_a = 1$ .

Theorem 2.19. Let  $(X, x)$  be a normal surface singularity with  $p_a = 1$ . Then  $(X, x)$  is Gorenstein if and only if  $(X, x)$  is minimal.

A resolution  $\tilde{X} \rightarrow X$  of a normal surface singularity  $(X, x)$  is good, if

(i) All the components of the exceptional set of  $\tilde{X} \rightarrow X$  are smooth and intersect transversely.

(ii) Not more than two components pass through any given point.

(iii) Two different components intersect at most once.

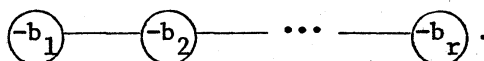
It is well-known (and easy to see) that there is a minimal resolution having these properties.

Now we give  $\delta_m$ -formula for the normal surface singularity whose minimal good resolution is star-shaped. In what follows we consider the normal surface singularity whose minimal good resolution is star-shaped. Let  $A_0$  be the center of the weighted graph. The branches of the star-shaped graph are indexed by  $i$ ,  $1 \leq i \leq n$ . The curves of the  $i$ -th branch are denoted by  $A_{ij}$ ,  $1 \leq j \leq r_i$ , where  $A_{i1}$  intersects  $A_0$  and  $A_{ij}$  intersects  $A_{i,j+1}$ . Let  $-b = A_0 \cdot A_0$ , and  $-b_{ij} = A_{ij} \cdot A_{ij}$ . Then  $b_{ij} \geq 2$  and  $b \geq 1$ . Finally, set

$$d_i/e_i = b_{i1} \frac{1}{b_{i2} \frac{1}{\dots \frac{1}{b_{ir_i}}}}$$

=  $[b_{i1}, b_{i2}, \dots, b_{ir_i}]$  with  $e_i < d_i$ , and  $e_i$  and  $d_i$  are relatively prime.

Lemma 2.20 (Brieskorn [3]). Let  $\tilde{X}$  be the minimal good resolution of a normal surface singularity  $(X, x)$  such that the weighted (dual) graph is



Let  $d/e = [b_1, b_2, \dots, b_r]$ ,  $e$  and  $d$  relatively prime. Then  $X$  is analytically isomorphic to the quotient of  $\mathbb{C}^2$  by the cyclic group  $G$  of order  $d$ , acting by  $(z_1, z_2) = (\zeta z_1, \zeta^e z_2)$ , where  $\zeta$  is a  $d$ -th root of unity.

We call this singularity the cyclic quotient singularity of type  $(d, e)$ .

For any  $k \geq 0$  and  $m \geq 1$  let  $D_m^{(k)}$  be the divisor on  $A_0$  :

$$D_m^{(k)} = kD - \sum_i [\{ke_i + m(d_i - 1)\}/d_i] P_i,$$

where  $D$  is any divisor such that  $\mathcal{O}_{A_0}(D)$  is the conormal sheaf of  $A_0$ ,  $P_i = A_0 \cap A_{i1}$ , and for any  $a \in \mathbb{R}$ ,  $[a]$  is the greatest integer less than, or equal to  $a$ .

Theorem 2.21. In case the minimal good resolution of  $(X, x)$  is star-shaped, the plurigenus  $\delta_m$  is not more than

$$\sum_{k \geq 0} \dim \Gamma(A_0, \mathcal{O}_{A_0}(mK_{A_0} - D_m^{(k)})).$$

Corollary 2.22. In the above situation the geometric genus  $p_g(X, x)$  is

not more than

$$\sum_{k \geq 0} \dim H^1(A_0, \mathcal{O}_{A_0}(D_1^{(k)})) .$$

Under the condition that  $(X, x)$  admits a  $C^*$  action

$$\delta_m(X, x) = \sum_{k \geq 0} \dim \Gamma(A_0, \mathcal{O}_{A_0}(mK_{A_0} - D_m^{(k)})) ,$$

and,

$$p_g(X, x) = \sum_{k \geq 0} \dim H^1(A_0, \mathcal{O}_{A_0}(D_1^{(k)})) ,$$

which was proved by Pinkham [17].

Proof of the Theorem. Let  $\{U_i\}$  be a cover of  $A$  such that  $P_i \in U_i$  for  $i = 1, \dots, n$ , and  $P_j \notin U_i$ ,  $i \neq j$ . Assume moreover there exist local coordinates  $(z_i, t_i)$  with  $A_0 \cap U_i = \{P \in U_i \mid t_i(P) = 0\}$  and  $P_i = \{P \in U_i \mid t_i(P) = z_i(P) = 0\}$ . Take a (sufficiently small) Stein neighborhood  $V$  of  $x$  so that  $\cup U_i \supset \pi^{-1}(V) = U$ . Let  $\omega$  be any  $m$ -ple holomorphic 2-form on  $U-A$ :  $\omega \in \Gamma(U-A, \mathcal{O}(mK))$ . On  $U_i$   $\omega$  is written as  $\omega|_{U_i} = \phi_i \{dz_i \wedge (dt_i/t_i)\}^m$ . Expand  $\phi_i$  in Laurent series on  $U_i$ :  $\phi_i = \sum \phi_i^{(k)} t_i^{-k}$ . The same argument as Theorem 1.7 works in this case, and so  $\{\phi_i^{(k)} (dz_i)^m\}$  becomes a meromorphic section of  $mK_{A_0} + kN$  where  $N$  is the normal bundle of  $A_0$  in  $U$ . Let  $v_i$  be the order of the pole of  $\phi_i^{(k)}$  at  $P_i$ . Then

$$v_i \leq \lfloor \{ke_i + m(d_i - 1)\} / d_i \rfloor .$$

In order to prove this, it is sufficient to prove the following lemma, since each branch is the cyclic quotient singularity.

Lemma. Given  $b_1, b_2, \dots, b_r$  with the  $b_i$  integers such that  $b_i \geq 2$ , the

manifold  $M = M(b_1, b_2, \dots, b_r)$  will be covered by  $r+1$  coordinate patches,  $W_i = (u^{(i)}, v^{(i)}) = \mathbb{C}^2$ ,  $0 \leq i \leq r$ , joined as follows.

$$\begin{array}{lll} W_0 \cap W_1 = \{ u \neq 0 \} & u' = 1/u & v' = u^{b_1} v \\ W_1 \cap W_2 = \{ v' \neq 0 \} & v'' = 1/v' & u'' = (v')^{b_2} u' \\ W_2 \cap W_3 = \{ u'' \neq 0 \} & u''' = 1/u'' & v''' = (u'')^{b_3} v'' \\ \vdots & \vdots & \vdots \end{array}$$

Let  $A_0 = \{ u = 0 \}$ ,  $A_1 = \{ v = 0 \} \cup \{ v' = 0 \}$ ,  $A_2 = \{ u' = 0 \} \cup \{ u'' = 0 \}$ ,  $A_3 = \{ v'' = 0 \} \cup \{ v''' = 0 \}$ , ...,  $A_r = \{ u^{(r-1)} = 0 \} \cup \{ u^{(r)} = 0 \}$ ,  $A_{r+1} = \{ v^{(r)} = 0 \}$  if  $r$  is even and  $A_r = \{ v^{(r-1)} = 0 \} \cup \{ v^{(r)} = 0 \}$ ,  $A_{r+1} = \{ u^{(r)} = 0 \}$  if  $r$  is odd. Then  $A' = A_1 \cup A_2 \cup A_3 \cup \dots \cup A_r$  is a compact analytic subset of  $M$ .

