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# On the Rational $K(\pi,1)$ - properties of Open Algebraic Varieties

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## § 1. Introduction

In this note we shall study the rational $K(\pi,1)$ - properties of a complement of a divisor. We shall say that a simplicial complex $X$ is rational $K(\pi,1)$ if its minimal algebra is generated by the elements of degree $\leq 1$.

By using the spectral sequence of Morgan [11], we give the explicit form of the minimal algebra of $P^2$ minus curves in § 3. The main theorem in this note will be the following.

Theorem : Let $X$ be $P^n$ minus a hypersurface $D$. Then the 1-minimal model of $X$, $\mathcal{M}_X(1)$, is formal. (i.e. there exists a quasi isomorphism $\psi : \mathcal{M}_X(1) \longrightarrow H^*(\mathcal{M}_X(1))$

Applying this theorem and results in § 3, we have an algorithm to study the gap between the 1-minimal model $\mathcal{M}_X(1)$ and the minimal model $\mathcal{M}_X$, which is closely related to the higher homotopy groups.

## 2. Preliminaries

In this section we review an outline of Sullivan's De Rham homotopy theory. For details, see [6], [8] and [13].

We denote by $\Lambda_n(V)$ the free algebra on a vector space V whose elements are of degree n . Then $\Lambda_n(V)$ is the polynomial algebra generated by V if n is even, and is the exterior algebra if n is odd.

**Definition (2.1).** By a Hirsch extension of a differential graded algebra ( d.g.a. ) A, we mean an inclusion $A \hookrightarrow B$ of d.g.a. such that B is isomorphic to $A \otimes \Lambda_k(V)$ and the differential of B sends $V \longrightarrow A_{k+1}$ , where $A_{k+1}$ is the degree k+1 part of A.

**Definition (2.2).** A d.g.a. M is a minimal algebra if :

a) M is connected. i.e. $M_0$ = ground field.

b) There is an increasing filtration :

$$\text{ground field} = M_0 \subset M_1 \subset M_2 \subset \ldots\ldots$$

such that $M_j$ is a subalgebra of M, $M_j \subset M_{j+1}$ is a Hirsch extention for each j, and $\bigcup_j M_j = M$.

c) The differential of M, d, is decomposable, i.e. $d : I(M) \longrightarrow I(M)$ is zero, where I(M) is indecomposable elements of M.

Definition (2.3). Let A be a differential algebra. An i- minimal model of A is a map $\rho : \mathcal{M} \longrightarrow A$ of d.g.a. such that :

a) $\mathcal{M}$ is a minimal algebra.

b $I(\mathcal{M}) = 0$ in degree $\geq i+1$.

c) $\rho^* : H(\mathcal{M}) \longrightarrow H(A)$ is an isomorphism in degree $\leq i$ and injective in degree $= i+1$.

By the theorem of Sullivan [8] an i- minimal model exists and is unique up to isomorphism.

Definition (2.4). Let K be a simplicial complex. The Q- polynomial forms of K, $\mathcal{A}^*_{PL}(|K|)$, are collections of forms, one on each simplex, $\omega_\sigma$ on $\sigma$, such that $\omega_\sigma|_\tau = \omega_\tau$ for $\tau$ a face of $\sigma$ ( $\tau < \sigma$ ). Each $\omega_\sigma$ can be written as:

$$\sum P(x_0, \ldots\ldots, x_k) dx_{i_1} \wedge \cdots \wedge dx_{i_t}$$

where $x_0, \ldots, x_k$ are the bary centric coordinates for $\sigma$ and P is a polynomial with $\mathbb{Q}$ - coefficients.

Definition (2.5). Let K be a simplicial complex. The minimal model of $X = |K|$, $\mathcal{M}_X$ is defined to be a minimal model of $\mathcal{A}^*_{PL}(X)$.

<u>Theorem ( Sullivan )</u> If X is nilpotent,

$$\pi_k(\mathcal{M}_X) \cong \pi_k(X) \otimes \mathbb{Q} \quad \text{for } k \geq 2,$$

where $\pi_k(\mathcal{M}_X)$ is the degree k part of the indecomposable elements of $\mathcal{M}_X$.

