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A Central Limit Theorem of Stationary Processes and the Parameter Estimation of Linear Processes

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§1 Some limit theorems on stationary processes

Let $\{z(n); n \in J\}$ be a s-vector-valued linear process generated as $z(n) = \sum_{j=0}^{\infty} G(j) e(n-j)$, $n \in J$, where the $z(n)$'s have s components and the $e(n)$'s are p-vectors such that $E(e(n)) = 0$ and $E(e(m)e(n)') = \delta(m,n)K$ (K is a nonsingular $p \times p$ matrix); $G(j)$'s are $s \times p$ matrices; the components of z, e, G are all real. If $\sum \mathrm{tr}\, G(j) K G(j)' < \infty$, the process $\{z(n)\}$ is a second order stationary process and has a spectral density matrix $f(\omega) = \frac{1}{2\pi} k(\omega) K k(\omega)^*$, $-\pi \le \omega \le \pi$ where $k(\omega) = \sum_{j=0}^{\infty} G(j) e^{i\omega j}$. Denote by $C_z(s)$ and $I_z(\omega)$ respectively the serial covariance and the periodogram matrices constructed from a partial realization $\{z(1), \dots, z(N)\}$; namely, $C_z(s) = \frac{1}{N} \sum_{m=1}^{N-s} z(m) z(m+s)'$, for $0 \le s \le N-1$, and $C_z(s) =$

$C_z'(-s)$ for $-N+1 \le s < 0$; $I_z(\omega) = F_z(\omega) F_z(\omega)^*$, where
$F_z(\omega) = \frac{1}{\sqrt{2\pi N}} \sum_{n=1}^{N} z(n) e^{in\omega}$.

Denote the (α,β) component of $G(j)$, C_z and I_z by $G_{\alpha\beta}(j)$, $C_{\alpha\beta}^z$ and $C_{\alpha\beta}^z$ respectively and denote the α-th component of $z(n)$ and $e(n)$ by $z_\alpha(n)$ and $e_\alpha(n)$. Assuming that the process $\{e(n)\}$ is fourth-order stationary, let $Q^e_{\alpha_1\cdots\alpha_4}(t_1,\cdots,t_4)$ be the fourth cumulant of $e_{\alpha_1}(t_1),\cdots,e_{\alpha_4}(t_4)$ and assume that

$$\sum_{t_2,\cdots,t_4=-\infty}^{\infty} | Q^e_{\alpha_1\cdots\alpha_4}(0, t_2-t_1,\cdots, t_4-t_1)| < \infty ;$$

then the process $\{e(n)\}$ has a fourth-order spectral density $\tilde{Q}^e_{\alpha_1\cdots\alpha_4}(\omega_1,\omega_2,\omega_3)$ such that

$$\tilde{Q}^e_{\alpha_1\cdots\alpha_4}(\omega_1,\omega_2,\omega_3) = \frac{1}{(2\pi)^3} \sum_{t_1,\cdots t_3=-\infty}^{\infty} \exp\{-i(\omega_1 t_1 + \cdots + \omega_3 t_3)\} \times Q^e_{\alpha_1\cdots\alpha_4}(0, t_1, t_2, t_3).$$

Denote by $Q^z_{q_1\cdots q_4}$ and $\tilde{Q}^z_{q_1\cdots q_4}$ respectively the fourth-order cumulant and spectral density of the process $z(n)$,

<u>Lemma 1.1.</u> If $\sum_{j=0}^{\infty} |G_{\alpha\beta}(j)|^2 < \infty$ for each α,β and if $\sum_{j_1,\cdots j_3=-\infty}^{\infty} | Q^e_{\alpha_1\cdots\alpha_4}(j_1,j_2,j_3)| < \infty$, the process $\{z(n)\}$ has a fourth-order spectral density $\tilde{Q}^z_{q_1,\cdots,q_4}(\omega_1,\omega_2,\omega_3)$ such that

(1.1) $\hat{Q}^{z}_{g_1 \cdots g_4}(\omega_1, \omega_2, \omega_3)$

$$= \sum_{\alpha_1, \ldots, \alpha_4 = 1}^{p} k_{g_1 \alpha_1}(\omega_1+\omega_2+\omega_3)\, k_{g_2 \alpha_2}(-\omega_1)\, k_{g_3 \alpha_3}(-\omega_2)\, k_{g_4 \alpha_4}(\omega_3)$$

$$\times \hat{Q}^{e}_{\alpha_1 \cdots \alpha_4}(\omega_1+\omega_2+\omega_3, \omega_2, \omega_3)$$

Lemma 1.2. Assume $\sum_{j_1, j_2, j_3 = -\infty}^{\infty} | Q^{z}_{\alpha_1 \cdots \alpha_4}(j_1, j_2, j_3) | < \infty$.

