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On relations between the Brown-Peterson cohomology theory and the ordinary mod p cohomology theory

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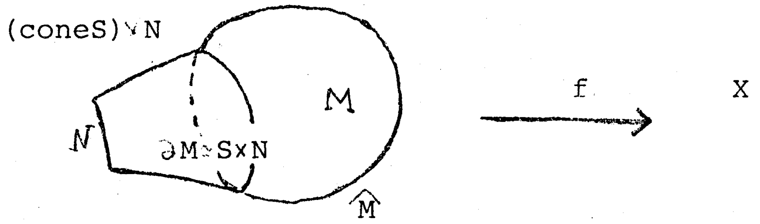
§1. Introduction. We recall the bordism theory with singularities Let S be a close (weakly complex) manifold. An S-manifold (manifold with singularities of type S,) means

$\hat{M} = M \cup (\text{cone}S)\times N$ where M is an open manifold with $\partial M \simeq S\times N$ and N is a close manifold.



The bordism group $MU(S)_*(X)$ is a group of bordism classes of maps $f : M \rightarrow X$.

Define a bordism operation Q_S by

$$Q_S[\hat{M},f] = [N,f|N].$$

We will show that this operation exhibits relationship between $BP_*(X)$ and $H_*(X;Z_p)$.

§2. $BP(S)_*(X)$. A main reference of this section is [1] [6] Let $MU_* \simeq Z[x_1,\ldots]$ and let $x_{p^i-1}=v_i$ (, where we take v_i as a Milnor manifold, i.e., $c_{\Delta p^i-1}(v_i)=p \bmod p^2$). For each sequence $S=(P_1,\ldots)$, P_i ; close manilod, Sullivan [5] also defined $MU(S)_*(X)$. The most important tool of this theory is the following Sullivan exact sequence.

$$\begin{tikzcd} MU(S)_*(X) \arrow[rr, "P"] & & MU(S,P)_*(X) \arrow[dl, "i"] \\ & MU(S,P)_*(X). \arrow[ul, "\delta"] & \end{tikzcd}$$

From this exact sequence, if S is regular, then

$MU(S)^*(S^0) \simeq MU^*/(S) = MU$ /ideal$(P_1,\ldots)$.

Hence $MU(\ldots,x_i,\ldots\ i=p^j-1)_*(X)_{(p)} \simeq BP_*(X)$.

By Quillen's splliting theorem, we can prove

$$MU(p,v_{i_1},\ldots)_*(X) \simeq MU_{*(p)} \otimes_{BP_*} BP(p,v_{i_1},\ldots)_*(X).$$

In particular,

$$\begin{aligned} MU(p,v_1,v_2,\ldots)_*(X) &\simeq MU_{(p)*} \otimes_{BP_*} BP(p,v_1,v_2,\ldots)_*(X) \\ &\simeq MU_{(p)*} \otimes_{BP_*} H_*(X;Z_p) \quad (\text{since } BP(p,v_1,.)=Z_p). \end{aligned}$$

This fact shows that $H_*(X;Z_p)$ is essentialy the bordism theory of type $(p,v_1,v_2,\ldots)$. (Of course, $H_*(X;Z_p)$ is the bordism theory of type $(p,x_1,x_2,\ldots)$ but the above fact shows singularities of x_i, $i \neq p^j-1$ do not appear.)

§3. Milnor operations. Taking Spanier-Whitehead dual, the bordism operation induces cohomology operation. (Quillen's geometric picture arguments [4] ,also induce cohomology operation.)

Recall the Milnor operation Q_i , namely, Q_0= the Bockstein operation and $Q_{i+1}=\mathcal{P}^{p^i} Q_i - Q_i \mathcal{P}^{p^i}$.

Theorem 1. [6] In $H^*(X;Z_p)=BP(p,v_1,\ldots)^*(X)$,the cobordism operation induced from a Milnor manifold is the Milnor operation, i.e., $Q_{v_i}=Q_i$.

Proof. Since Q_{v_i} is defined from taking the boundaries, Q_{v_i} is a derivation. Recall that the product of Lens spaces $L_p^m \times \ldots \times L_p^m$ is a retract of the Eilenberg MacLane spectrum KZ_p.

Hence we have only to prove $Q_{v_i}=Q_i$ in $H^*(L_p\,;Z_p)$.

It is well known

$$H^*(L_p\,;Z_p)\simeq Z_p[x]/(x^m)\otimes\Lambda(\alpha),\quad Q_i\alpha=x^{p^i}\ \text{and}\ Q_ix=0.$$

By the Gysin sequence, it is also well known

$$BP^*(L_p)\simeq BP^*[x]/([p],(x^m))$$

where $[p]=c_1(\xi_1\otimes\cdots\otimes\xi_1)=px+a_1x^2+\dots\ =px+v_1x^{p-1}+\dots$,

$a_{p^i-1}=v_i \mod(p,\dots,v_{i-1})$.