We define positive integers  $\{ \lambda_r, \lambda_{r-1}, \dots, \lambda_0 \}$  as follows :

$$\begin{array}{ll} \lambda_r = 1 & \vdots \\ & \lambda_2 = b_3 \lambda_3 - \lambda_4 \\ \lambda_{r-1} = b_r \lambda_r & e = \lambda_1 = b_2 \lambda_2 - \lambda_3 \\ \lambda_{r-2} = b_{r-1} \lambda_{r-1} - \lambda_r & d = \lambda_0 = b_1 \lambda_1 - \lambda_2 \\ \vdots & \end{array}$$

Now consider a meromorphic  $m$ -ple 2-form

$$\omega = (du \wedge dv)^m / u^{k+m} v^v$$

on  $W_0$ . Then  $\omega$  is holomorphic on  $M - (A_0 \cup A')$  if and only if

$$v \leq \lceil \{k+m(d-1)\} / d \rceil .$$

Proof. Let  $(\omega) = \sum_{i=0}^{r+1} -a_i A_i$  be the divisor of  $\omega$  on  $M$ , where  $a_0 = k + m$  and  $a_1 = v$ . Since  $p(A_i) = 0$  for  $i = 1, \dots, r$ ,

$$m(-2 + b_i) = -a_{i-1} + b_i a_i - a_{i+1} .$$

Hence, by the definition of  $\{ \lambda_i \}$

$$\begin{aligned} -a_{r+1} &= a_{r-1} - b_r a_r + m(-2 + b_r) \\ &= \lambda_r (a_{r-1} - m) + m(\lambda_{r-1} - 1) - a_r \lambda_{r-1} \\ &= \lambda_{r-1} (a_{r-2} - m) + m(\lambda_{r-2} - 1) - a_{r-1} \lambda_{r-2} \\ &\vdots \\ &= \lambda_1 (a_0 - m) + m(\lambda_0 - 1) - a_1 \lambda_0 \\ &= ek + m(d - 1) - vd . \end{aligned}$$

Thus  $\omega$  is holomorphic on  $M-(A_0 \cup A')$  if and only if  $ek+m(d-1)-vd \geq 0$ , i.e.,

$$v \leq \lfloor \{ek + m(d - 1)\} / d \rfloor . \quad \square$$

Remark. If  $k < 0$ , then  $a_0 = k + m < m$ . Since

$$\lambda_1 (a_0 - m) + \lambda_0 (m - a_1) - m \geq 0 ,$$

$\lambda_1 (a_0 - m) > \lambda_0 (a_1 - m)$ , and so  $a_1 < m$ . Hence by induction, it is true that  $a_i < m$  for  $i = 2, 3, \dots, r$ .

Now we continue our proof of Theorem 2.21. Since  $v_i \leq \lfloor \{ke_i + m(d_i - 1)\} / d_i \rfloor$ ,  $\{ \phi_i^{(k)} (dz_i)^m \}$  is a holomorphic section of

$$mK_{A_0} + kN + \sum_i \lfloor \{ke_i + m(d_i - 1)\} / d_i \rfloor P_i .$$

Therefore we have a homomorphism

$$\Gamma(U-A, \mathcal{O}(mK)) \rightarrow \bigoplus_{k \geq 0} \Gamma(A_0, \mathcal{O}_{A_0}(mK_{A_0} - D_m^{(k)})) .$$

By the above remark the kernel of this mapping is

$$\Gamma(U, \mathcal{O}(mK+(m-1)A)) \cong L^{2/m}(U-A) .$$

Thus

$$\Gamma(U-A, \mathcal{O}(mK))/L^{2/m}(U-A) \rightarrow \bigoplus_{k \geq 0} \Gamma(A_0, \mathcal{O}_{A_0}(mK_{A_0} - D_m^{(k)}))$$

is injective.

Next consider the case where  $(X, x)$  admits a  $\mathbb{C}^*$ -action. For a holomorphic section  $\{ \phi_i^{(k)} (dz_i)^m \}$  of  $mK_{A_0} - D_m^{(k)}$ ,

$$\phi_i^{(k)} t_i^{-k} \{ dz_i \wedge (dt_i/t_i) \}^m$$

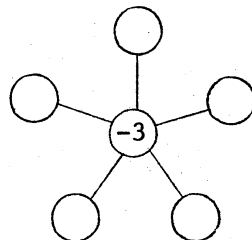
becomes a meromorphic section of  $\mathcal{O}(mK)$ , which is defined in the neighborhood of  $A_0$ , and extends to be a holomorphic section of  $\mathcal{O}(mK)$  on  $U-A$ . Thus

$$\Gamma(U-A, \mathcal{O}(mK))/L^{2/m}(U-A) \cong \bigoplus_{k \geq 0} \Gamma(A_0, \mathcal{O}_{A_0}(mK_{A_0} - D_m^{(k)})) . \quad \square$$

Example 2.23. Let  $(X, x)$  be defined by a quasihomogeneous polynomial  $z_0^2 + z_1^5 + z_2^5$  :

$$X = \{ (z_0, z_1, z_2) \in \mathbb{C}^3 \mid z_0^2 + z_1^5 + z_2^5 = 0 \} .$$

Then  $\delta_m$  can be calculated by Theorem 1.13. The minimal resolution of  $(X, x)$  is as follows, so  $\delta_m$  can also be computed directly from the above theorem.



Since  $c_1(mK_{A_0} - D_m^{(k)}) = -2m - 3k + 5[(k+m)/2]$ ,



$$\delta_m = (m^2 + 3)/4, \quad \text{if } m \text{ is odd,}$$

$$\delta_m = (m^2 + 8)/4, \quad \text{if } m \text{ is even.}$$

by the Riemann-Roch theorem. In particular

$$\delta = \limsup_{m \rightarrow \infty} \delta_m/m^2 = 1/4 .$$

Using Theorem 2.21, T. Tomaru classified, by the behavior of  $\delta_m$ , the singularities with  $\mathbb{C}^*$  action.

Theorem 2.24 (Tomaru [21]). Normal surface singularities with  $\mathbb{C}^*$  action are classified as follows :  $g = \text{genus}(A_0)$ ,  $L = \text{L.C.M.}(d_1, d_2, \dots, d_n)$ .

$\delta$	$\delta_m$	structure
$> 0$	$\delta_m$ diverges with second order as $m \rightarrow \infty$	(i) $g \geq 2$
		(ii) $g = 1$ and $n \geq 1$
		(iii) $g = 0$ and $\sum_{i=1}^n (d_i - 1)/d_i > 2$
0	(I) $\delta_m = 1$ for any $m \geq 1$	$g = 1$ and $n = 0$
	(II) $\delta_m = \begin{cases} 0 & \text{if } m \not\equiv 0 \pmod{L} \\ 1 & \text{if } m \equiv 0 \pmod{L} \end{cases}$	$g = 0$ and $\sum_{i=1}^n (d_i - 1)/d_i = 2$
	(III) $\delta_m = 0$ for any $m \geq 1$	$g = 0$ and $\sum_{i=1}^n (d_i - 1)/d_i < 2$ , or cyclic quotient singularities

Corollary 2.25. If  $\limsup_{m \rightarrow \infty} \delta_m/m^2 > 0$ , then

$$\limsup_{m \rightarrow \infty} \delta_m/m^2 = (1/2)\{2g-2 + \sum_{i=1}^n (d_i - 1)/d_i\}^2 / (b - \sum_{i=1}^n e_i/d_i) .$$

## §3. Classification

Next we study the normal surface singularities such that  $\delta_m$  is either 0 or 1 for all  $m \geq 1$ .

Lemma 3.1. Let  $(a_{ij})$  be negative definite matrix of rank  $n$ . Assume  $a_{ij} \geq 0$  for  $i \neq j$  and  $a_{ii} < 0$  for all  $i$ . Consider  $n$  linear equations

$$b_j = \sum_{i=1}^n x_i a_{ij} \quad \text{for } j = 1, \dots, n$$

in  $n$  unknowns  $x_1, x_2, \dots, x_n$ . If  $b_j \geq 0$  for all  $j$ , then any  $x_i \leq 0$ .

Proof. The proof will be by induction on  $n$ . When  $n = 1$ , the equation is

$$b_1 = x_1 a_{11},$$

and hence the lemma trivially holds. Therefore assume that the lemma holds for  $n-1$ . By the negative definiteness of  $(a_{ij})$ , for any  $(x_j) \neq 0$ ,

$$\sum_{j=1}^n b_j x_j = \sum_{j=1}^n \sum_{i=1}^n x_i a_{ij} x_j < 0.$$

Therefore, if  $x_i > 0$  for all  $i$ , then  $b_{j_0} < 0$  for some  $j_0$ . This contradicts the assumption. Thus we can assume  $x_{i_0} \leq 0$  for some  $i_0$ . Consider  $n-1$  linear equations

$$b_j + (-x_{i_0}) a_{i_0 j} = \sum_{i \neq i_0} x_i a_{ij} \quad \text{for } j \neq i_0.$$

By the induction hypothesis,  $x_i \leq 0$  for  $i \neq i_0$ , thus proving the theorem.  $\square$

Lemma 3.2. In the above situation we assume moreover that for any  $i$  there exists  $j$  such that  $a_{ij} > 0$  in the case  $n \geq 2$ . If  $b_j \geq 0$  for all  $j$

and  $b_{j_0} > 0$  for some  $j_0$ , then any  $x_i < 0$ .

Proof. Quite similar to the above case. □

Let  $\pi : \tilde{X} \rightarrow X$  be the minimal resolution of the normal surface singularity  $(X, x)$ . The exceptional set  $A = \pi^{-1}(x)$  is decomposed into irreducible components ;  $A = \cup_{i=1}^n A_i$ .