Definition (2.5) We shall say that $X$ is rational $K(\pi,1)$ if $\mathcal{M}_X(1) = \mathcal{M}_X$ , where we denote by $\mathcal{M}_X(1)$ the 1 - minimal model of $X$ .

Let $X$ be a plyhedron. We form the lower central series for $\pi_1(X)$ :

$$\pi_1(X) \supset \Gamma_2 \supset \Gamma_3 \quad \cdots\cdots$$

where $\Gamma_2 = [\ \pi_1(X)\ ,\ \pi_1(X)\ ]$

and we define inductively $\Gamma_{i+1} = [\ \pi_1(X), \Gamma_i\ ]$

We get the tower of nilpotent groups :

$$\rightarrow \pi_1(X) / \Gamma_3 \rightarrow \pi_1(X) / \Gamma_2 \longrightarrow e$$

It is a central extensinn of $\pi_1(X) / \Gamma_{n-1}$ by the abelian group $\Gamma_n / \Gamma_{n+1}$ .

Then it is possible to " tensor " these nilpotent groups with Q. This gives a tower of rational nilpotent groups, and is called a rational nilpotent completion of $\pi_1(X)$.

The 1 - minimal model of $X$ , $\mathcal{M}_X(1)$ has the following canonical filtration ;

$$Q = \mathcal{M}_X(1)^0 \subset \mathcal{M}_X(1)^1 \subset \mathcal{M}_X(1)^2 \subset \cdots\cdot$$

where $\mathcal{M}_X(1)^1$ is the subalgebra generated by closed 1 - forms and $\mathcal{M}_X(1)^2$ is the subalgebra generated by the elements whose image under $d$ is contained in $\mathcal{M}_X(1)^1$ , and so on.

By dualizing, we get a tower of Q- Lie algebras ;

$$\cdots \to \mathcal{L}_2 \to \mathcal{L}_1 \to 0$$

From the Sullivan's theorem [13] , this tower of rational Lie algebra is the tower of nilpotent Lie algebra associated to the rational nilpotent completion of $\pi_1(X)$.

<u>Proposition (2.6)</u> If X is rational $K(\pi,1)$, X has a rational principal Postonikov decomposition :

$$\begin{array}{ccc} & & \downarrow \\ & \overset{\rho_{k+1}}{\nearrow} & K(\pi_1(X)/\Gamma_{k+1}\otimes Q.\,1) \\ X & & \downarrow \\ & \underset{\rho_k}{\searrow} & K(\pi_1(X)/\Gamma_k\otimes Q.\,1) \\ & & \downarrow \end{array}$$

which induces: $H^*(X) \xleftarrow{\sim} \varinjlim_k H^*(\pi_1(X)/\Gamma_k\otimes Q\ ,Q) = H^*(\widehat{\pi_1}(X),\ Q)$
where $\widehat{\pi_1}(X)$ is a rational nilpotent completion of $\pi_1(X)$.

This is the direct consequence of Sullivan's de Rham homotopy theory and we omitt the proof.

## § 3 The structure of the minimal algebras of affine algebraic varieties

In this section we consider the following situation.
Let V be a smooth projective variety ,and let D be a divisor with normal crossings. We shall study the minimal algebra of $X = V - D$. First, we filter D in the following way.
We denote by $D^p$ the set of poits $x \in D$ such that $\text{mult}_x D \geqq p$.
Let us denote by $D^0$ the variety V itself.

Let $\tilde{D}^p \rightarrow D^p$ be the normalization of $D^p$ and let $\varepsilon^p$ be the $\mathbb{Q}$ - local system over $\tilde{D}^p$ defined by the numbering of the divisors.