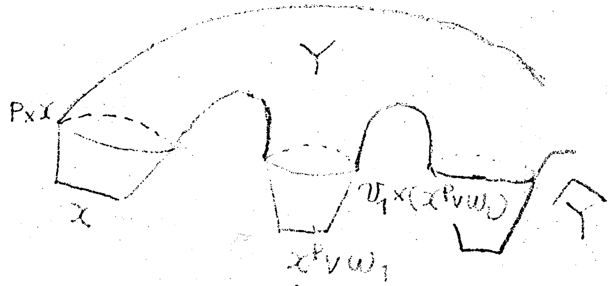
In $BP^*(L_p)$, that $[p]=0$ means there is a manifold Y such that

$$\partial Y = p\times x\vee a_1\times x^2\vee\ \dots\vee a_{p^i-1}\times x^{p^i}\ \dots$$
$$= p\times x\vee(v_1\times x^p\vee v_1\times w_1)\vee(v_2\times x^{p^2}\vee v_2\times w_2)\vee\dots$$

where $w_i\in BP^+\cdot BP^*(L_p)$.

Attach the cones.

$$\hat{Y} = Y\bigcup_i \mathrm{cone}\,v_i\times(x^{p^i}\vee w_i).$$



Then we have

$$Q_{v_i}\hat{Y}=x^{p^i}\vee w_i.$$

In $H^*(X;Z_p)$, $BP^+BP^*(L_p)=0$ and hence $Q_{v_i}\hat{Y}=x^{p^i}$. In particular, $Q_{p\text{-points}}\hat{Y}=x$. It is immediate seen that $Q_{p\text{-points}}=$ the Bockstein Q_0. Since there is only one α such that $Q_0\alpha=x$, we have $\hat{Y}=\alpha$.

Hence $Q_{v_i}\alpha=Q_{v_i}\hat{Y}=x^{p^i}=Q_i\alpha$.

Since x is a closed manifold and x has no singularities, $Q_ix=0$.

Hence we have the theorem. q.e.d.

§4. Relations $BP^*(X)$ to $H^*(X;Z_p)$.

Theorem 2. If $pb_0+v_1b_1+\dots+v_nb_n=0$ in $BP^*(X)$, then there is $y\in H^*(X;Z_p)$, such that $Q_j(y)=i(b_j)$ where $i : BP\to KZ_p$ is the natural inclusion map (Thom map).

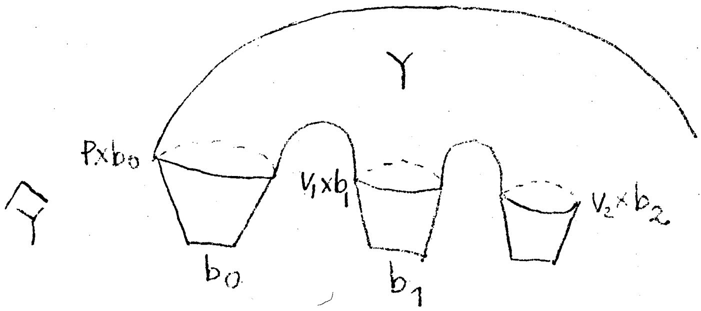
Proof. Because there is Y such that

$\partial Y = p\times b_0 \vee v_1\times b_1 \vee \ldots \vee v_n\times b_n$.

Construct $\hat{Y}$ as in the proof of Theorem 1. Then

$$Q_{v_i}\hat{Y} = b_i.$$

q.e.d.



Example 1. Let X be a finite H-space. By [2], there is a system of even dimensional indecomposable elements in $H(X;Z_p)$

$(y_k,\ldots,y_1)$, $|y_i| = (p^{k+1}-1)/(p-1)-p^i$, (*)

such that there is $i_{\ell,\ell'}$ with $y_\ell = Q_{\ell'} i_{\ell,\ell'}$ $(1\leq \ell \neq \ell' \leq k)$.

Conjecture 1. If there is a finite H-space of type (*), then there are y_ℓ' such that $i(y_\ell') = y_\ell$ and

$$v_h y_\ell' + v_\ell y_h' = 0 \quad \text{in } BP^*(X).$$

This conjecture is true for Lie groups, for example, we have

(1) $H^*(F_4;Z_3) \simeq Z_3[x_8]/(x_8^3)\otimes\Lambda(x_3,\ldots)$, $Q_1x_3=0$.

In $BP^*(F_4)$, $v_1x_8'=0$.

(2) $H^*(E_8;Z_3) \simeq Z_3[x_8,x_{20}]/(x_8^3,x_{20}^3)\otimes\Lambda(x_3,x_7,x_{15},\ldots)$.

In $BP^*(E_8)$, $v_1x_8'+v_2x_{20}'=0$, $v_1x_{20}'=0$.