For any square-integrable functions W_1 and W_2 defined on $[-\pi, \pi]$, $\lim_{N \to \infty} N \operatorname{Cov}\left\{ \int_{-\pi}^{\pi} W_1(\omega) I^{z}_{\alpha_1 \alpha_2}(\omega)\, d\omega, \int_{-\pi}^{\pi} W_2(\lambda) I^{z}_{\alpha_3 \alpha_4}(\lambda)\, d\lambda \right\}$

(1.2)
$$= 2\pi \int_{-\pi}^{\pi} W_1(\omega) \overline{W_2(\omega)}\, f_{\alpha_1 \alpha_3}(\omega) \overline{f_{\alpha_2 \alpha_4}(\omega)}\, d\omega$$
$$+ 2\pi \int_{-\pi}^{\pi} W_1(\omega) \overline{W_2(\omega)}\, f_{\alpha_1 \alpha_4}(\omega) \overline{f_{\alpha_2 \alpha_3}(\omega)}\, d\omega$$
$$+ 2\pi \iint_{-\pi}^{\pi} W_1(\omega_1) W_2(-\omega_2)\, \tilde{Q}^{z}_{\alpha_1 \cdots \alpha_4}(\omega_1, \omega_2, -\omega_2)\, d\omega_1 d\omega_2 .$$

Lemma 1.3. If $\sum |G_{\alpha\beta}(j)|^2 < \infty$ for each α, β and if $\sum_j | Q^{e}_{\alpha_1 \cdots \alpha_4}(j_1, j_2, j_3) | < \infty$,

$$\lim_{N \to \infty} N \operatorname{Cov}\{ C^{z}_{\alpha_1 \alpha_2}(m), C^{z}_{\alpha_3 \alpha_4}(n) \}$$
$$= 2\pi \int_{-\pi}^{\pi} \left\{ f_{\alpha_1 \alpha_3}(\omega) \overline{f_{\alpha_2 \alpha_4}(\omega)}\, e^{-i(n-m)\omega} + f_{\alpha_1 \alpha_4}(\omega) \overline{f_{\alpha_2 \alpha_3}(\omega)}\, e^{i(n+m)\omega} \right\} d\omega$$
$$+ \sum_{\beta_1 \cdots \beta_4 = 1}^{p} \iint_{-\pi}^{\pi} \exp\{ i m \omega_1 + i n \omega_2 \}\, k_{\alpha_1 \beta_1}(\omega_1)\, k_{\alpha_2 \beta_2}(-\omega_1)\, k_{\alpha_3 \beta_3}(\omega_2)\, k_{\alpha_4 \beta_4}(-\omega_2)$$
$$\times \hat{Q}^{e}_{\beta_1 \cdots \beta_4}(\omega_1, -\omega_2, \omega_2)\, d\omega_1 d\omega_2 .$$

Let $\{W_m(n), \mathcal{F}_m(n) ; n = 0, 1, \ldots, n(m)\}$, $m = 1, 2, \ldots$, be a zero-mean square-integrable martingale for each m where $\{\mathcal{F}_m(n) ; n = 1, 2, \ldots, n(m)\}$ is a sequence of increasing σ-fields and $n(m) \to \infty$ as $m \to \infty$. Let $U_m(1) = W_m(1)$ and $U_m(k) = W_m(k)$ and $U_m(k) = W_m(k) - W_m(k-1)$ $(W_m(0) = 0)$.