We denote by $\mathcal{A}^n_X$ the $\mathbb{Q}$ - vector space :

$$\bigoplus_{q-p=n} H^{q-2p}(\tilde{D}^p ; \varepsilon^p)$$

We introduce the Q- differential graded algebra structure in the direct sum :

$$\mathcal{A}_X = \bigoplus_n \mathcal{A}^n_X$$

Namely, $d_1 : \mathcal{A}^n_X \rightarrow \mathcal{A}^{n+1}_X$ is defined to be the d.g.a. homomorphism such that the following diagram is commutative.

$$\begin{array}{ccc}
H^{q-2p}(D_{i_1} \cap \cdots \cap D_{i_p}) & \xrightarrow{d_1} & \bigoplus_k H^{q-2p+2}(D_{i_1} \cap \cdots \hat{k} \cdots \cap D_{i_p}) \\
\downarrow \alpha & & \nearrow j^* \\
\bigoplus_k H^{q-2p+2}(\mathcal{N}_k, \mathcal{N}_k - 0) & & \\
\wr\wr & & \\
\bigoplus_k H^{q-2p+2}(D_{i_1} \cap \cdots \hat{k} \cdots \cap D_{i_p} , & D_{i_1} \cap \cdots \cap D_{i_p}) &
\end{array}$$

where $j: D_{i_1} \cap \cdots \cap D_{i_p} \longrightarrow D_{i_1} \cap \cdots \hat{k} \cdots \cap D_{i_p}$ is the inclusion map with the tubular neighbourhood $\mathcal{N}_k$ and Thom class $\tau_k$, and $\alpha$ is a $\mathbb{Q}$- homomorphism defined by :

$$\alpha(x) = \Sigma_k (-1)^{q-2p} x \cup \tau_k$$

for $x \in H^{q-2p}(D_{i_1} \cap \cdots \cap D_{i_p})$.

The product structure is induced from the wedge product of PL forms, namely:

for $[\omega_1] \in H^{q-2p}(D_{i_1} \cap \cdots \cap D_{i_p})$ and

$[\omega_2] \in H^{q'-2p'}(D_{i'_1} \cap \cdots \cap D_{i'_{p'}})$

the product $[\omega_1]\cdot[\omega_2]$ is defined to be

$$[\, j_1^*\omega_1 \wedge j_2^*\,\omega_2] \in H^{(q+q')-2(p+p')}(D_{i_1}\cap \cdots \cap D_{i_{p'}})$$

where $j_1$ and $j_2$ are inclusions :

$$D_{i_1}\cap \cdots \cap D_{i_p} \xleftarrow{\;j_1\;} D_{i_1}\cap\cdots\cap D_{i_p}\cap D_{i'_1}\cap\cdots\cap D_{i'_p} \xrightarrow{\;j_2\;} D_{i'_1}\cap \cdots \cap D_{i'_p}$$

By caluulating the Morgan's spectral sequence [3] explicitly, we have the following structure theorem for $\mathcal{M}_X$.

Theorem 1 [11]
Let $\mathcal{M} \longrightarrow \mathcal{A}_X$ be the minimal model of $\mathcal{A}_X$.
Then $\mathcal{M}$ is isomorphic to the minimal algebra of X as $\mathbb{Q}$-differential graded algebras.

By using these methods we shall study the minimal algebra of $P^2$ minus curves. Let C be an algebraic curve in $P^2$. Let $\mu : (\hat{P}^2, \hat{C}) \longrightarrow (P^2, C)$ be its minimal resolution. In this case $\mathcal{A}_X$ can be calculated in the following way :

$$\mathcal{A}_0 = H^0(\hat{P}^2)$$

$$\mathcal{A}_1 = (\oplus H^0(\hat{C}j)) \oplus (\bigoplus_{k=1}^{l} H^0(\varepsilon_k))$$

$$\mathcal{A}_2 = H^2(\hat{P}^2) \oplus (\oplus H^0(\hat{C}_i \cap \hat{C}_j)) \oplus (\oplus H^0(\hat{C}_k \cap \varepsilon_m)) \oplus (\oplus H^1(\hat{C}_j))$$

$$\mathcal{A}_3 = \oplus H^2(\hat{C}_j)$$

$$\mathcal{A}_4 = H^4(\hat{P}^2)$$

$\hat{C}_j$ : proper transform of the irreducible component $C_j$

$\varepsilon_k$ : exceptional divisor

Let $\{b_j\}$ be basis of $H^0(\hat{C}_j)$ and let $\{\varepsilon_k\}$ be a basis of $H^0(\mathcal{E}_k)$. $H^2(\hat{P}^2)$ has $(\ell+1)$ bases :

$$\alpha \, , \, \beta_1, \cdots \beta_\ell$$

where $\beta_k$ corresponds to an exceptional divisor $\mathcal{E}_k$.