Proposition 3.3. Let  $\sum_{i=1}^n k_i A_i$  be the cycle which is numerically equivalent to the canonical line bundle  $K$  of  $\tilde{X}$ . Then  $k_i \leq 0$  for  $i = 1, \dots, n$ . If, moreover,  $K$  is not numerically trivial (i.e.,  $(X, x)$  is not a rational double point), then  $k_i < 0$  for  $i = 1, \dots, n$ .

Proof. The virtual genus of  $A_j$  is

$$p(A_j) = (1/2)(A_j \cdot A_j + K \cdot A_j) + 1 .$$

Then we obtain  $n$  linear equations

$$-2 + 2p(A_j) - A_j \cdot A_j = \left( \sum_{i=1}^n k_i A_i \right) \cdot A_j, \quad 1 \leq j \leq n ,$$

in  $n$  unknowns  $k_1, k_2, \dots, k_n$ . Since  $\pi$  is the minimal resolution,

$$-2 + 2p(A_j) - A_j \cdot A_j \geq 0 \quad \text{for all } j .$$

Moreover the intersection matrix is negative definite. Hence the rest part of the proof is obvious by Lemma 3.1 and Lemma 3.2. □

Let  $A'$  be a connected proper analytic subset of  $A$ . Then  $A'$  is also exceptional ; and so there exists a strongly pseudoconvex neighborhood  $U'$  of  $A'$ . We may assume, without loss of generality, that  $A' = \cup_{i=1}^m A_i$ ,  $m < n$ .

Theorem 3.4. Let  $A$  be the exceptional set of the minimal resolution. Let  $\sum_{i=1}^n k_i A_i$  (resp.  $\sum_{i=1}^m k'_i A_i$ ) be the cycle which is numerically equivalent to the canonical line bundle  $K$  of  $\tilde{X}$  (resp.  $U'$ ). Suppose that  $K$  is not numerically trivial. Then  $k_i < k'_i$  for  $i = 1, \dots, m$ . If  $K$  is numerically trivial, then  $k'_i = 0$  for all  $i$ .

Proof. First suppose that  $K$  is not numerically trivial. Then we have two systems of linear equations. One is

$$-2 + 2p(A_j) - A_j \cdot A_j = \left( \sum_{i=1}^n k_i A_i \right) \cdot A_j \quad \text{for } j = 1, \dots, n, \quad (*)$$

and the other is

$$-2 + 2p(A_j) - A_j \cdot A_j = \left( \sum_{i=1}^m k'_i A_i \right) \cdot A_j \quad \text{for } j = 1, \dots, m. \quad (**)$$

From (\*) and (\*\*)

$$\left( \sum_{i=m+1}^n k_i A_i \right) \cdot A_j = \sum_{i=1}^m (k'_i - k_i) A_i \cdot A_j \quad \text{for } j = 1, \dots, m.$$

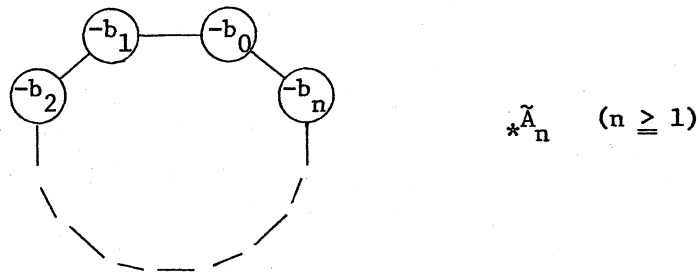
Since  $k_i < 0$  for all  $i$  by Proposition 3.3,  $A_i \cdot A_j \geq 0$  for all  $i \geq m+1 > j$ , and  $A_i \cdot A_j > 0$  for some  $i \geq m+1 > j$  by the connectivity of  $A$ , we have  $k'_i - k_i > 0$  for  $i = 1, \dots, m$ , by Lemma 3.2. The latter statement is clear. □

Proposition 3.5. Let  $(X, x)$  be a normal surface singularity with the minimal resolution  $\pi : \tilde{X} \rightarrow X$ . Let  $\pi^{-1}(x) = A$ . Denote by  $A_i$  ( $i = 1, \dots, n$ ) the irreducible component of  $A$ . Let  $\sum_{i=1}^n k_i A_i$  be the cycle which is numerically equivalent to the canonical line bundle of  $\tilde{X}$ . We have  $\delta_m = 0$  for all  $m \geq 1$  if and only if  $-1 < k_i \leq 0$  for all  $i$ .

Proof. Since  $p_g = \delta_1 = 0$ ,  $(X, x)$  is a rational singularity. Then  $hK$  is defined by an integral cycle for some  $h$ , say  $hK = \sum_{i=1}^n \mu_i A_i$ . Hence there

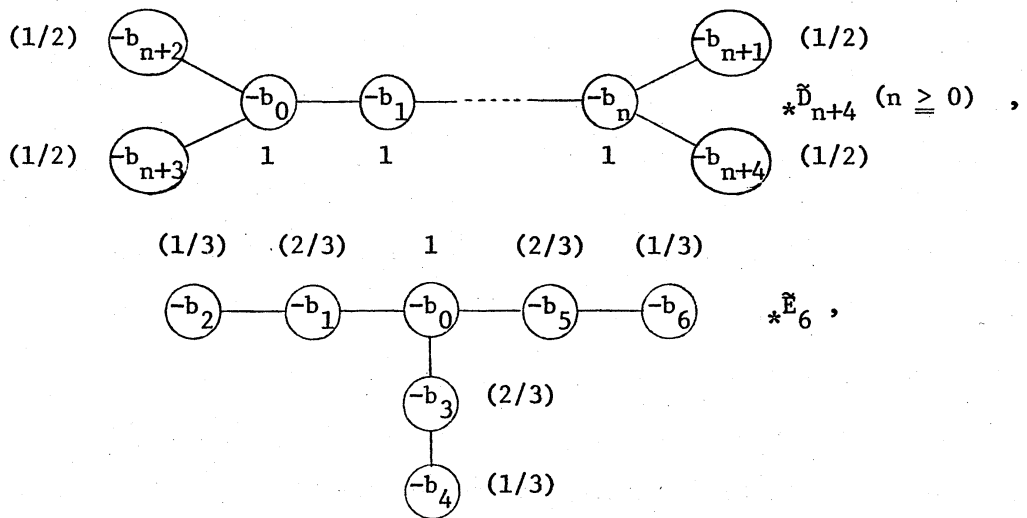
exists a meromorphic  $h$ -ple 2-forms  $\omega$  such that  $(\omega) = \sum_{i=1}^n \mu_i A_i$ .  $\omega$  is an element of  $\Gamma(U-A, \mathcal{O}(hK))$  and  $0 = \delta_h = \dim \Gamma(U-A, \mathcal{O}(hK)) / L^{2/h}(U-A)$ . Thus  $\omega$  is an element of  $L^{2/h}(U-A)$ .  $(X, x)$  is rational, and so  $A$  is an integral cycle of normal crossings. Hence  $\omega$  is the element of  $\Gamma(U, \mathcal{O}(hK + (h-1)A))$ , i.e.,  $-h < \mu_i$  for all  $i$ . By Proposition 3.3,  $k_i \leq 0$ . Therefore  $-1 < \mu_i/h = k_i \leq 0$ . □

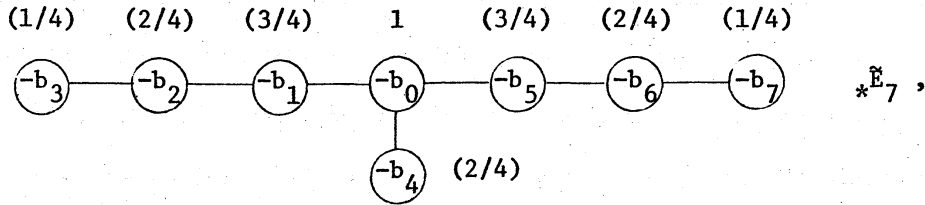
Example 3.6. Suppose  $G$  is of the form



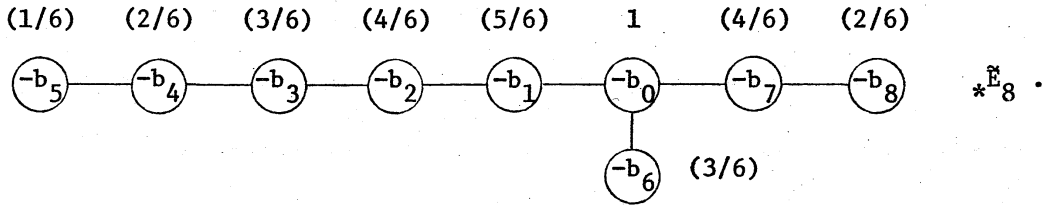
Then  $K \sim \sum_{i=0}^n -A_i$ . In fact  $K$  is linearly equivalent to  $\sum_{i=0}^n -A_i$ , i.e.,  $(X, x)$  is Gorenstein (cf. Theorem 2.15).

