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A Class of Recurrence Relations on Acyclic Digraphs of Poset Type

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We shall talk about a systematic study on a class of elementary combinatorial functions related to the number of pathes (chains) on an acyclic digraph (poset). Let D be an acyclic digraph. Then, for each arc (s,t) in D we say that s is adjacent to t, written $s \to t$. In this talk, each vertex in the vertex-set V(D) of D is regarded as a path of length 0. Let R and R[x] be the real field and the polynomial ring of one variable, respectively. Then, for each a and b in R we define a map $f^{(a,b)}:V(D) \longrightarrow R[x]$ by

$$f^{(a,b)}(s) = \begin{cases} a & \text{if } s \text{ is a sink} \\ (\sum_{s\to t} f^{(a,b)}(t))x + b & \text{otherwise.} \end{cases}$$

Let $\mathcal{L}(D)$ denote the set $\{f^{(a,b)} \mid a \text{ and } b \text{ in } R\}$ of all such maps over R. Then, for each f and g in $\mathcal{L}(D)$ and each a in R the sum + and the scalar multiple af are defined as follows:

(1) $(f + g)(s) = f(s) + g(s)$ (2) $(af)(s) = a(f(s))$,

where s is any vertex in V(D).

Theorem 1. Under the sum + and the scalar multiple, $\mathcal{L}(D)$ is a linear space over R isomorphic to the 2-dimensional linear space R^2.

Therefore, $\{f^{(1,0)}, f^{(0,1)}\}$ is a base of the linear space $\mathcal{L}(D)$, and each $f^{(a,b)}$ in $\mathcal{L}(D)$ is uniquely representable as follows:

$$f^{(a,b)} = af^{(1,0)} + bf^{(0,1)}.$$

Remark 1. For $f^{(1,0)}$, $f^{(0,1)}$ and $f^{(1,1)}$ in $\mathcal{L}(D)$ and each s in V(D),

(1) the coefficient of x^i in $f^{(1,0)}(s)$ is the number of pathes of length i from s to sinks in D,

(2) the coefficient of x^i in $f^{(0,1)}(s)$ is the number of pathes of length i from s to vertices but sinks in D,

(2) the coefficient of x^i in $f^{(1,1)}(s)$ is the number of pathes of length i from s in D.

For each f in $\mathcal{L}(D)$, we define $\tilde{f}$ by $\tilde{f} = \sum_{s \in V(D)} f(s)$, that is, $\tilde{}$ is a map from $\mathcal{L}(D)$ into R[x].

Theorem 2. $\tilde{\mathcal{L}}(D)$ is the linear subspace $\langle \tilde{f}^{(1,0)}, \tilde{f}^{(0,1)} \rangle$ of R[x] and each $\tilde{f}^{(a,b)}$ in $\tilde{\mathcal{L}}(D)$ is uniquely representable as follows:

$$\tilde{f}^{(a,b)} = a\tilde{f}^{(1,0)} + b\tilde{f}^{(0,1)}.$$

Remark 2. For $\tilde{f}^{(1,0)}$, $\tilde{f}^{(0,1)}$ and $\tilde{f}^{(1,1)}$ in $\tilde{\mathcal{L}}(D)$, we have the following facts:

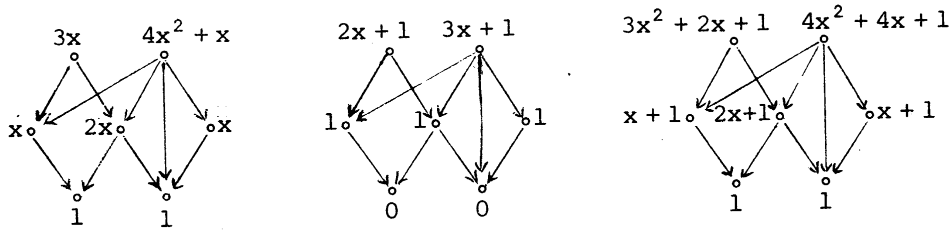
(1) the coefficient of x^i in $\tilde{f}^{(1,0)}$ is the number of all pathes of length i to sinks in D,

(2) the coefficient of x^i in $\tilde{f}^{(0,1)}$ is the number of all pathes of length i to vertices but sinks in D,

(3) the coefficient of x^i in $\tilde{f}^{(1,1)}$ is the number of all pathes of length i in D.