Let $\{c_{j_1} \ldots\ldots c_{j_{2g}}\}$ be a basis of $H^1(C_j)$.

We observe that :

i) $\mathcal{A}$ is generated by :

$$\{ b_j\}, \ \{\varepsilon_k\}, \ \alpha \, , \{\beta_k\}, \ \{ c_{j_i} \ldots c_{j_{2g}}\}$$

ii) The differential d satisfies ;

$$d\varepsilon_k = \beta_k$$

$$d\alpha = d\beta_1 = \cdots = d\beta_\ell = 0$$

$$db_j = \delta_j \alpha - m_1\beta_1 - m_2\beta_2 - \cdots - m_\ell \beta_\ell$$

where $\delta_j = \deg C_j$ and $m_j$ is the multiplicity of an infinitely near singular point.

iii) The product structure is induced from the intersection forms. Only non trivial parts are :

$$b_i . b_j \neq 0 \ \text{ iff } \ \hat{C}_i \cap \hat{C}_j \neq \emptyset$$

$$b_k . \varepsilon_m \neq 0 \ \text{ iff } \ \hat{C}_k \cap \mathcal{E}_m \neq \emptyset$$

$$\alpha^2 + \beta_j^2 = 0$$

Let $X$ be $\mathbb{C}^2 - L_1 \cup \cdots \cup L_m = P^2 - L_1 \cup \cdots \cup L_m \cup L_\infty$

where $\{ L_j \}$ are lines and $L_\infty$ is a line at infinity.

By blowing up at the poits $p$ such that $\text{mult}_p \ L_j > 2$

we may assume that each singular point is a node.,

Let $\hat{L}_j$ be the proper transform of $L_j$ and we denote by $b_j$ the corresponding basis of $\mathcal{A}^1$.

Let $m_{ij}$ be the multiplicity of $L_i \cap L_j$ and we put

$$a_i = b_i + \Sigma_j \, m_{ij} \cdot \varepsilon_{ij}$$

where $\varepsilon_{ij}$ is the corresponding generator of the exoeptional divisor of the blowing up at $L_i \cap L_j$.

Let $f_i$ be a defining equation of $L_j$ and let $\omega_i$ be

$$\frac{1}{2\pi\sqrt{-1}} \, d \log f_i$$

The following theorem describes the structure of the 1- minimal algebra of X.

<u>Theorem (3.1)</u>

The 1- minimal model of X , $\mathcal{M}_X(1)$ is constructed in the following way :

$$\mathcal{M}_X(1) = \varinjlim_k \mathcal{M}_X(1)^k$$

where $\mathcal{M}_X(1)^0 = \mathbb{Q}$

$$\mathcal{M}_X(1)^1 = \wedge(\omega_1, \cdots, \omega_m)$$

$$\mathcal{M}_X(1)^2 = \mathcal{M}_X(1)^1 \otimes_d \wedge(\{\omega^{(2)}\})$$

$d\omega^{(2)}$ equals one of the elements of

$$\omega_p \wedge \omega_q + \omega_q \wedge \omega_r + \omega_r \wedge \omega_p$$

$$( \text{ for } L_p \cap L_q \cap L_r \neq \emptyset )$$

$$\omega_a \wedge \omega_b \qquad ( \text{ for } L_a \cap L_b = \emptyset )$$

$$\mathcal{M}_X(1)^{k+1} = \mathcal{M}_X(1)^k \otimes_d ( \{ \omega^{(k+1)} \} )$$

$d\omega^{(k+1)}$ = closed form of degree 2 in $\mathcal{M}_X(1)^k$ not in $\mathcal{M}_X(1)^{k-1}$ $(k \geq 2)$