Proposition 3.7. Suppose  $G =$





or



Let  $\sum k_i A_i$  be the cycle which is numerically equivalent to the canonical line bundle. Then  $\sum k_i A_i \leq \sum -\lambda_i A_i$ , where  $\lambda_i$ 's are the numbers written by the side of each vertex.

Proof. Let  $Z$  be defined by  $Z = \sum \lambda_i A_i$ . Then  $Z \cdot A_j = \lambda_j (2 + A_j \cdot A_j)$  for any  $j$ . Since  $A_j$  is a non-singular rational curve,  $-2 - A_j \cdot A_j = K \cdot A_j$ , and hence

$$(\lambda_j - 1)(2 + A_j \cdot A_j) = (Z + K) \cdot A_j = \sum (\lambda_i + k_i) A_i \cdot A_j .$$

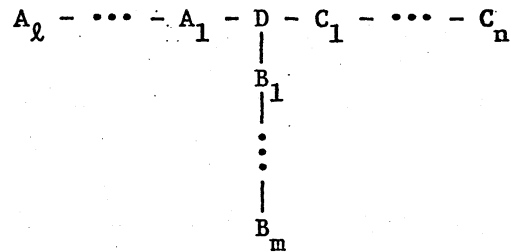
$\lambda_j \leq 1$  and  $A_j \cdot A_j \leq -2$  and so  $(\lambda_j - 1)(2 + A_j \cdot A_j) \geq 0$ . Thus it follows from Lemma 3.1 that  $\lambda_i + k_i \leq 0$ , i.e.,  $k_i \leq -\lambda_i$  for any  $i$ . □

Theorem 3.8. Suppose that  $(X, x)$  is a normal surface singularity and  $G$  is the weighted dual graph associated with the minimal resolution  $\pi : \tilde{X} \rightarrow X$ . If  $\delta_m = 0$  for all  $m \geq 1$ , then  $G$  is either chain-shaped or star-shaped with three branches.

Proof. Let  $A = \pi^{-1}(x) = \bigcup_{i=1}^n A_i$ . Since  $p_g = \delta_1 = 0$ ,  $A_i \cong \mathbb{P}^1$  and  $\pi$  is a good resolution. Let  $\sum_{i=1}^n k_i A_i$  be the cycle which is numerically equivalent to the canonical line bundle of  $\tilde{X}$ . Then by Proposition 3.5,

$-i < k_i \leq 0$  for all  $i$ . By Example 3.6 and Theorem 3.4,  $G$  can not contain  $*\tilde{A}_n$  as a subgraph. Hence  $G$  is tree-shaped. Similarly  $G$  does not contain  $*\tilde{D}_{n+4}$  as a subgraph. Thus there is at most one center in  $G$ , and so  $G$  is either chain-shaped or star-shaped with three branches. Suppose not, then  $G$  would contain  $*\tilde{D}_4$  as a subgraph. This is a contradiction.  $\square$

Let  $(X, x)$  be a normal surface singularity. Suppose that the weighted dual graph of the minimal resolution of  $\tilde{X}$  is star-shaped as follows :



where  $A_i, B_j, C_k$  and  $D$  denote non-singular rational curves. Let

$$E = dD + \sum_{i=1}^{\ell} \lambda_i A_i + \sum_{j=1}^m \mu_j B_j + \sum_{k=1}^n \nu_k C_k$$

be the cycle which is numerically equivalent to the canonical line bundle  $K$  of  $\tilde{X}$  :

$$(*) \quad \left\{ \begin{array}{ll}
 -2 \cdot D \cdot D = K \cdot D = E \cdot D & \\
 -2 \cdot A_i \cdot A_i = K \cdot A_i = E \cdot A_i & i = 1, \dots, \ell, \\
 -2 \cdot B_j \cdot B_j = K \cdot B_j = E \cdot B_j & j = 1, \dots, m, \\
 -2 \cdot C_k \cdot C_k = K \cdot C_k = E \cdot C_k & k = 1, \dots, n.
 \end{array} \right.$$

Let  $Z_A = \sum_{i=1}^{\ell} \alpha_i A_i$  be the cycle which is uniquely determined by the following equation :

$$(**) \quad \begin{cases} 1 + (-2 - A_1 \cdot A_1) = 1 + K \cdot A_1 = Z_A \cdot A_1 \\ -2 - A_i \cdot A_i = K \cdot A_i = Z_A \cdot A_i \quad 2 \leq i \leq \ell . \end{cases}$$

Then (\*\*) is equivalent to

$$\begin{cases} 0 = (Z_A + A) \cdot A_i \quad 1 \leq i \leq \ell - 1 , \\ -1 = (Z_A + A) \cdot A_\ell \end{cases}$$

where  $A = \sum_{i=1}^{\ell} A_i$ . So by Lemma 3.2,  $0 < \alpha_i + 1$ , i.e.,  $-1 < \alpha_i$ . (In the case where  $\ell = 1$ ,  $-1 = \alpha_1$ ). Moreover, (\*\*) is equivalent to

$$\begin{cases} 1 + d = (Z_A - E|_A) \cdot A_1 \\ 0 = (Z_A - E|_A) \cdot A_i \quad 2 \leq i \leq \ell . \end{cases}$$

By Lemma 3.2, hence

$$1 + d > 0 \implies \alpha_i - \lambda_i < 0$$

$$1 + d = 0 \implies \alpha_i - \lambda_i = 0$$

$$1 + d < 0 \implies \alpha_i - \lambda_i > 0 ,$$

i.e.,

$$d > -1 \implies \alpha_i < \lambda_i$$

$$d = -1 \implies \alpha_i = \lambda_i$$

$$d < -1 \implies \alpha_i > \lambda_i .$$

It is the same with  $Z_B$  and  $Z_C$ . So we obtain a cycle  $Z = -D + Z_A + Z_B + Z_C$ ,

and (\*) is equivalent to



$$\left\{ \begin{array}{l} -2 - \alpha_1 - \beta_1 - \gamma_1 = (E - Z) \cdot D \\ 0 = (E - Z) \cdot A_i \quad 1 \leq i \leq \ell \\ 0 = (E - Z) \cdot B_j \quad 1 \leq j \leq m \\ 0 = (E - Z) \cdot C_k \quad 1 \leq k \leq n \end{array} \right. .$$

Then, by Lemma 3.2,

$$-2 - \alpha_1 - \beta_1 - \gamma_1 < 0 \implies d > -1$$

$$-2 - \alpha_1 - \beta_1 - \gamma_1 = 0 \implies d = -1$$

$$-2 - \alpha_1 - \beta_1 - \gamma_1 > 0 \implies d < -1 .$$

We must still express  $\alpha_1$  in terms of the selfintersection number of the  $A_i$ ,  $1 \leq i \leq \ell$ . Let  $-a_i = A_i \cdot A_i$ , then (\*\*\*) is equivalent to

$$\left\{ \begin{array}{l} -1 + a_1 = -a_1 \alpha_1 + \alpha_2 \\ -2 + a_i = \alpha_{i-1} - a_i \alpha_i + \alpha_{i+1} \quad 2 \leq i \leq \ell-1, \\ -2 + a_\ell = \alpha_{\ell-1} - a_\ell \alpha_\ell . \end{array} \right.$$

Hence

$$\begin{aligned} (\alpha_2+1)/(\alpha_1+1) &= a_1, \quad (\alpha_3+1)/(\alpha_1+1) = a_1 a_2 - 1, \quad \dots, \quad (\alpha_{i+1}+1)/(\alpha_1+1) = \\ &a_i (\alpha_i+1)/(\alpha_1+1) - (\alpha_{i-1}+1)/(\alpha_1+1), \quad \dots, \quad 1/(\alpha_1+1) = a_\ell (\alpha_\ell+1)/(\alpha_1+1) - \\ &(\alpha_{\ell-1}+1)/(\alpha_1+1) . \end{aligned}$$

Let  $p_\ell = 1/(\alpha_1+1)$ . Moreover, an easy induction proof shows that

$$p_\ell = 1/(\alpha_1+1) \geq \ell+1 .$$

Then the first few  $p_\ell$ 's are

$$p_1 = a_1 \geq 2 ,$$

$$p_2 = a_1 a_2 - 1 \geq 3 ,$$

$$p_3 = a_1 a_2 a_3 - a_1 - a_3 \geq 4 ,$$

$$p_4 = a_1 a_2 a_3 a_4 - a_1 a_4 - a_3 a_4 - a_1 a_2 + 1 \geq 5 ,$$

$$p_5 = a_1 a_2 a_3 a_4 a_5 - a_1 a_4 a_5 - a_3 a_4 a_5 - a_1 a_2 a_5 \\ - a_1 a_2 a_3 + a_1 + a_3 + a_5 \geq 6 .$$

Remark. Setting  $p_0 = 1$ , then  $p_\ell/p_{\ell-1}$  has the continued fraction expansion :

$$p_\ell/p_{\ell-1} = [a_\ell, a_{\ell-1}, \dots, a_1] .$$

Thus, letting  $q_m = 1/(\beta_1+1)$  and  $r_n = 1/(\gamma_1+1)$ , we obtain the following theorem.