<u>Example</u> 1. $f^{(1,0)}$, $f^{(0,1)}$, $f^{(1,1)}$ and each $\tilde{f}$ are illustrated.

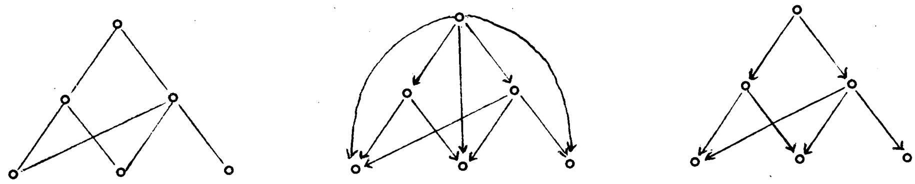
(1) $f^{(1,0)}$ (2) $f^{(0,1)}$ (3) $f^{(1,1)}$



$\tilde{f}^{(1,0)} = 7x^2 + 5x + 2$ $\tilde{f}^{(0,1)} = 5x + 5$ $\tilde{f}^{(1,1)} = 7x^2 + 10x + 7$

If a digraph D represents the incidence relation of a poset P, then D is said to be <u>of poset type</u> P and identified with P. If a digraph D represents the Hasse diagram H(P) of a poset P, then D is said to be <u>of Hasse diagram type</u> H(P) and identified with H(P).

<u>Example</u> 2. A digraph of poset type P and a digraph of Hasse diagram type H(P) are illustrated.



The Hasse diagram H(P) of a poset P — A digraph of poset type P — A digraph of Hasse diagram type H(P)

<u>Remark</u> 3. Each $f^{(a,b)}$ in $\mathcal{L}(P)$ is rewritten as follows:

$$f^{(a,b)}(s) = \begin{cases} a & (s = \text{a minimal element}) \\ (\sum_{s>t} f^{(a,b)}(t))x + b & \text{otherwise.} \end{cases}$$

Each $f^{(a,b)}$ in $\mathcal{L}(H(P))$ is rewritten as follows:

$$f^{(a,b)}(s) = \begin{cases} a & (s = \text{a minimal element}) \\ (\sum_{s\downarrow t} f^{(a,b)}(t))x + b & \text{otherwise,} \end{cases}$$

where $s \downarrow t$ denotes "s covers t".

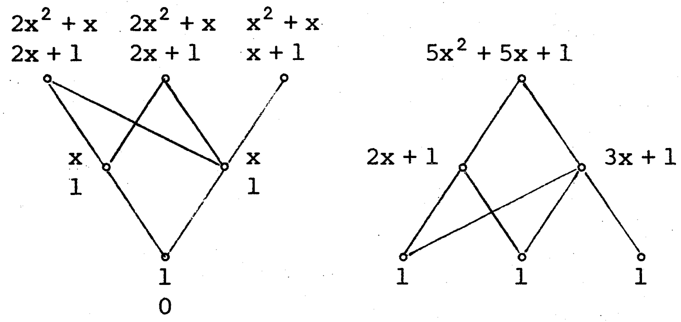
A poset P is said to be <u>connected</u> if the incidence relation of P is represented by a connected digraph.

<u>Theorem</u> 3. Let P be a connected poset. Let $f^{(1,0)}$ and $f^{(0,1)}$ be in $\mathcal{L}(P)$. Then, $f^{(1,0)}(s) = (f^{(0,1)}(s))x$ for all s but minimal elements in P if and only if P has a unique minimal element.

<u>Theorem</u> 4. Let P be a poset with a unique maximal element $\mathbf{1}$, of which cardinality ≥ 2. Then, for $\tilde{f}^{(a,b)}$ in $\tilde{\mathcal{L}}(P)$ and $f^{(a,b)}$ in $\mathcal{L}(P)$, the following identity holds:

$$\tilde{f}^{(a,b)} = ((x + 1)f^{(a,b)}(\mathbf{1}) - b)/x.$$

<u>Example</u> 3. Theorem 3 and 4 are illustrated.



Each upper poly. is $f^{(1,0)}(s)$. $f^{(1,1)}(\mathbf{1}) = 5x^2 + 5x + 1$.

Each lower poly. is $f^{(0,1)}(s)$. $\tilde{f}^{(1,1)} = ((x + 1)f^{(1,1)}(\mathbf{1}) - 1)/x$

$= 5x^2 + 10x + 6.$

<u>Remark</u> 4. Let P be a poset. Then, in [1] the following notations are used. For any s and t in P such that $s \geq t$,

C(s,t;x): the command flow polynomial from s to t.