Example 2. Let K be the Eilenberg MacLane space $K(Z,3)$. The mod p cohomology ring is

$$H^*(K;Z_p) \simeq Z_p[\delta\mathcal{P}\iota,\ldots]\otimes\Lambda(\iota,\mathcal{P}\iota,\mathcal{P}^p\mathcal{P}\iota,\ldots]$$

For simplicity of notations, let denote $\mathcal{P}^{p^{n-1}}\cdots\mathcal{P}\iota = c_n$, $\delta c_n = b_n$. Then $Q_m\iota = b_m$, $Q_mc_n = (b_{n-m})^{p^m}$ for $n>m\geq 0$, $Q_mc_n=(b_{m-n})^{p^n}$ for $m>n\geq 0$, $Q_mc_m=0$ and $Q_mb_n=0$.

Conjecture 2. There are b_j' in $BP^*(K)$ such that $i(b_j')=b_j$ and

$$v_1b_1'+v_2b_2'+\ldots=0.$$

Proposition 1. In $BP^*(K^{2p^4+2p^3};Z_p)=P(1)^*(K^{2p^4+2p^3})$, there are b_j' for $1 \leq j \leq 4$ such that $i(b_j')=b_j$ and mod $(p,v_1,..)^2$

$$v_1b_1'+\dots+v_4b_4'=0$$

$$v_1b_3'^{p}+v_2b_2'^{p^2}+v_3b_1'^{p^3}=0,\ v_1b_2'^{p}+v_2b_1'^{p^2}=0,\ v_1b_1'^{p^3}=0.$$

Proof. We will prove only the first relation. We notice that $|b_i|=2(p^n+1)$ and $|c_i|=2(p^n+1)-1$. The degrees of the differentials of Atiyah-Hirzebruch spectral sequence ${}_{P(1)}E_r$ which convergents $P(1)^*(X)$ are 4m-1. Hence we can prove $b_1,..,b_4$ are permanent in ${}_{P(1)}E_r$. Since $d_{2p-1}=v_1b_1$, we have $v_1b_1=0$ in ${}_{P(1)}E$. This means $v_1b_1 \in F^s$(the associated filtration), $s>|b_1|$. But 4m-dimensional elements are generated, as an $P(1)^*$-module, by $b_1^{\alpha_1}..b_4^{\alpha_4}$.

Therfore there is a relation

$$v_1b_1+\dots=0 \text{ in } P(1)^*(K^{2p^4+2p^3}).$$

The fact that there is only one ι with $Q_1\iota=b_1$ implies $i(b_j)=Q_j\iota$.

q.e.d.

§5. Relations between A-H spectral sequence and the Sullivan exact sequence.

Lemma. Let $wx=0$ in $P(1)^*(X)=BP^*(X;Z_p)=BP(p)^*(X)$ for $0\neq w\in P(1)^*$, and let $i(x)=x'\neq 0$ in $H^*(X;Z_p)$. From Sullivan exact sequence there is $y\in BP(p,w)^*(X)$. Then in A-H spectral sequence ${}_{P(1)}E_r^{*,*}$,

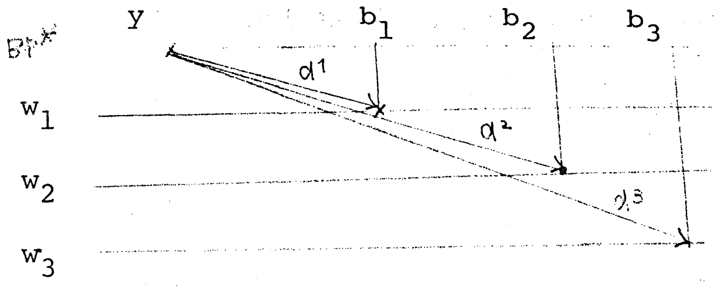
$$d_r(y')=\lambda wx\ ,$$

where $\lambda \neq 0\in Z_p$, $y'\in {}_{P(1)}E_r^{*,*}$ corresponds to y.

Theorem 3. Let $(w_1,\dots,w_s)=J_s$ $|w_i|>|w_{i+1}|$, regular sequence in $P(1)^*=BP^*/p$. Let $b_j\in P(1)^*$, $0\neq i(b_j)$ in $H^*(X;Z_p)$. Suppose $w_1b_1+\dots+w_sb_s=0$ in $P(1)^*(X)$. Then there is $y\in {}_{P(1)}E_2^{*,*}$ such that

$$d_{r_t}(y)=\lambda_t w_t i(b_t) \quad \text{in} \quad {}_{BP(p,J_{t-1})}E_{r_t}^{*,*},\ 0<t\leq s$$

where $\lambda_t\neq 0$ in Z_p.



d^1 in $BP(p)^E$

d^2 in $BP(p,w_1)^E$

d^3 in $BP(p,w_1,w_2)^E$

When we study relations in $BP^*(X)$ with decomposable elements of BP^*, Theorem 3 is useful. For example, there is a relation in $BP(p)^*(K^{2p^4+2p^3})$

$$v_1{}^p b_2 + v_2 b_1{}^p + v_3 b_2{}^p + v_4 b_3{}^p = 0 \quad \text{mod}(\ (p,v_1,v_2,..)^2 - \{v_1{}^2\}).$$

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