Theorem 3.8.B. Let  $(X, x)$  be a normal surface singularity. Let  $G$  be the weighted graph which is associated with the minimal resolution. Suppose that  $\delta_m = 0$  for all  $m \geq 1$  and  $G$  has three branches, then  $G$  is of the form

$$\begin{array}{ccccccc} A_\ell & - & \cdots & - & A_1 & - & D & - & C_1 & - & \cdots & - & C_n \\ & & & & & & | & & & & & & \\ & & & & & & B_1 & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & \vdots & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & B_m & & & & & & \end{array}$$

such that

$$(*) \quad 1 < 1/p_\ell + 1/q_m + 1/r_n .$$

The systems of positive integers  $p_\ell$ ,  $q_m$ ,  $r_n$  which satisfy the condition (\*) of Theorem 3.8.B are the following four types :

$$(2,2,n), n \geq 2, (2,3,3), (2,3,4), (2,3,5) .$$

Hence from the result of Brieskorn [3], these singularities are quotient singularities. Thus we have the following.

Theorem 3.9. Let  $(X,x)$  be a normal surface singularity. Then  $(X,x)$  is a quotient singularity if and only if  $\delta_m(X,x) = 0$  for all  $m \geq 1$  (see [25]).

Theorem 3.10. Let  $(X,x)$  be a normal surface rational singularity with the minimal resolution  $\pi : \tilde{X} \rightarrow X$ . Let  $\pi^{-1}(x) = A$ . Denote by  $A_i$  ( $i = 1, \dots, n$ ) the irreducible component of  $A$ . Let  $\sum_{i=1}^n k_i A_i$  be the cycle which is numerically equivalent to the canonical line bundle of  $\tilde{X}$ . Suppose that  $0 \leq \delta_m \leq 1$  for all  $m \geq 1$ . Then  $-1 \leq k_i \leq 0$  for  $i = 1, \dots, n$ .

Proof. Suppose not, then there would exist  $k_{i_0}$  with  $k_{i_0} < -1$ . Take a holomorphic function  $f$  on  $U$  which vanishes on  $A_{i_0}$ , and let  $\alpha_{i_0}$  be its order. Then there is a positive integer  $m$  such that  $-m - mk_{i_0} > \alpha_{i_0}$  and such that all  $mk_i$  are integers.  $(X,x)$  is rational, and so  $mK$  is linearly equivalent to the integral cycle  $\sum mk_i A_i$ ; there exists  $\omega \in \Gamma(U-A, \mathcal{O}(mK))$  so that  $(\omega) = \sum mk_i A_i$ . Since  $\alpha_{i_0} + mk_{i_0} + (m-1) < -1$ , it follows that

$$(f\omega) + (m-1)A \not\leq 0,$$

and so  $\omega$  and  $f\omega$  are not  $L^{2/m}$ -integrable. Thus  $\delta_m \geq 2$ , a contradiction.  $\square$

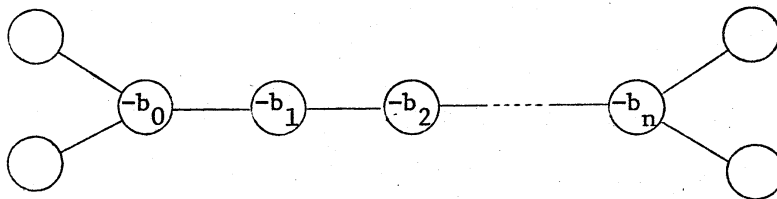
By Theorem 3.4 and 3.10, we have

Corollary. Let  $(X, x)$  be a normal surface rational singularity with the minimal resolution  $\pi : \tilde{X} \rightarrow X$ . Let  $A = \pi^{-1}(x)$  be the exceptional set. If  $\delta_m(X, x)$  is either 0 or 1 for  $m \geq 1$ , then every connected proper subvariety of  $A$  is the exceptional set for a quotient singularity.

Lemma 3.11. Let  $(X, x)$  be a normal surface rational singularity. Let  $G$  be the weighted graph associated with the minimal resolution of  $(X, x)$ . Then  $G$  does not contain  ${}^* \tilde{A}_n$  as a subgraph.

Proof. In general  $p_a(X, x) \leq p_g(X, x)$  and  $p(Z) \leq p_a(X, x)$  for a positive integral cycle  $Z$ . Since  $p({}^* \tilde{A}_n) = 1$ ,  ${}^* \tilde{A}_n$  can not be a subgraph of  $G$ . □

Proposition 3.12. Let  $(X, x)$  be a normal surface rational singularity. Let  $G$  be the weighted graph which is associated with the minimal resolution. Suppose that  $0 \leq \delta_m \leq 1$  and  $G$  has at least two centers. Then  $G$  is of the form



Proof. From Lemma 3.11  $G$  is tree-shaped and so  $G$  contains  ${}^* \tilde{D}_{n+4}$  as a subgraph. Let  $\sum \kappa_i A_i$  be the cycle which is numerically equivalent to the canonical line bundle of the minimal resolution  $\tilde{X}$  of  $X$ . Let  $\sum_{i=0}^{n+4} k_i A_i$  and  $\{ \lambda_i \}$  be as in Proposition 3.7. Then by Theorem 3.4,  $-1 \leq \kappa_i \leq k_i \leq -\lambda_i$ . If  ${}^* \tilde{D}_{n+4}$  is a proper subgraph of  $G$ , then  $\kappa_i < k_i$  by Lemma 3.2. This

contradicts the fact that a certain  $k_i$  is equal to  $-1$ , and hence  $G = *D_{n+4}$ .  
 Moreover if  $\lambda_i = 1$ , then  $\kappa_i = k_i = -1$ . Let  $Z$  be defined by  $Z = \sum_{i=0}^n A_i$ .  
 Since  $Z$  is a reduced cycle,  $p(Z) \geq 0$ . Hence  $0 = p_a(X, x) \geq p(Z) \geq 0$ , i.e.,  
 $0 = p(Z) = (1/2)(Z \cdot Z + K \cdot Z) + 1$ . Therefore

$$-2 = Z \cdot Z + (-Z + \sum_{i=1}^4 k_{n+i} A_{n+i}) \cdot Z$$

and

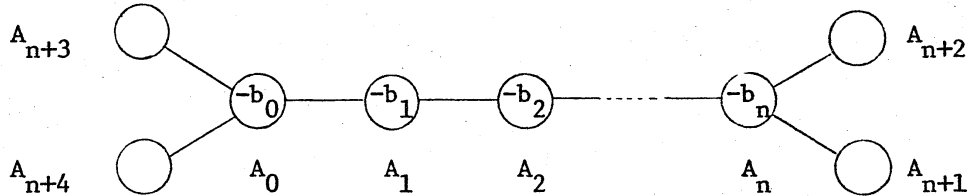
$$-2 = \sum_{i=1}^4 k_{n+i}$$

$k_{n+i} \leq -(1/2)$  for  $i = 1, 2, 3$  and  $4$ , then  $k_{n+i} = -(1/2)$  for  $i = 1, 2, 3$  and  $4$ . Thus, for  $i = 1, 2, 3$  and  $4$

$$0 = p(A_{n+i}) = (1/2)(A_{n+i} \cdot A_{n+i} + K \cdot A_{n+i}) + 1$$

and  $-2 = A_{n+i} \cdot A_{n+i}$ ; the result follows. □

Finally we show that  $\delta_m$  of  $G =$



is either 0 or 1 for all  $m \geq 1$ . Consider  $A_0$  as the center of  $G$ . Then we get the analogous formula as in Theorem 2.21.

