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The Initial Value Problems for the Equations of Viscous Compressible and Perfect Compressible Fluids

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§ 1. Introduction

We consider the following system of equations with a parameter $\varepsilon \in [0,1]$:

$$(1.1)\quad \begin{cases} \rho_t + (\rho u^j)_{x_j} = 0 , \\ u^i_t + u^j u^i_{x_j} + \frac{1}{\rho} p_{x_i} = \frac{\varepsilon}{\rho}\{(\mu(u^i_{x_j} + u^j_{x_i}))_{x_j} + (\mu' u^j_{x_j})_{x_i}\} + f^i , \\ \qquad i = 1,2,3, \\ \theta_t + u^j\theta_{x_j} + \frac{\theta p_\theta}{\rho c_v} u^j_{x_j} = \frac{\varepsilon}{\rho c_v}\{(\kappa\theta_{x_j})_{x_j} + \Psi\} , \end{cases}$$

where $t \geq 0$, $x=(x_1,x_2,x_3) \in \mathbb{R}^3$. The system (1.1) with $\varepsilon=0$ and $\varepsilon=1$ describes the motion of a perfect compressible fluid and that of a viscous compressible fluid respectively. Here the unknown functions ρ, $u=(u^1,u^2,u^3)$ and θ represent the density, velocity and absolute temperature of the fluid ; $f=(f^1,f^2,f^3)$ is the external force and

$\Psi = \frac{\mu}{2}(u^i_{x_j} + u^j_{x_i})^2 + \mu'(u^j_{x_j})^2$ is the dissipation function. It is assumed that the pressure p, the heat capacity at constant volume c_v, the coefficients of viscosity μ and μ' and the coefficient of heat conduction κ are smooth functions of the thermodynamic quantities $\rho>0$ and $\theta>0$, and that

$$(1.2)\quad \begin{cases} p_\rho = \dfrac{\partial p}{\partial \rho}, \; p_\theta = \dfrac{\partial p}{\partial \theta}, \; c_v, \; \mu, \; \kappa > 0, \\ \\ \mu' + \dfrac{2}{3}\mu \geq 0 . \end{cases}$$

The relations between perfect fluids and viscous fluids have been considered by Golovkin [4], Swann [14] and Kato [7] under the hypothesis that the fluids are incompressible. In the present paper these relations are investigated in the case of compressible fluids. The results obtained are the following : a unique smooth solution of the initial value problem for the system (1.1), $\varepsilon \in (0,1]$, of a viscous compressible fluid exists on the time interval $[0,T_0]$ independent of ε, and as $\varepsilon \to 0$ it converges to the smooth solution of the system (1.1), $\varepsilon=0$, of a perfect compressible fluid on the interval $[0,T_0]$. The proof is based on the facts that (1.1), $\varepsilon=0$, is a symmetrizable hyperbolic system and that for $\varepsilon \in (0,1]$ the equations of u and θ in (1.1) can be considered as a parabolic system. However we don't know whether similar results are also true for large t, when we have to consider weak solutions for the system (1.1), $\varepsilon=0$.

As to the initial value problem for the system (1.1) of a compressible fluid, the case of $\varepsilon=1$ (viscous fluid) has been solved locally in time by Nash [13], Itaya [5], [6] and Vol'pert and Hudjaev [18], and globally in time by Matsumura and Nishida [10], [11] for the small initial data, while the case of $\varepsilon=0$ (perfect fluid) has been solved locally in time by [18] and Kato [8]. As to the initial boundary value problem for (1.1), the case of $\varepsilon=1$ has been solved locally in time by Tani [15], [16] and globally in time by Matsumura and Nishida [12] for the small initial data, and for $\varepsilon=0$ the studies are in progress on the existence of the local solution by Ebin [3], Veiga [17] and Agemi [1].

Another interesting problem on the relations between compressible fluids and incompressible fluids is analyzed by Ebin [2] and Klainerman and Majda [9] when the fluids are perfect, the results of which we knew after completion of the present work. The latter also studies the case of viscous fluids under the hypothesis that the flows are barotropic.