Lemma 1.4. [essentially due to Brown (1971)]. Suppose that (i) $\lim_{m\to\infty} \frac{1}{n(m)} \sum_1^{n(m)} E[U_m(k)^2 I\{|U_m(k)| \geq \varepsilon n(m)\}] = 0$ for any $\varepsilon > 0$,

(ii) $[\sum_{k=1}^{n(m)} E\{U_m(k)^2 / \mathcal{F}_m(k-1)\}] / \sum_{k=1}^{n(m)} E\{U_m(k)^2\}$

tends to 1 in probability as $m \to \infty$; then

$\sum_1^{n(m)} U_m(k) / \sqrt{\sum E\{U_m(k)^2\}}$ is asymptotically normally distributed with mean 0 and variance 1.

Theorem 1.1. If a zero-mean vector-valued second order stationary process $\{x(t) ; t \in J\}$ satisfies that

(i) $\mathrm{Var}\{E(x_\alpha(t+\tau) | \mathcal{F}_t)\} = O\left(\frac{1}{\tau^{2+\varepsilon}}\right)$ $(\varepsilon > 0)$

(ii) For a positive constant $\eta (> 0)$,

$$E\left| E(x_\alpha(l) x_\beta(m) | \mathcal{F}_t) - E\{x_\alpha(l) x_\beta(m)\}\right|$$

$$= O\left\{\frac{1}{(\min\langle |l-t|, |m-t|\rangle)^{1+\eta}}\right\} \quad l, m > t.$$

(iii) $\{x(t)\}$ has a spectral density matrix $f(\omega)=\{f_{\alpha\beta}(\omega)\}$ such that each element is continuous at the origin and $f(0)$ is non-degenerate, then $\sum_{n=1}^{N} x(n)/\sqrt{N}$ is asymptotically normally distributed with mean zero and covariance matrix $2\pi f(0)$, where $\mathcal{F}_t$ is the σ-field generated by the set of random vectors $\{x(m);\ m\le t\}$.

Let $\{z(n)\}$ be the linear process $z(n)=\sum_{j=0}^{\infty} G(j)e(n-j)$, and denote by $\mathcal{B}(t)$ the σ-field generated by $\{e(m);\ m\le t\}$.

<u>Theorem 1.2.</u> Suppose (i) $\mathrm{Var}\{E\{e_{\beta_1}(n)e_{\beta_2}(n+m)/$

$\mathcal{B}(n-\tau)\}-\delta(m)K_{\beta_1\beta_2}] = O\left(\frac{1}{\tau^{2+\varepsilon}}\right) \quad (\varepsilon>0)$

(ii) $E\,|\,E\{e_{\beta_1}(n_1)e_{\beta_2}(n_2)e_{\beta_3}(n_3)e_{\beta_4}(n_4)\,/\,\mathcal{B}(n_1-\tau)\}$

$-E(e_{\beta_1}(n_1)e_{\beta_2}(n_2)e_{\beta_3}(n_3)e_{\beta_4}(n_4))\,|=O\left(\frac{1}{\tau^{1+\eta}}\right)$

$(\eta>0,\ n_1\le n_2\le n_3\le n_4)$,

(iii) $f_{\beta\beta}\ (\beta=1,\dots,s)$ is square-integrable,

(iv) $\sum_{j_1,j_2,j_3=-\infty}^{\infty} |\,Q^{e}_{\beta_1\cdots\beta_4}(j_1,j_2,j_3)\,|<\infty$, then $\sqrt{N}\,(C^{z}_{\alpha_1\alpha_2}(m) - \gamma^{z}_{\alpha_1\alpha_2}(m))$ $(\alpha_1, \alpha_2 = 1, \cdots, s,\ 0 \le n \le L)$ have a joint asymptotic normal distribution with mean 0 and the asymptotic covariance between $\sqrt{N}\,\{C^{z}_{\alpha_1\alpha_2}(m_1) - \gamma^{z}_{\alpha_1\alpha_2}(m_1)\}$ and $\sqrt{N}\,(C^{z}_{\alpha_3\alpha_4}(m_2) - \gamma^{z}_{\alpha_3\alpha_4}(m_2))$ is given as

$$(1.4) \quad 2\pi \int_{-\pi}^{\pi} \{ f_{\alpha_1\alpha_3}(\omega)\, \overline{f_{\alpha_2\alpha_4}(\omega)}\, e^{-i(m_2 - m_1)\omega}$$

$$+ f_{\alpha_1\alpha_4}(\omega)\, \overline{f_{\alpha_2\alpha_3}(\omega)}\, e^{i(m_1+m_2)\omega} \} d\omega$$