Theorem 3.13. Let  $P_0, P_1$  and  $P_2$  be the intersection points on  $A_0$  which is defined respectively by  $P_0 = A_0 \cap A_1, P_1 = A_0 \cap A_{n+3}$  and  $P_2 = A_0 \cap A_{n+4}$ . Set  $d/e = [b_1, b_2, \dots, b_{n-1}]$  with  $1 \leq e \leq d$ , and  $e$  and  $d$  relatively prime. Let

$$\sum_{k \geq 0} \dim \Gamma(A_0, \mathcal{O}_{A_0}(mK_{A_0} + kN + [(k+m)d]/d]P_0 + [(k+m)/2]P_1 + [(k+m)/2]P_2))$$

be denoted by  $\Delta_m$ . Then  $\delta_m \leq \Delta_m$ .

Proof. The same argument as in Theorem 2.21 works in the almost part of the proof. Following Theorem 2.21 we define a homomorphism

$$\phi : \Gamma(U-A, \mathcal{O}(mK)) \rightarrow \bigoplus_{k \geq 0} \Gamma(A_0, \mathcal{O}_{A_0}(mK_{A_0} + kN + [(k+m)d]/d]P_0 + [(k+m)/2]P_1 + [(k+m)/2]P_2))$$

Let  $U'$  be a strongly pseudoconvex neighborhood of  $\bigcup_{i=1}^{n+2} A_i$ . Set  $A'_0 = A_0 \cap U'$ . Suppose  $\omega \in \Gamma(U-A, \mathcal{O}(mK))$ . Then  $(\omega|_{U'}) = (-\mu_0)A'_0 + \sum_{i=1}^{n+2} (-\mu_i)A_i + B$  where  $B$  is effective and does not involve any of the  $A_i$ ,  $i = 1, 2, \dots, n+2$ .

Let  $Z = (-\mu_0)A'_0 + \sum_{i=1}^{n+2} (-k_i)A_i$  be the divisor which is numerically equivalent to  $mK_{U'}$ , i.e.,  $mK_{U'} \cdot A_i = Z \cdot A_i$  for  $i = 1, 2, \dots, n+2$ . By easy computation  $k_1 = \{(\mu_0 - m)e + dm\}/d$ .  $Z \cdot A_j = mK_{U'} \cdot A_j = (\omega|_{U'}) \cdot A_j$ , and so

$$\sum_{i=1}^{n+2} (\mu_i - k_i)A_i \cdot A_j = B \cdot A_j .$$

Since  $B \cdot A_j \geq 0$ ,  $\mu_i - k_i \leq 0$  by Lemma 3.1. In particular

$$\mu_1 \leq k_1 = \{(\mu_0 - m)e + dm\}/d .$$

Hence  $\phi$  is well-defined. If  $\mu_0 < m$ , then  $\mu_1 < m$ , and by induction  $\mu_i < m$  for  $i = 2, 3, \dots, n+2$ . Therefore the kernel of  $\phi$  is

$$\Gamma(U, \mathcal{O}(mK + (m-1)A)) \cong L^{2/m}(U-A) .$$

Thus the proof is complete. □

Lemma 3.14. If we are as above,

$$\Delta_m = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases}$$

Proof. We compute the degree of the line bundle ;

$$\begin{aligned} & \text{degree}(mK_{A_0} + kN + [(ke+md)/d]P_0 + [(k+m)/2]P_1 + [(k+m)/2]P_2) \\ &= -2m - b_0k + [(ke+md)/d] + [(k+e)/2] + [(k+e)/2] \\ &\leq -2m - b_0k + (ke+md)/d + (k+e)/2 + (k+e)/2 = k(-b_0 + 1 + e/d) . \end{aligned}$$

By the definition of  $e/d$ ,  $e/d \leq 1$  and  $e/d = 1$  if and only if  $b_i = 2$  for  $i = 1, 2, \dots, n$ . Then  $-b_0 + 1 + e/d \leq 0$ . If  $-b_0 + 1 + e/d = 0$ ,  $b_i = 2$  for all  $i$ , and so  $G$  is not negative definite, a contradiction. Hence degree = 0 if and only if  $k = 0$  and  $-2m + [m/2] + [m/2] + m = 0$ , i.e.,  $k = 0$  and  $m$  is even. Thus our result follows from the Riemann-Roch theorem.  $\square$

Let  $Z$  be defined by

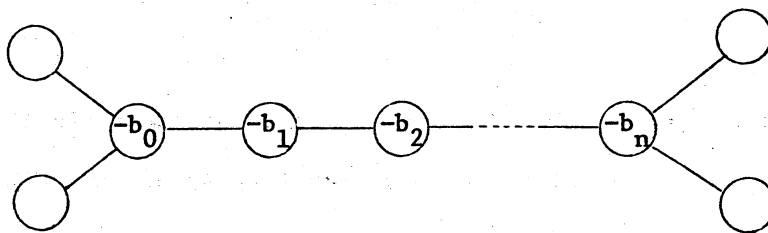
$$Z = 2(A_0 + A_1 + \dots + A_n) + A_{n+1} + A_{n+2} + A_{n+3} + A_{n+4} .$$

Then  $-Z$  is numerically equivalent to  $2K$ . By the above lemma the singularity with graph  $G$  is rational, and so  $-Z$  is linearly equivalent to  $2K$ . Hence  $\delta_{2m} \geq 1$ . Then it follows from Theorem 3.13 and Lemma 3.14 that

$$\delta_m = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases}$$

Thus we obtain the following.

Proposition 3.15. Suppose  $G$  is of the form

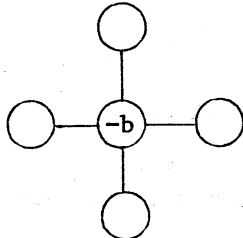


Then

$$\delta_m = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases}$$

Similarly for  $n = 0$  we have

**Proposition 3.15.B.** Let  $(X, x)$  be a normal surface rational singularity. Let  $G$  be the weighted graph which is associated with the minimal resolution. Suppose  $0 \leq \delta_m \leq 1$  ( $\delta_{m_0} = 1$  for some  $m_0 \geq 2$ ) and  $G$  is star-shaped with at least four branches. Then  $G$  is of the form



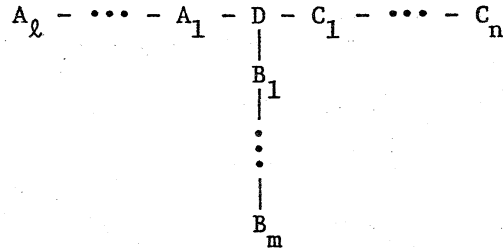
In fact  $\delta_m$  of the singularity with the above graph is

$$\delta_m = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases}$$

Moreover, from Theorem 3.10 and the proof of Theorem 3.8.B we obtain the following.



Proposition 3.15.C. Let  $(X,x)$  be a normal surface rational singularity. Let  $G$  be the weighted graph which is associated with the minimal resolution. Suppose that  $0 \leq \delta_m \leq 1$  ( $\delta_{m_0} = 1$  for some  $m_0 \geq 2$ ) and  $G$  is star-shaped with three branches. Then  $G$  is of the form



such that

$$(*) \quad 1 = 1/p_\ell + 1/q_m + 1/r_n .$$

The possible solutions of  $(*)$  are easily enumerated. They are depicted as follows :

$$(p_\ell, q_m, r_n) \in \{ (2,3,6), (2,4,4), (3,3,3) \} .$$

Hence it follows from Theorem 2.21, Theorem 2.24 and Theorem 3.9 that  $0 \leq \delta_m \leq 1$  for the singularities with the condition  $(*)$ .

Next we recall a few results about minimally elliptic singularities, which was examined by Laufer [15]. Karras [9] and Saito [18] have studied some of particular elliptic singularities.

A normal surface singularity  $(X,x)$  is called a simple elliptic singularity if the exceptional set of the minimal resolution consists of a single non-singular elliptic curve  $A$ .  $(X,x)$  up to analytic isomorphism is uniquely determined by the analytic structure of the curve  $A$ ,  $j(A) =$

$g_2^3/(g_2^3 - 27g_3^2)$ , where  $w^2 = 4z^3 - g_2z - g_3$  is the equation of  $A$  in  $\mathbb{C}^2$ , see [5] and [18].

Cusp singularities are characterized as follows. Let  $(X, x)$  be a normal surface singularity and  $\pi : \tilde{X} \rightarrow X$  be the minimal resolution of  $(X, x)$ . Let  $A = \pi^{-1}(x)$  be the exceptional set. Then  $(X, x)$  is a cusp singularity if and only if  $A$  is an irreducible rational curve with a node singularity or  $A$  is a "cycle" of non-singular rational curve  $A_i$ . The configuration is illustrated in Example 3.6. Moreover, the associated cycle

$$\{ (-b_0, -b_1, \dots, -b_n) \}$$

of selfintersection numbers determines the singularity  $(X, x)$  up to complex-analytic equivalence (see [8, 9, 14]).