§ 2. Main results

For precise formulations of the main results of the paper we introduce some function spaces. H^ℓ ($\ell=0,1,2,\cdots$) denotes the L^2-Sobolev space of order ℓ with the norm $\|\cdot\|_\ell$. For $\ell=0$, we simply write $\|\cdot\|$. We define for $\ell=1,2,\cdots$

$$(2.1)\quad \begin{cases} V^\ell = \{f(x)\ ;\ f \in L^\infty \text{ and } Df \in H^{\ell-1}\} \\ \|f\|_{V^\ell} = \|f\|_{L^\infty} + \|Df\|_{\ell-1}\ , \end{cases}$$

where L^∞ denotes the space of bounded measurable functions on $\mathbb{R}^3$. By the Sobolev's lemma, there is an imbedding of H^2 into $\mathcal{B}^0$:

(2.2) $\quad |f|_0 \le C_0 \| f \|_2 \quad$ for $\quad f \in H^2$,

where C_0 is a constant independent of f, and $\mathcal{B}^0$ denotes the space of bounded continuous functions on $\mathbb{R}^3$, with the norm $|\cdot|_0$. By (2.2), for $\ell \ge 2$ there is an imbedding of H^ℓ into V^ℓ.

Let B be a Banach space with the norm $\| \cdot \|_B$ and T>0. $\mathcal{C}(0,T;B)$ denotes the Banach space of continuous functions u(t) on [0,T] with the values in B, with the norm $\| u \|_{\mathcal{C}(0,T;B)} = \sup_{0\le t\le T} \| u(t) \|_B$. Banach spaces $L^\infty(0,T;B)$ and $L^2(0,T;B)$ are defined similarly.

The solution for (1.1) is sought in the following spaces. Set $Q_T = [0,T] \times \mathbb{R}^3$.

(2.3) $\quad X^s(Q_T) = \{(\rho,u,\theta) \in \mathcal{C}(0,T;H^s)\ ;\ \rho_t \in \mathcal{C}(0,T;H^{s-1}),$

$\qquad (u_t,\theta_t) \in \mathcal{C}(0,T;H^{s-2})\},$

(2.4) $\quad X_V^s(Q_T) = \{(\rho,u,\theta) \in \mathcal{C}(0,T;V^s)\ ;\ \rho_t \in \mathcal{C}(0,T;H^{s-1}),$

$\qquad (u_t,\theta_t) \in \mathcal{C}(0,T;H^{s-2})\}.$

We denote by $Y^s(Q_T)$ and $Y_V^s(Q_T)$ the spaces defined by exchanging $\mathcal{C}(0,T;\cdot)$ for $L^\infty(0,T;\cdot)$ in (2.3) and (2.4) respectively.

The following two theorems are the principal contents of the paper.

Theorem 1 $\quad$ Let $s \ge 3$ and T>0. Assume that the initial data and the external force satisfy the conditions

(2.5) $\quad (\rho,u,\theta)(0) \in V^s, \quad \inf_x \{\rho(0),\theta(0)\} > 0,$

(2.6) $\quad f \in L^2(0,T;H^s) \cap \mathcal{C}(0,T;H^{s-1})$

uniformly with respect to $\varepsilon \in (0,1]$. Then there exists a positive constant T_0 ($\le T$) independent of ε, such that the initial value problem for (1.1), $\varepsilon \in (0,1]$, has a unique solution (ρ,u,θ) which belongs to $X_V^s(Q_{T_0})$ and satisfies $\sqrt{\varepsilon}D(u,\theta) \in L^2(0,T_0;H^s)$ and $\inf_{Q_{T_0}} \{\rho,\theta\} > 0$ uniformly in ε :

(2.7) $\quad \| (\rho,u,\theta)(t) \|_{L^\infty} \le 2 \| (\rho,u,\theta)(0) \|_{L^\infty},$

(2.8) $$\| D(\rho,u,\theta)(t) \|_{s-1}^2 + \varepsilon \int_0^t \| D(u,\theta)(\tau) \|_s^2 d\tau$$

$$\le C_1 (\| D(\rho,u,\theta)(0) \|_{s-1}^2 + \| f \|_{L^2(0,T_0;H^s)}^2),$$

(2.9) $\quad \inf_x \{\rho(t),\theta(t)\} \ge \frac{1}{2} \inf_x \{\rho(0),\theta(0)\}$

for any $t \in [0,T_0]$ and a constant $C_1 > 1$ independent of ε. Here T_0 depends only on $\| (\rho,u,\theta)(0) \|_{V^s}$, $\inf_x \{\rho(0),\theta(0)\}$ and f, and C_1 depends only on $\| (\rho,\theta)(0) \|_{L^\infty}$ and $\inf_x \{\rho(0),\theta(0)\}$.