Note that the coefficient of x^i in C(s,t;x) is the number of covering chains of length i from s to t.

$C(s,t)$: the command flow number from s to t (= $C(s,t;1)$).

Note that $C(s,t)$ is the number of covering chains from s to t.

$C(s;x)$: the command flow polynomial from s.

Note that the coefficient of x^i in $C(s;x)$ is the number of covering chains of length i from s.

$C(s)$: the command flow number from s (= $C(s;1)$).

Note that $C(s)$ is the number of covering chains from s.

$C(P;x)$: the command flow polynomial of P.

Note that the coefficient of x^i in $C(P;x)$ is the number of covering chains of length i in P.

C_P: the command flow number of P (= $C(P;1)$).

Note that C_P is the number of covering chains in P.

A: the adjacency matrix of $H(P)$.

$[t,s]$: the closed interval.

Then, for $f^{(1,0)}$ in $\mathcal{L}(H([t,s]))$, $f^{(1,1)}$ in $\mathcal{L}(H(P))$, we have the following facts.

(1) $f^{(1,0)}(s) = C(s,t;x)$

(2) $f^{(1,0)}(s)|_{x=1} = C(s,t) = (E - A)^{-1}(t,s)$

(3) $f^{(1,0)}(s)|_{x=-1} = (E + A)^{-1}(t,s)$

(4) $f^{(1,1)}(s) = C(s;x)$

(5) $f^{(1,1)}(s)|_{x=1} = C(s) = \sum_{t\in P}(E - A)^{-1}(t,s)$

(6) $f^{(1,1)}(s)|_{x=-1} = \sum_{t\in P}(E + A)^{-1}(t,s)$

(7) $\tilde{f}^{(1,1)} = C(P;x)$

(8) $\tilde{f}^{(1,1)}|_{x=1} = C_P = \sum_{s\in P}\sum_{t\in P}(E - A)^{-1}(t,s)$

(9) $\tilde{f}^{(1,1)}|_{x=-1} = \sum_{s\in P}\sum_{t\in P}(E + A)^{-1}(t,s)$

Remark 5. Let P be a poset. Then, in [1] the following notations are used. For any s and t in P such that $s \gneqq t$,

$C^*(s,t;x)$: the total command flow polynomial from s to t.

Note that the coefficient of x^i in $C^*(s,t;x)$ is the number of chains of length i from s to t.

$C^*(s,t)$: the total command flow number from s to t ($= C^*(s,t;1)$).

Note that $C^*(s,t)$ is the number of chains from s to t.

$C^*(s;x)$: the total command flow polynomial from s.

Note that the coefficient of x^i in $C^*(s;x)$ is the number of chains of length i from s.

$C^*(s)$: the total command flow number from s ($= C^*(s;1)$).

Note that $C^*(s)$ is the number of chains from s.

$C^*(P;x)$: the total command flow polynomial of P.

Note that the coefficient of x^i in $C^*(P;x)$ is the number of chains of length i in P.

C_P^* : the total command flow number of P ($= C^*(P;1)$).

Note that C_P^* is the number of chains in P.

μ: the Möbius function of P ($= \zeta^{-1} = (\delta + n)^{-1}$, ζ:the zeta function, n:the incidence matrix, δ:the delta function).

$[t,s]$: the closed interval.

Then, for $f^{(1,0)}$ in $\mathcal{L}([t,s])$ and $f^{(1,1)}$ in $\mathcal{L}(P)$, we have the following facts.

(1) $f^{(1,0)}(s) = C^*(s,t;x)$

(2) $f^{(1,0)}(s)|_{x=1} = C^*(s,t) = (\delta - n)^{-1}(t,s)$

(3) $f^{(1,0)}(s)|_{x=-1} = \mu(t,s) = (\delta + n)^{-1}(t,s)$

(4) $f^{(1,1)}(s) = C^*(s;x)$

(5) $f^{(1,1)}(s)|_{x=1} = C^*(s) = \sum_{t \in P} (\delta - n)^{-1}(t,s)$

(6) $f^{(1,1)}(s)|_{x=-1} = \sum_{t \in P} \mu(t,s) = \sum_{t \in P} (\delta + n)^{-1}(t,s)$

(7) $\tilde{f}^{(1,1)} = C^*(P;x)$

(8) $\tilde{f}^{(1,1)}|_{x=1} = C_P^* = \sum_{s\in P} \sum_{t\in P} (\delta - n)^{-1}(t,s)$

(9) $\tilde{f}^{(1,1)}|_{x=-1} = \sum_{s\in P} \sum_{t\in P} \mu(t,s) = \sum_{s\in P} \sum_{t\in P} (\delta + n)^{-1}(t,s)$

The Command Flow Number Theory on Boolean Lattices

For a Boolean lattice B_n of n atoms, we use the following notations on the command flow polynomials.