<u>proof</u> We study the $\mathbb{Q}$- d.g.a. $\mathcal{A}_X$. $\mathcal{A}_X$ is generated by;

$$\{ b_j \} \quad \{ \varepsilon_k \} \quad \alpha \ , \quad \{ \beta_k \}$$

and they satisfy the following equations :

$$d\varepsilon_k = \beta_k$$

$$d\alpha = d\beta_1 = \cdots = d\beta_\ell = 0$$

$$db_j = \alpha - \Sigma_j m_{ij} \beta_j$$

Let $\mathcal{M}_X(1)^1 = \wedge(\omega_1, .. \omega_m)$, $d\omega_1 = \cdots = d\omega_m = 0$ .

We define $\rho : \mathcal{M}_X(1)^1 \to \mathcal{A}_X$

by $\rho(\omega_j) = \alpha_j = a_j - a_\infty$

To show that $\rho$ is a d. g. a. map we shall compute $d\rho(\omega_j)$

$$
\begin{aligned}
d\rho(\omega_j) &= da_j - da_\infty \\
&= d(b_j + \Sigma_i m_{ij}\varepsilon_i) - \alpha \\
&= \alpha - \Sigma m_{ij}\beta_i + \Sigma m_{ij}\beta_i - \alpha \\
&= 0
\end{aligned}
$$

Therefore $\rho$ is a d.g.a. map.

We define $\mathcal{M}_X(i)^2$ as in the statement of the theorem and we define $\rho(\omega^{(2)}) = 0$. To prove that $\rho$ is a d.g.a. map , we claim that $\rho(d\omega^{(2)}) = 0$.

If $L_p \cap L_q \cap L_r \neq \emptyset$ ,

$$\rho(d\omega^{(2)}) = \omega_p \wedge \omega_q + \omega_q \wedge \omega_r + \omega_r \wedge \omega_p$$

which is zero because we have the relation

$$(\alpha_p - \alpha_r)\cdot(\alpha_q - \alpha_r) = 0$$

In this way we define $\rho(\omega^{(k+1)}) = 0$ and we have a d.g.a. homomorphism :

$$\rho : \varinjlim \mathcal{M}_X(1)^k \longrightarrow \mathcal{A}_X$$

From the calculation of betti numbers it can be shown that $\rho$ is a quasi isomorphism up to dimension 2, which completes the proof.

## § 5. On the formality of 1- minimal models

The main theorem in this section is the following:

Theorem(5.1) Let $X$ be $P^n$ minus a hypersurface $D$. Then the 1 - minimal model of $X$, $\mathcal{M}_X(1)$ is formal. (i.e. there exists a quasi isomorphism :

$$\psi : \mathcal{M}_X(1) \longrightarrow H^*(\mathcal{M}_X(1)) .$$

By the theorem of Zariski, [14] , we can take general $P^2$ such that :

$$\pi_1(P^2 - C) \longrightarrow \pi_1(P^n - D)$$

is bijective, where $C = P^2 \cap D$ . Applying the theorem of Sullivan (§ 2), We have the following Lefshetz type theorem.

Theorem (5.2) Let $X$ be $P^n - D$. For a general $P^2$

$$\mathcal{M}_{P^2 - C}(1) \xleftarrow{\cong} \mathcal{M}_X(1),$$

where $C = P^2 \cap D$.

Therefore it is sufficient to prove theorem (5.1) in the case of $P^2$ minus curves.

Cororally (5.3) Let $X$ be $C^n$ minus hyperplanes $H_j$. Let $\omega_j$ be $\frac{1}{2\pi\sqrt{-1}} d \log f_j$ where $f_j$ is a defining equation of $H_j$. Then the cohomology ring of the rational nilpotent completion of $\pi_1(X)$, $H^*(\widehat{\pi_1}(X);Q)$, is generated by $[\omega_j]$.

proof of Cor. (5.3)  Since $\mathcal{M}_X(1)$ is formal, we have a formal structure :

$$\psi : \mathcal{M}_X(1) \longrightarrow H^*(\mathcal{M}_X(1))$$

such that $\psi(\omega_j) = [\omega_j]$, and $\psi(x) = 0$ for x such that $x \notin \mathcal{M}_X(1)^1$.