$$+ 2\pi \sum_{\beta_1 \cdots \beta_4 = 1}^{p} \iint_{-\pi}^{\pi} \exp\{ i m_1\omega_1 + i m_2\omega_2 \}\, k_{\alpha_1\beta_1}(\omega_1)\, k_{\alpha_2\beta_2}(-\omega_1)$$

$$\times\, k_{\alpha_3\beta_3}(\omega_2)\, k_{\alpha_4\beta_4}(-\omega_2)\, \tilde{Q}^{\varepsilon}_{\beta_1 \cdots \beta_4}(\omega_1, -\omega_2, \omega_2)\, d\omega_1 d\omega_2 .$$

§2. Asymptotic properties of quasi-Gaussian maximum likelihood estimates for a linear process

Let P_0 be a family of spectral densities $f_\theta(\omega)$ $\theta \in \Theta \subset R^q$. Then a functional T defined on P_0 is determined by the requirement that, for P, the set of all density such that $\sum \operatorname{tr} G(j) K G(j)' < \infty$,

$$D(f_{T(f)}, f) = \min_{t \in \Theta} D(f_t, f)$$

where $D(f_t, f) = \int_{-\pi}^{\pi} \{\log \det f_t(\omega) + \operatorname{tr} f_t(\omega)^{-1} f(\omega)\} d\omega$,

if there is a unique $T(f)$ in Θ, Let $f_N, f \in \mathcal{P}$ be such that for every continuous $s \times s$ matrix-valued function ψ

$$\int_{-\pi}^{\pi} \operatorname{tr} \psi(\omega) f_N(\omega) d\omega \longrightarrow \int_{-\pi}^{\pi} \operatorname{tr} \psi(\omega) f(\omega) d\omega$$

as $N \to \infty$; then f_N is called weakly converge to f ($f_N \xrightarrow{W} f$).

<u>Lemma 2.1.</u> Suppose that Θ is a compact subset of R^q, $\theta_1 \neq \theta_2$ implies that $f_{\theta_1} \neq f_{\theta_2}$ on a set of positive definite and that $f_\theta(\omega)$ is continuous in θ and ω. Then

(1) For every $f \in \mathcal{P}$, there is a value $T(f)$.

(2) Assume $T(f)$ is unique; then if $f_N \xrightarrow{W} f$ then

$$T(f_N) \longrightarrow T(f).$$

(3) $T(f_\theta) = \theta$ uniquely for every $\theta \in \Theta$.

<u>Lemma 2.2.</u> Assume that every component of

$f_\theta(\omega)$ is twice continuously differentiable function of $\theta \in \Theta$ and that the second derivatives of these component are continuous in ω; moreover, suppose that $T(f)$ exists uniquely and in $\mathrm{Int}\,\Theta$ and that $M_f = \int_{-\pi}^{\pi} \{ \frac{\partial^2}{\partial\theta\partial\theta'} \mathrm{tr}\, f_\theta(\omega)^{-1} f(\omega) + \frac{\partial^2}{\partial\theta\partial\theta'} \log \det f_\theta(\omega) \} d\omega \Big|_{\theta=T(f)}$ is a non-singular matrix. Then for every sequence of spectral density matrices $\{f_N\}$ such that $f_N \xrightarrow{w} f$;

$$(2.1) \quad T(f_N) = T(f) - \int_{-\pi}^{\pi} M_f^{-1} \frac{\partial}{\partial\theta} \{ \mathrm{tr}\, f_\theta(\omega)^{-1} (f_N(\omega) - f(\omega)) \}_{\theta=T(f)} d\omega$$

$$+ a_N \int_{-\pi}^{\pi} \frac{\partial}{\partial\theta} \{ \mathrm{tr}\, f_\theta(\omega)^{-1} (f_N(\omega) - f(\omega)) \}_{\theta=T(f)} d\omega$$

where a_N is a $q\times q$ matrix which tends to zero as $N \to \infty$.