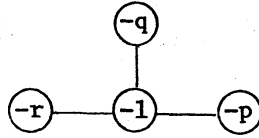
Then by Example 1.9 and Theorem 1.16 we have

**Theorem 3.16.** Let  $(X, x)$  be a simple elliptic singularity or a cusp singularity. Then  $\delta_m(X, x) = 1$  for all  $m \geq 1$ .

**Definition 3.17.** Let  $(X, x)$  be a normal surface singularity.  $(X, x)$  is purely elliptic if  $\delta_m(X, x) = 1$  for  $m \geq 1$ .

In the following we shall review the resolutions of minimally elliptic singularities and with a few exceptions classify those graphs which can arise from the purely elliptic singularities.

**Lemma 3.18.** Let  $(X, x)$  be a normal surface singularity. If one of the good resolutions of  $(X, x)$  has the following weighted dual graph :



then  $\limsup_{m \rightarrow \infty} \delta_m / m^2 > 0$ .

Proof. Since  $(X, x)$  is minimally elliptic (see Theorem 3.19, (3)-(5)), there is some neighborhood  $V$  of  $x$  in  $X$  and a holomorphic 2-form  $\omega$  on  $V - \{x\}$  such that  $\omega$  has no zeros on  $V - \{x\}$ . Let  $\pi : \tilde{X} \rightarrow X$  be the minimal good resolution. Then the weighted dual graph of  $\tilde{X}$  is of the form mentioned above. Let  $(\pi^*\omega)$  be the divisor of  $\pi^*\omega$ . An easy computation shows that  $-(\pi^*\omega)$  is not reduced. In fact the multiplicity of  $-(\pi^*\omega)$  at the central curve is equal to two. Let  $\rho : X \rightarrow \Delta \subset \mathbb{C}^2$  be an admissible representation and let  $f = \rho^*z$  and  $g = \rho^*\omega$ , where  $(z, \omega)$  is a coordinate system for  $\Delta$ . Denote by  $a$  (resp.  $b$ ) the order of zeros of  $\pi^*f$  (resp.  $\pi^*g$ ) at the central curve. If  $m \geq \lambda a + \mu b$ , then

$$(\pi^*(f^\lambda g^\mu \omega^m)) + (m-1)A \not\leq 0,$$

i.e.,  $f^\lambda g^\mu \omega^m$  is not  $L^{2/m}$ -integrable. Hence

$$\delta_m \geq \#\{(\lambda, \mu) \in \mathbb{N}^2 \mid m \geq \lambda a + \mu b\}.$$

Thus  $\limsup_{m \rightarrow \infty} \delta_m / m^2 > 0$ . □

About the resolutions of the minimally elliptic singularities the following fact was proved by Laufer [15].

Theorem 3.19 (Laufer [15]). Let  $\pi : \tilde{X} \rightarrow X$  be the minimal resolution

for a minimally elliptic singularity  $(X, x)$ . Let  $\pi' : \tilde{X} \rightarrow X$  be the minimal reresolution such that  $A'_i$  are non-singular and have normal crossings, i.e., the  $A'_i$  meet transversely and no three meet at a point. Then  $\pi = \pi'$  and all the  $A_i$  are rational curves except for the following cases :

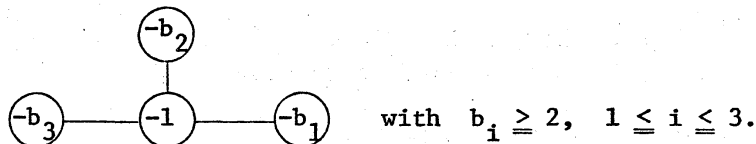
- (1)  $A$  is an elliptic curve.  $\pi = \pi'$ .
- (2)  $A$  is a rational curve with a node singularity.
- (3)  $A$  is a rational curve with a cusp singularity.
- (4)  $A$  is two non-singular rational curves which have first order tangential contact at one point.

(5)  $A$  is three non-singular rational curves all meeting transversely at the same point.

In case (2),  $\pi'$  has the following weighted dual graph :



In case (3) - (5),  $\pi'$  has the following weighted dual graph :



Remark. In case (1),  $(X, x)$  is a simple elliptic singularity. In case (2),  $(X, x)$  is one of the simplest cusp singularity.

Theorem 3.20. If a purely elliptic singularity  $(X, x)$  is Gorenstein, then  $(X, x)$  is a simple elliptic singularity or a cusp singularity.

Proof. Let  $\pi : \tilde{X} \rightarrow X$  be the minimal resolution of the singularity. Denote  $\pi^{-1}(x)$  by  $A$ . Let  $A = \cup A_i, 1 \leq i \leq n$ , be the decomposition of the exceptional set  $A$  into irreducible components. We assume that  $U$  is a

strongly pseudoconvex neighborhood of  $A$ . Let  $K$  be the canonical line bundle of  $U$ . Since  $p_g = \delta_1 = 1$  and  $(X, x)$  is Gorenstein, there exists  $\omega \in \Gamma(U-A, \mathcal{O}(K))$  such that

$$K = (\omega) = \sum -\lambda_i A_i$$

with  $\lambda_i \geq 1$ , that is,  $K$  is defined by an integral cycle. By Corollary 2.12  $(X, x)$  is a minimally elliptic singularity. Lemma 3.18 implies that  $(X, x)$  is none of (3), (4) and (5) of Theorem 3.19. In case (1),  $(X, x)$  is a simple elliptic singularity. In case (2),  $(X, x)$  is one of the simplest cusp singularity. Thus we may assume that  $\pi$  is the minimal good resolution and any  $A_i$  is a non-singular rational curve. For any holomorphic function  $f$  which vanishes at  $x$

$$(\pi^*f) + m(\omega) + (m-1)A \geq 0$$

as  $\delta_m = 1$  for any  $m \geq 1$ . Let  $\alpha_i$  be the order of zeros of  $f$  at  $A_i$ . Then  $\alpha_i \geq 1$  and

$$\alpha_i + m(-\lambda_i) + (m-1) \geq 0 \quad \text{for } m \geq 1.$$

Hence  $\lambda_i = 1$  and so  $K = -A$ . Since  $p(A_j) = 0$ ,

$$0 = (1/2)\{A_j \cdot A_j + (\sum -A_i) \cdot A_j\} + 1.$$

Then  $2 = (\sum_{i \neq j} A_i) \cdot A_j$ . This implies that  $A_j$  meets two other irreducible components of  $A$ . Thus  $(X, x)$  is a cusp singularity.  $\square$

Let  $\pi : \tilde{X} \rightarrow X$  be the minimal resolution for a purely elliptic singularity  $(X, x)$ . Let  $A = \pi^{-1}(x)$ . Suppose that  $A'$  is a connected proper analytic subvariety of  $A$ . Then the singularity  $(X', x')$  obtained by blowing

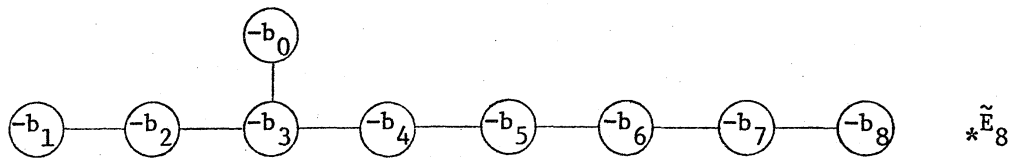
down  $A'$  is also a purely elliptic if  $p(A') = 1$ . For  $\delta_1(X', x') = p_g(X', x')$   
 $\geq p(A') = 1$  and  $1 = \delta_m(X, x) \geq \delta_m(X', x') \geq 1$  by the second fundamental theorem  
 If any connected proper subvariety of  $A$  is the exceptional set of a rational  
 singularity, then  $(X, x)$  is minimally elliptic. Therefore  $(X, x)$  is Gorenstein  
 and so  $(X, x)$  is a simple elliptic singularity or a cusp singularity by  
 Theorem 3.20.

Now suppose that there exists a connected proper subvariety  $A'$  of  $A$   
 such that  $p(A') = 1$ . Let  $A_0$  be the minimal one. Such a cycle always exists  
 by [15, Proposition 3.2, p.1261]. Since  $A_0$  has the minimality,  $A_0$  is the  
 exceptional set of the minimal resolution for a simple elliptic singularity  
 or a cusp singularity. Hence, applying the second fundamental theorem we  
 get the followings.