$\psi$ is a quasi isomorphism, hence $H^*(\mathcal{M}_X(1))$ is generated by $[\omega_j]$.

Let X be $C^2$ minus lines.

Cororally (5.4) X is rational $K(\pi,1)$ if and only if $\omega_{i_1} \wedge \omega_{i_2} \wedge \omega_{i_3}$ is exact for each $\omega_{i_1}, \omega_{i_2}, \omega_{i_3} \in \mathcal{M}_X(1)^1$.

proof . As we proved in the previous section

$$\rho : \mathcal{M}_X(1) \longrightarrow \mathcal{A}_X$$

induces an isomorphism up to dim. $\leqq 2$. Therefore $\mathcal{M}_X(1) \longrightarrow \mathcal{A}_X$ is a minimal model if $\mathcal{M}_X(1)$ is acyclic in dim. $\geqq 3$.

Cororally (5.5) We assume that if three lines $L_p, L_q, L_r$ are in general position , there exists a line $L_s$ such that :

$$L_s \cap L_r = \emptyset \quad \text{and} \quad L_s \supset L_p \cap L_q .$$

Then X is rational $K(\pi,1)$.

proof First we consider the case that three lines $L_p$, $L_q$, $L_r$ are not in general postinn. Then the following two cases occur.

case1 : $L_p \cap L_q \cap L_r \neq \emptyset$

In this case :

$$\omega_p \wedge \omega_q \wedge \omega_r = d(\omega_{pqr} \wedge \omega_r).$$

case 2 : There exist $L_p$ and $L_q$ such that $L_p \cap L_q = \emptyset$.

In this case ;

$$\omega_p \ \omega_q \ \omega_r = d(\omega_{pq} \wedge \omega_r)$$

If $L_p$, $L_q$, $L_r$ are in general position , from hypothesis we have $\omega_{pqs}$, and $\omega_{sr}$ such that :

$$d\,\omega_{pqs} = \omega_p \wedge \omega_q + \omega_q \wedge \omega_s + \omega_s \wedge \omega_p$$

$$d\,\omega_{sr} = \omega_s \wedge \omega_r$$

Therefore we have the following equation :

$$d(\omega_{pqs} \wedge \omega_r + \omega_q \wedge \omega_{sr} - \omega_p \wedge \omega_{sr}) = \omega_p \wedge \omega_q \wedge \omega_r.$$

This completes the proof of the corollally.

Remark :If X is an $S^1 \vee \cdots \vee S^1$ -bundle over $S^1 \vee \cdots \vee S^1$, X is $K(\pi,1)$ and it is rational $K(\pi,1)$ by this corollally.

We divide the proof of the main theorem into several steps.

Step 1 By [11], we have a mixed Hodge structure on the complexified minimal model $\mathcal{M}_X(1)_{\mathbb{C}}$ such that the differential d, preserves the bidegrees. In particular $\mathcal{M}_X(1)^1_{\mathbb{C}} = \Lambda_1(A)$ where $A = H^1(V;\mathbb{C}) \oplus \mathrm{Ker}\,(H^0(\tilde{D}^1;\mathbb{C}) \longrightarrow H^2(V;\mathbb{C}))$ and A has the following decomposition in the category of mixed Hodge structure :

$$A = A^{1,0} \oplus A^{0,1} \oplus A^{1,1}$$

The dual Lie algebra of $\mathcal{M}_X(1)$, $\hat{\pi}$, has the following presentation:

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{F}(A^*) \longrightarrow \hat{\pi} \longrightarrow 0$$

where $\mathcal{F}(A^*)$ is a free Lie algebra generated by the dual of A, and $\mathcal{J}$ is a homogeneous ideal generated by the elements of type : (-1, -1), (-2, -1), (-1, -2), (-2, -2).