$C_B^{(1,0)}(n;x)$: the command flow polynomial from the top $\mathbb{1}$ to the bottom $\mathbb{0}$, i.e., $C(\mathbb{1},\mathbb{0};x)$.

$C_B^{(1,1)}(n;x)$: the command flow polynomial from the top $\mathbb{1}$, i.e., $C(\mathbb{1};x)$.

$\tilde{C}_B^{(1,1)}(n;x)$: the command flow polynomial of B_n, i.e., $C(B_n;x)$.

Then, we have the following formulas.

(1) $C_B^{(1,0)}(n;x) = \begin{cases} 1 & (n=0) \\ nC^{(1,0)}(n-1;x)x & (n \geqq 1) \end{cases}$

(2) $C_B^{(1,0)}(n;x) = n!x^n$

(3) $C_B^{(1,0)}(n;1) = n!$ (the CF-number from $\mathbb{1}$ to $\mathbb{0}$)

(4) $C_B^{(1,0)}(n;-1) = (-1)^n n!$

(5) $C_B^{(1,1)}(n;x) = \begin{cases} 1 & (n=0) \\ nC_B^{(1,1)}(n-1;x)x + 1 & (n \geqq 1) \end{cases}$

(6) $C_B^{(1,1)}(n;x) = \sum_{k=0}^{n} {}_nP_k x^k$

(7) $C_B^{(1,1)}(n;1) = (\sum_{k=0}^{n} \frac{1}{k!})n! = (e)_n \cdot n!$ (the CF-number from $\mathbb{1}$),

where $(e)_n$ denotes the first n+1 terms of Maclaurin expansion of the constant e.

(8) $C_B^{(1,1)}(n;-1) = (\sum_{k=0}^{n} (-1)^{n-k}\frac{1}{k!})n! = (-1)^n(e^{-1})_n \cdot n!$

$= (-1)^n D(n)$,

where $(e^{-1})_n$ denotes the first n+1 terms of Maclaurin expansion of the constant e^{-1} and D(n) denotes "the well-known number of permutations admitting no coincidences (derangements) of n objects.

(9) $\tilde{C}_B^{(1,1)}(n;x) = \sum_{k=0}^{n} \binom{n}{k} C_B^{(1,1)}(k;x)$

Let $G(\tilde{C}_B^{(1,1)}(n;x);z)$ denote the exponential generating function of $\tilde{C}_B^{(1,1)}(n;x)$.

(10) $G(\tilde{C}_B^{(1,1)}(n;x);z) = \dfrac{e^{2Z}}{1-xz}$

(11) $\tilde{C}_B^{(1,1)}(n;x) = \sum_{k=0}^{n} 2^{n-k}\, {}_nP_k x^k = (\sum_{k=0}^{n} \frac{2^k}{k!} x^{n-k}) \cdot n!$

(12) $\tilde{C}_B^{(1,1)}(n;1) = (\sum_{k=0}^{n} \frac{2^n}{k!}) \cdot n! = (e^2)_n \cdot n!$ (the CF-number of B_n)

(13) $\tilde{C}_B^{(1,1)}(n;-1) = (-1)^n (e^{-2})_n \cdot n!$

(14) $e \cdot n! - C_B^{(1,1)}(n;1) = O(\frac{1}{n})$

(15) $e^2 \cdot n! - \tilde{C}_B^{(1,1)}(n;1) = O(\frac{2^n}{n})$

We use the following notations on the total command flow polynomials.

$C_B^{*(1,0)}(n;x)$: the total command flow polynomial from $\mathbb{1}$ to $\mathbb{0}$, i.e., $C^*(\mathbb{1},\mathbb{0};x)$.

$C_B^{*(1,1)}(n;x)$: the total command flow polynomial from $\mathbb{1}$, i. e., $C^*(\mathbb{1},x)$.