Theorem 3.21. Suppose that  $(X, x)$  is a purely elliptic,  $\pi : \tilde{X} \rightarrow X$  is  
 a minimal resolution of the singularity,  $A = \pi^{-1}(x)$ . If  $(X, x)$  is not  
 Gorenstein and there exists a connected proper analytic subvariety  $A_0$  of  $A$   
 such that  $A_0$  is the exceptional set of a cusp singularity, then any  
 connected proper analytic subvariety of  $A$ , not containing  $A_0$ , is the  
 exceptional set of a quotient singularity.

Corollary 3.22. In the above situation the number of the irreducible  
 components of  $A_0$  is at most eight.

Proof. Suppose not, then  $A$  would contain



as a proper connected subvariety of  $A$ . Let  $(X', x')$  be the singularity obtained by blowing down  ${}^* \tilde{E}_8$ , which is not a quotient singularity. Then  $\delta_{m_0}(X', x') \geq 1$  for some  $m_0 \geq 1$  by Theorem 3.9. Hence the second fundamental theorem says that  $\delta_{m_0}(X, x) \geq 2$ , a contradiction.  $\square$

## §4. Appendix

In this section we prove that Theorem 2.1 is generalized to the case of arbitrary dimensions  $n \geq 2$ , which was suggested to us by the referee. Then the first fundamental theorem of  $\{\delta_m\}$  holds in the case of arbitrary dimensions  $n \geq 2$ .

Theorem 4.1. There is a positive constant  $c$  such that  $\gamma_m \leq cm^n$  for an  $n$ -dimensional normal isolated singularity  $(X^n, x)$ .

Proof. By a theorem of M. Artin [30],  $(X^n, x)$  can be realized as a Zariski open subset  $\underline{U}$  of a projective variety  $\underline{V}$  with  $x \in \underline{U}$  as its only singularity. Let  $\pi : U \rightarrow \underline{U}$  be a resolution of the singular point. Then, in a natural manner, we get a desingularization  $p : V \rightarrow \underline{V}$  of  $\underline{V}$  by letting  $V$  to be  $(\underline{V} - \{x\}) \cup U$ . Let  $A = \pi^{-1}(x) = p^{-1}(x)$  and consider local cohomologies on  $V$  and  $U$  with the support  $A$ . Since

$$\Gamma(U-A, \mathcal{O}(mK_U)) / \Gamma(U, \mathcal{O}(mK_U)) \hookrightarrow H_A^1(U, \mathcal{O}(mK_U)) \simeq H_A^1(V, \mathcal{O}(mK_V)),$$

it suffices to show that  $h_A^1(V, \mathcal{O}(mK_V)) \leq am^n$  for some  $a > 0$ . By the exact sequence

$$0 \rightarrow H^0(V, \mathcal{O}(mK_V)) \rightarrow H^0(V-A, \mathcal{O}(mK_V)) \rightarrow H_A^1(V, \mathcal{O}(mK_V)) \rightarrow H^1(V, \mathcal{O}(mK_V)) \rightarrow \dots,$$

we have

$$h_A^1(V, \mathcal{O}(mK_V)) \leq h^0(V-A, \mathcal{O}(mK_V)) + h^1(V, \mathcal{O}(mK_V)).$$

From the compactness of  $V$ ,  $h^1(V, \mathcal{O}(mK_V)) \leq a_1 m^n$  follows for some  $a_1 > 0$ . Since  $V-A$  is strongly pseudoconcave, we have  $h^0(V-A, \mathcal{O}(mK_V)) < \infty$  by a theorem of Andreotti-Grauert [31]. Hence it remains to prove that



$h^0(V-A, \mathcal{O}(mK_V)) \leq a_2 m^n$  holds for some  $a_2 > 0$ . Let  $H_1$  be a very ample line bundle on  $V$  such that  $H = K_V + H_1$  is also very ample. Since

$$H^0(V-A, \mathcal{O}(mK_V)) \hookrightarrow H^0(V-A, \mathcal{O}(mH)) ,$$

it is enough to show that  $h^0(V-A, \mathcal{O}(mH)) \leq a_3 m^n$  ( $a_3 > 0$ ) holds for any very ample line bundle  $H$ . We shall prove this by the induction on  $n$ .

Suppose that  $n = 2$ . Let  $m_0 = |\det(A_i \cdot A_j)|$ , where  $(A_i \cdot A_j)$  is the intersection matrix of the exceptional set  $A = \cup A_j$ . Let  $H_1$  be another ample line bundle on  $V$  and  $H' = m_0 H + m_0 H_1$ . We can choose  $H_1$  so that  $(m_0 - 1)H + m_0 H_1$  has a global non-zero section, and moreover, the restriction  $H'_U$  of  $H'$  to the open subset  $U$  satisfies  $H'_U \cdot A_j \geq K \cdot A_j$  for any curve  $A_j$  of  $A$ . Then, by the exact sequence

$$0 \rightarrow H^0(U, \mathcal{O}(mH'_U)) \rightarrow H^0(U-A, \mathcal{O}(mH'_U)) \rightarrow H^1_A(U, \mathcal{O}(mH'_U)) \rightarrow H^1(U, \mathcal{O}(mH'_U)) \rightarrow \dots ,$$

and [10, Vanishing Theorem], we have

$$h^1_A(U, \mathcal{O}(mH'_U)) = \dim \Gamma(U-A, \mathcal{O}(mH'_U)) / \Gamma(U, \mathcal{O}(mH'_U)) .$$

Note that  $mH'_U = mm_0(H + H_1)_U$  is numerically equivalent to an integral divisor for any  $m \geq 1$ . Therefore from [10, Theorem 2], it follows that

$$\dim \Gamma(U-A, \mathcal{O}(mH'_U)) / \Gamma(U, \mathcal{O}(mH'_U)) \leq (1/2)(mK_U \cdot H'_U - m^2 H'_U \cdot H'_U) - m_0 K_U \cdot K_U .$$

Hence we get  $h^1_A(U, \mathcal{O}(mH'_U)) \leq a_3 m^2$  ( $a_3 > 0$ ). It is clear that  $h^0(V, \mathcal{O}(mH')) \leq a_4 m^2$  ( $a_4 > 0$ ), since  $V$  is compact. Thus

$$\begin{aligned} h^0(V-A, \mathcal{O}(mH)) &\leq h^0(V-A, \mathcal{O}(mH + m((m_0 - 1)H + m_0 H_1))) \\ &= h^0(V-A, \mathcal{O}(mH')) \leq h^0(V, \mathcal{O}(mH')) + h^1_A(V, \mathcal{O}(mH')) \\ &= h^0(V, \mathcal{O}(mH')) + h^1_A(U, \mathcal{O}(mH'_U)) \leq (a_3 + a_4)m^2 . \end{aligned}$$

Thus the case  $n = 2$  is proved. Next suppose that  $n \geq 3$ . Let  $V_1 = (s)$  be the non-singular divisor for a general element  $s \in H^0(V, \mathcal{O}(H))$ . We can assume that  $A_1 = V_1 \cap A \neq \emptyset$ . On  $V-A$ , consider the exact sequence

$$0 \rightarrow \mathcal{O}((k-1)H) \rightarrow \mathcal{O}(kH) \rightarrow \mathcal{O}_{V_1-A_1}(kH) \rightarrow 0$$

for  $k = 1, \dots, m$ . Then summing up the inequalities

$$h^0(V-A, \mathcal{O}(kH)) \leq h^0(V-A, \mathcal{O}((k-1)H)) + h^0(V_1-A_1, \mathcal{O}_{V_1-A_1}(kH))$$

for  $k = 1, \dots, m$ , we have

$$h^0(V-A, \mathcal{O}(mH)) \leq h^0(V-A, \mathcal{O}) + \sum_{k=1}^m h^0(V_1-A_1, \mathcal{O}_{V_1-A_1}(kH)).$$

Since  $V_1-A_1$  is strongly pseudoconcave,  $h^0(V_1-A_1, \mathcal{O}_{V_1-A_1}(kH)) \leq \alpha_5 k^{n-1}$  holds by the induction assumption. Hence

$$h^0(V-A, \mathcal{O}(mH)) \leq h^0(V-A, \mathcal{O}) + \alpha_5 \sum_{k=1}^m k^{n-1} \leq \alpha_6 m^n$$

holds for some  $\alpha_6 > 0$ . □

Thus we obtain the first fundamental theorem of  $\{\delta_m\}$ :

**Theorem 4.2.** For any  $n$ -dimensional normal isolated singularity, We have

$$\delta = \limsup_{m \rightarrow \infty} \delta_m / m^n < \infty.$$

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