If we assume that $H^1(V) = 0$, then $A = A^{1,1}$ and $\mathcal{J}$ is generated by the elements of type; ( -2, -2 ) .
Moreover $\mathcal{M}_X(1)_c$ has the bigrading :

$$\mathcal{M}_X(1)_c \cong \bigoplus_{p \geq 0} \mathcal{M}_X(1)_c^{p,p}$$

Step2 By using the fact that , if $\mathcal{L}$ is a free Lie algebra over k, $H^j(\mathcal{L};V) = 0$ for any k - module V and $j \geq 2$, [9] [10] , we have a vanishing :

$$H^k(\mathcal{M}^{p,p}) = 0 \quad \text{for} \quad p > k .$$

where $\mathcal{M}$ is the dual of the free Lie algebra $\mathcal{F}(A^*)$.

We have an injective homomorphism :

$$\mathcal{M}_X(1)_c^{p,p} \longrightarrow \mathcal{M}^{p,p}$$

Let $\mathcal{A}^{p,p}$ be its cokernel.

We have an exact sequence:

$$0 \longrightarrow \mathcal{M}_X(1)_C^{p,p} \longrightarrow \mathcal{M}^{p,p} \longrightarrow \mathcal{A}^{p,p} \longrightarrow 0$$

Since d preserves the bidegrees , we have the following commutative diagram :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}_X(1)_C{}^{p,p}_k & \longrightarrow & \mathcal{M}^{p,p}_k & \longrightarrow & \mathcal{A}^{p,p}_k & \longrightarrow & 0 \\
 & & d\downarrow & & d\downarrow & & d\downarrow & & \\
0 & \longrightarrow & \mathcal{M}_X(1)_C{}^{p,p}_{k=1} & \longrightarrow & \mathcal{M}^{p,p}_{k+1} & \longrightarrow & \mathcal{A}^{p,p}_{k+1} & \longrightarrow & 0
\end{array}$$

From the long exact sequence we have an isomorphism:

$$H^k(\mathcal{M}_X(1)_C^{p,p}) \cong H^{k-1}(\mathcal{A}^{p,p}) \qquad p > k$$

Step 3 We have a following proposition :

Proposition (5.6)

If $H^k(\mathcal{M}_X(1)_C^{p,p}) = 0$ for $p > k$, $\mathcal{M}_X(1)$ is formal.

proof We defime $V_k$ by the extention:

$$\mathcal{M}_X(1)^k = \mathcal{M}_X(1)^{k-1} \otimes \wedge (V_k)$$

In particular $V_1 = A$. Let $N = \bigoplus_{j \geq 2} V_j$ .

Let x be the element of degree k in the ideal $\mathcal{J}$ (N) ( ideal generated by N ). Then x has a bidegree (p,p) such that $p > k$ . Therefore under the assumption every closed form if degree k in $\mathcal{J}$ N is exact.

Hence $\mathcal{M}_X(1)_C$ is formal. The decent from $\mathbb{C}$ to Q is due to Sullivan [13] .

Step 4 We first claim that :

$$H^2(\mathcal{M}_X(1)_C^{p,p}) = 0 \quad \text{for} \quad p > 2$$

Let $x \in \mathcal{M}_X(1)_C^{p,p}$ be a closed form of degree 2 such that $p > 2$. Then $x$ has a bidegree $(3,3)$, $(4,4)$.....
By definition of $\mathcal{M}_X(1)$,

$$H^2(\mathcal{M}_X(1)) \longrightarrow H^2(X)$$

is injective. But $\mathcal{A}(X)_2 = H^2(v) \oplus H^0(\tilde{D}^2) \oplus H^1(\tilde{D}^1)$ therefore $H^2(X)$ has the following decomposition.

$$H^2(X) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \oplus H^{2,2} \oplus H^{1,2} \oplus H^{2,1}$$

Hence $x$ must be exact.

Finally we have the following reduction lemma:

Lemma (5.7) Let $V$ be a smooth projective variety such that $H^1(V;Q) = 0$. Let $D$ be a divisor with normal crossings The 1-minimal model of $X = V-D$, $\mathcal{M}_X(1)$ is formal if each closed form $x \in \mathcal{M}_X(1)^2$ such that $x \notin \mathcal{M}_X(1)^I$ is exact.

We can show directly in the case of the non-singular model of plane curves, the above $x$ is exact! which completes the proof of the main theorem.

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