$\tilde{C}_B^{*(1,1)}(n;x)$: the total command flow polynomial of B_n, i.e., $C^*(B_n;x)$.

Then, we have the following formulas.

(1) $C_B^{*(1,0)}(n;x) = \begin{cases} 1 & (n=0) \\ \sum_{k=0}^{n-1} \binom{n}{k} C_B^{*(1,0)}(k;x)x & (n \geqq 1) \end{cases}$

(2) $G(C_B^{*(1,0)}(n;x);z) = \dfrac{1}{1+x-xe^z}$,

where $G(C_B^{*(1,0)}(n;x);z)$ is the exponential generating function of $C_B^{*(1,0)}(n;x)$.

(3) $C_B^{*(1,0)}(n;x) = \sum_{j=0}^{\infty} \frac{x^j}{(x+1)^{j+1}} j^n = \sum_{k=0}^{n} M(n,k,0)x^k$,

$M(n,k,0) = \sum_{i+j=k} (-1)^i \binom{k}{i} j^n$ (the number of surjections from A ($|A| = n$) to B ($|B| = k$)) $= k!S(n,k)$, where $S(n,k)$ is the Stirling number of second kind.

(4) $C_B^{*(1,0)}(n;1) = \sum_{j=0}^{\infty} \frac{j^n}{2^{j+1}} = \sum_{k=0}^{n} M(n,k,0)$

Remark 6. In p.15 and 149 of [4], the following number is defined: for $N_n = \{1, 2, \cdots, n\}$,

S_n: the number of mappings f from N_n into itself such that if f takes a value i then it also takes each value j, $1 \leqq j \leqq i$, $S_0 = 1$). Also, in [5], the following number is dealt:

P(n): the number of total preorders on a n-set.

Each recurrence relation for S_n and P(n) is equal to $(1)|_{x=1}$, and therefore we have the following equality:

$$C_B^{*(1,0)}(n;1) = S_n = P(n).$$

(5) $C_B^{*(1,0)}(n;-1) = \sum_{k=0}^{n} (-1)^k M(n,k,0) = \sum_{k=0}^{n} (-1)^k k!S(n,k)$

$= (-1)^n$ (the Möbius function of B_n)

(6) $C_B^{*(1,1)}(n;x) = \begin{cases} 1 & (n=1) \\ \sum_{k=0}^{n-1} \binom{n}{k} C_B^{*(1,1)}(k;x)x + 1 & (n \geqq 1) \end{cases}$

(7) $G(C_B^{*(1,1)}(n;x);z) = \frac{e^z}{1 + X - xe^z}$

(when $x = 1$, $G(C_B^{*(1,1)}(n;1);z) = \frac{e^z}{2 - e^z}$, by H. Enomoto), hereafter G(f;z) denotes the exponential generating function of f.

(8) $C_B^{*(1,1)}(n;x) = \sum_{j=0}^{\infty} \frac{x^j}{(x+1)^{j+1}} (j+1)^n = \sum_{k=0}^{n} M(n,k,1)x^k,$

$M(n,k,1) = \sum_{i+j=k} (-1)^i \binom{k}{i} (j+1)^n$ (with M. Tsuchiya).

(9) $C_B^{*(1,1)}(n;1) = \sum_{j=0}^{\infty} \frac{(j+1)^n}{2^{j+1}} = \sum_{k=0}^{n} M(n,k,1)$

Note that for $n \neq 0$, from Theorem 1 and 3 or comparing with (4),

$C_B^{*(1,1)}(n;1) = 2C_B^{*(1,0)}(n;1).$

(10) $\tilde{C}_B^{*(1,1)}(n;x) = \sum_{k=0}^{n} \binom{n}{k} C_B^{*(1,1)}(k;x)$

(11) $G(\tilde{C}_B^{*(1,1)}(n;x);z) = \frac{e^{2z}}{1 + x - xe^z}$

(12) $\tilde{C}_B^{*(1,1)}(n;x) = \sum_{j=0}^{\infty} \frac{x^j}{(x+1)^{j+1}} (j+2)^n = \sum_{k=0}^{n} M(n,k,2)x^k,$

$M(n,k,2) = \sum_{i+j=k} (-1)^i \binom{k}{i} (j+2)^n.$

(13) $\tilde{C}_B^{*(1,1)}(n;1) = \sum_{j=0}^{\infty} \frac{(j+2)^n}{2^{j+1}} = \sum_{k=0}^{n} M(n,k,2),$