Remark By this theorem we can show that the set of solutions $\{(\rho^\varepsilon,u^\varepsilon,\theta^\varepsilon)\}_{\varepsilon\in(0,1]} \subset \mathcal{C}(0,T_0;V^{s-1})$ is uniformly bounded and equi-continuous. Therefore by the Ascoli-Arzelà lemma we can choose a convergent subsequence $\{(\rho^{\varepsilon_j},u^{\varepsilon_j},\theta^{\varepsilon_j})\}$, such that $(\rho^{\varepsilon_j},u^{\varepsilon_j},\theta^{\varepsilon_j}) \to (\rho^0,u^0,\theta^0)$ strongly in $\mathcal{C}(0,T_0;V^{s-1})$ as $\varepsilon_j \to 0$. The limit function (ρ^0,u^0,θ^0) belongs to $L^\infty(0,T_0;V^s)$ and satisfies the system (1.1),

$\varepsilon=0$. The convergence of $(\rho^{\varepsilon},u^{\varepsilon},\theta^{\varepsilon})$ to (ρ^0,u^0,θ^0) as $\varepsilon \to 0$ is valid without taking a subsequence, because the solutions of (1.1), $\varepsilon=0$, are unique in $L^{\infty}(0,T_0;V^s)$. Thus the system of a perfect compressible fluid is obtained as the limit of the system of a viscous compressible fluid as the coefficients μ, μ' and κ tend to zero.

On the other hand we obtain the system of a viscous fluid as a perturbation of the system of a perfect fluid.

<u>Theorem 2</u> Assume that the initial value problem for (1.1), $\varepsilon=0$, has a solution $(\rho_0,u_0,\theta_0)(t,x)$ such as

$$(2.10)\quad \begin{cases} (\rho_0,u_0,\theta_0) \in \mathcal{C}(0,T;V^{s+2}), \ (\rho_{0,t},u_{0,t},\theta_{0,t}) \in \mathcal{C}(0,T;H^{s+1}), \\ \inf_{Q_T} \{\rho_0,\theta_0\} > 0 \end{cases}$$

for some $s\geq 3$ and a fixed $T>0$. If the initial data for (1.1), $\varepsilon \in (0,1]$, satisfy

$$(2.11)\quad \frac{1}{\varepsilon}(\rho-\rho_0,u-u_0,\theta-\theta_0)(0) \equiv (\eta,v,\zeta)(0) \in H^s$$

uniformly in ε, then there exists a constant $\varepsilon_0 \in (0,1]$ depending only on (ρ_0,u_0,θ_0), $\|(\eta,v,\zeta)(0)\|_s$ and T such that the initial value problem for (1.1), $\varepsilon \in (0,\varepsilon_0]$, has a unique solution $(\rho,u,\theta)(t,x)$ in Q_T. The solution satisfies $\frac{1}{\varepsilon}(\rho-\rho_0,u-u_0,\theta-\theta_0) \in X^s(Q_T)$ and $\frac{1}{\sqrt{\varepsilon}}(u-u_0,\theta-\theta_0) \in L^2(0,T;H^{s+1})$ uniformly in $\varepsilon \in (0,\varepsilon_0]$:

$$(2.12)\quad \|(\rho-\rho_0,u-u_0,\theta-\theta_0)(t)\|_s^2 + \varepsilon\int_0^t \|(u-u_0,\theta-\theta_0)(\tau)\|_{s+1}^2\, d\tau$$

$$\leq \varepsilon^2 C_2(\|(\eta,v,\zeta)(0)\|_s^2 + C_3)$$

for any $t \in [0,T]$ and constants $C_2>1$ and $C_3>0$ independent of ε. Here C_2 depends only on $\|\rho_0,u_0,\theta_0\|_{\mathcal{C}(0,T;V^{s+1})}$, $\|\rho_{0,t},\theta_{0,t}\|_{\mathcal{C}(0,T;\mathcal{B}^0)}$, $\inf_{Q_T}\{\rho_0,\theta_0\}$ and T, and C_3 depends only on $\|\rho_0,u_0,\theta_0\|_{\mathcal{C}(0,T;V^{s+2})}$, $\inf_{Q_T}\{\rho_0,\theta_0\}$ and T.