Lemma 2.3. Assume that $\{z(n)\}$ satisfies the conditions (i)–(iv) of Theorem 1.2 and (v) $\sum_{j=-\infty}^{\infty} |j|^{\frac{1}{2}} |\gamma_{\alpha\beta}^{z}(j)| < \infty$. Let $\phi_j(\omega)$, $j = 1, \ldots, q$ be $s\times s$ matrix-valued continuous functions on $[-\pi, \pi]$ such that $\phi_j(\omega) = \phi_j(\omega)^*$. Then,

(1) $$p\text{-}\lim \int_{-\pi}^{\pi} \mathrm{tr}\, I_z(\omega) \phi_j(\omega) d\omega = \int_{-\pi}^{\pi} \mathrm{tr}\, f(\omega) \phi_j(\omega) d\omega.$$

(2) $\sqrt{N}\int_{-\pi}^{\pi} \operatorname{tr}\{I_z(\omega)-f(\omega)\}\phi_j(\omega)\,d\omega$, $j=1,\dots,g$ have asymptotically a normal distribution with zero mean-vector and covariance matrix V whose (j,l) element is

$$(2.2)\quad 4\pi\int_{-\pi}^{\pi} \operatorname{tr} f(\omega)\phi_j(\omega)f(\omega)\phi_l(\omega)\,d\omega + 2\pi\sum_{r,t,u,v=1}^{s}\iint_{-\pi}^{\pi}$$

$$\phi_{rt}^{(j)}(\omega_1)\phi_{uv}^{(l)}(\omega_2)\,\hat{Q}^{z}_{rtuv}(-\omega_1,\omega_2,-\omega_2)\,d\omega_1 d\omega_2$$

where $\phi_{rt}^{(j)}(\omega)$ is the (r,t)th element of $\phi_j(\omega)$.

Theorem 2.1. Suppose that in addition to the conditions of Lemma 2.2, the conditions (i)–(iv) in Theorem 1.2 and (v) in Lemma 2.3 are satisfied. Then $\operatorname{p-lim}_{N\to\infty} T(I_z)=T(f)$ and the vector $\sqrt{N}(T(I_z)-T(f))$, under f, tends to the normal distribution $N(0_g, M_f^{-1}\tilde{V}M_f^{-1})$ where $\tilde{V}=\{\tilde{V}_{jl}\}$ is a $g\times g$ matrix such that

$$(2.3)\quad \tilde{V}_{jl} = 4\pi\int_{-\pi}^{\pi}\operatorname{tr}\left\{f(\omega)\frac{\partial}{\partial\theta_j}f_\theta(\omega)^{-1}f(\omega)\frac{\partial}{\partial\theta_l}f_\theta(\omega)^{-1}\right\}\Big|_{\theta=T(f)}d\omega$$

$$+\,2\pi\sum_{r,t,u,v=1}^{s}\iint_{-\pi}\frac{\partial}{\partial\theta_j}f_\theta^{(r,t)}(\omega_1)^{-1}\frac{\partial}{\partial\theta_l}f_\theta^{(u,v)}(\omega_2)^{-1}\Big|_{\theta=T(f)}\tilde{Q}_{rtuv}(\,$$

$-\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2$

and $f_\theta^{(r,t)}(\omega)^{-1}$ is the (r,t)th element of $f_\theta(\omega)^{-1}$.

Corollary

$$\tilde{V}_{j\ell} = 4\pi \int_{-\pi}^{\pi} \mathrm{tr}\left\{f(\omega) \frac{\partial}{\partial\theta_j} f_\theta(\omega)^{-1} f(\omega) \frac{\partial}{\partial\theta_\ell} f_\theta(\omega)^{-1}\right\}_{\theta=\underline{T}(f)} d\omega$$

$$+ 2\pi \sum_{a,b,c,d=1}^{p} \sum_{r,t,u,v=1}^{s} \iint_{-\pi}^{\pi} \frac{\partial}{\partial\theta_j} f_\theta^{(r,t)}(\omega_1)^{-1} \frac{\partial}{\partial\theta_\ell} f_\theta^{(u,v)}(\omega_2)^{-1}\Big|_{\theta=\underline{T}(f)}$$

$$\times k_{ra}(-\omega_1) k_{tb}(\omega_1) k_{uc}(-\omega_2) k_{vd}(\omega_2) \tilde{Q}^{e}_{abcd}(-\omega_1, \omega_2, -\omega_2) d\omega_1 d\omega_2.$$

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