Note that from Theorem 4 or comparing with (9),

$\tilde{C}_B^{*(1,1)}(n;1) = 2C_B^{*(1,1)}(n;1) - 1.$

We now stand on a stage of introducing the following polynomial with respect to t:

$$M(n,k,t) = \sum_{i+j=k} (-1)^i \binom{k}{i} (j+t)^n.$$

This polynomial has the following property:

$$M(n,k,t) = \begin{cases} M(n,k,t-1) + M(n,k+1,t-1) & (0 \leqq k \leqq n-1) \\ n! & (k = n) \\ 0 & (k \geqq n+1). \end{cases}$$

From $M(n,n,t) = n!$, we obtain the following formula:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^i = \begin{cases} 0 & (0 \leqq i \leqq n-1) \\ (-1)^n n! \ (= C_B^{(1,0)}(n;-1)) & (i = n). \end{cases}$$

Also, by putting $C^*(n,t;x) = \sum_{k=0}^{n} M(n,k,t)x^k$, we obtain the following formulas:

(i) $C^*(n,t;x) = \sum_{k=0}^{n} \binom{n}{k} C^*(n,t-1;x)$

(ii) $C^*(n,t;x) = (C^*(n,t-1;x)(1+x) - (t-1)^n)/x$

(iii) $G(C^*(n,t;x);z) = \dfrac{e^{tz}}{1+x-xe^z}$.

(14) $\lim_{n\to\infty}(C_B^{*(1,0)}(n;1)/\frac{1}{2}(\frac{1}{\log 2})^{n+1}\cdot n!) = 1$ (with T. Ohya)

($\lim_{n\to\infty}(S_n/\frac{1}{2}(\frac{1}{\log 2})^{n+1}\cdot n!) = 1$ in Lovász [4])

($\lim_{n\to\infty}(P(n)/\frac{1}{2}(\frac{1}{\log 2})^{n+1}\cdot n!) = 1$ in Barthelemy [5])

(15) $\limsup_{n\to\infty} (C_B^{*(1,0)}(n;1) - \frac{1}{2}(\frac{1}{\log 2})^{n+1}\cdot n!) = \infty$

$\liminf_{n\to\infty} (C_B^{*(1,0)}(n;1) - \frac{1}{2}(\frac{1}{\log 2})^{n+1}\cdot n!) = -\infty$

(with T. Hilano)

(16) $C_B^{*(1,0)}(n;1) = n!(\frac{1}{2}(\frac{1}{\log 2})^{n+1} + \sum_{k=1}^{\infty} \mathrm{Re}(z_k^{n+1}))$

$\mathrm{Re}(z_k^{n+1}) = \dfrac{1}{(\sqrt{(\log 2)^2 + (2\pi k)^2})^{n+1}} \cdot \cos(n+1)\theta_k$

$\tan\theta_k = (2\pi k)/\log 2$

($P(n) = \dfrac{n!}{2(\log 2)^{n+1}} + \sum_{p=0}^{\infty} (-1)^p \dfrac{B_{n+p+1}}{(n+p+1)\cdot p!}(\log 2)^p$,

B_{n+p+1} is the Bernoullis number, in Barthelemy [5])

The following lemma is useful in obtaining a generating function.

Lemma. Let $F(z)$ denote the exponential generating function $\Sigma_{n=0}^{\infty} f(n)(z^n/n!)$ of $f(n)$ and $G(z)$ denote the exponential generating function $\Sigma_{n=0}^{\infty} g(n)(z^n/n!)$ of $g(n) = \Sigma_{k=0}^{n}\binom{n}{k} f(k)$.

Then, the following identity holds.

$$G(z) = e^{z}F(z).$$

Remark 7. Generally speaking, the so-called computational complexity of the well-known method with matrix operations for computing the number of pathes in a given acyclic digraph is $O(n^{2+\alpha})$ for n = the number of vertices. But, the complexity of our method is $O(\ell)$ for ℓ = the number of arcs. Note that $\ell \leqq n^2$.

The author thinks that "the command flow complexity of a social system" with an order relation is evaluated by the command flow numbers on the system.

Finally, we restate the following open problem.

Open Problem. Decide whether or not there exists $n \geqq 17$ such that $|C_B^{*(1,0)}(n;1) - \frac{1}{2}(\frac{1}{\log 2})^{n+1} \cdot n!| < \frac{1}{2}$.

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