§ 3. Linearized equations

We consider the linearized system of (1.1) :

$$
(3.1)\quad \begin{cases}
P^0_{\rho,u}(\eta;v) \equiv \eta_t + K^0_{\rho,u}(\eta,v) = g^0 , \\
P^i_{\rho,u,\theta}(v;\eta,\zeta) \equiv v^i_t + K^i_{\rho,u,\theta}(\eta,v,\zeta) - \varepsilon B^i_{\rho,u,\theta}(v;\eta,\zeta) = g^i , \\
\qquad i = 1,2,3, \\
P^4_{\rho,u,\theta}(\zeta;\eta,v) \equiv \zeta_t + K^4_{\rho,u,\theta}(v,\zeta) - \varepsilon B^4_{\rho,u,\theta}(\zeta;\eta,v) = g^4 ,
\end{cases}
$$

where ρ, $u=(u^1,u^2,u^3)$, θ and g^0, $g=(g^1,g^2,g^3)$, g^4 are given functions, and

$$
(3.2)\quad \begin{cases}
K^0_{\rho,u}(\eta,v) = u^j\eta_{x_j} + \rho v^j_{x_j} , \\
K^i_{\rho,u,\theta}(\eta,v,\zeta) = u^j v^i_{x_j} + \dfrac{p_\rho}{\rho}\eta_{x_i} + \dfrac{p_\theta}{\rho}\zeta_{x_i} , \\
K^4_{\rho,u,\theta}(v,\zeta) = u^j\zeta_{x_j} + \dfrac{\theta p_\theta}{\rho c_v} v^j_{x_j} ,
\end{cases}
$$

$$(3.3)\quad \begin{cases} B^i_{\rho,u,\theta}(v;\eta,\zeta) = \frac{1}{\rho}\{\mu v^i_{x_jx_j} + (\mu+\mu')v^j_{x_jx_i} + (u^i_{x_j}+u^j_{x_i})\cdot \\ \qquad \cdot(\mu_\rho\eta_{x_j}+\mu_\theta\zeta_{x_j}) + u^j_{x_j}(\mu'_\rho\eta_{x_i}+\mu'_\theta\zeta_{x_i})\}, \\ B^4_{\rho,u,\theta}(\zeta;\eta,v) = \frac{1}{\rho c_v}\{\kappa\zeta_{x_jx_j} + \theta_{x_j}(\kappa_\rho\eta_{x_j}+\kappa_\theta\zeta_{x_j}) \\ \qquad + \frac{\mu}{2}(u^i_{x_j}+u^j_{x_i})(v^i_{x_j}+v^j_{x_i}) + \mu' u^i_{x_i}v^j_{x_j}\}, \end{cases}$$

where $p_\rho=p_\rho(\rho,\theta)$ etc.

First we regard the linear system (3.1) with a fixed $\varepsilon \in (0,1]$ as a single hyperbolic equation in η and as a parabolic system in (v,ζ). Then we have :

<u>Proposition 3.1</u> Let $s\geq 3$, $2\leq \ell \leq s$ and $T>0$. Assume that $(\rho,u,\theta) \in \mathcal{C}(0,T;V^s)$, $\inf_{Q_T}\{\rho,\theta\} > 0$ and $g^0 \in L^2(0,T;H^\ell) \cap \mathcal{C}(0,T;H^{\ell-1})$, $(g,g^4) \in \mathcal{C}(0,T;H^{\ell-1})$ and that $(\eta,v,\zeta)(0) \in H^\ell$. Then the initial value problem for the linear system (3.1), $\varepsilon \in (0,1]$, has a unique solution $(\eta,v,\zeta) \in X^\ell(Q_T)$, $(v,\zeta) \in L^2(0,T;H^{\ell+1})$ which satisfies the energy estimate

$$(3.4)\quad \|(\eta,v,\zeta)(t)\|^2_\ell + \varepsilon\nu\int_0^t \|(v,\zeta)(\tau)\|^2_{\ell+1}\,d\tau$$
$$\leq e^{C_4 t/\varepsilon}\{\|(\eta,v,\zeta)(0)\|^2_\ell + \int_0^t \|g^0(\tau)\|^2_\ell\,d\tau$$
$$+ \frac{C_5}{\varepsilon}\int_0^t \|(g,g^4)(\tau)\|^2_{\ell-1}\,d\tau\}$$

for any $t \in [0,T]$ and positive constants ν, C_4 and C_5 independent of ε. Here ν and C_5 depend only on $\|\rho,\theta\|_{\mathcal{C}(0,T;L^\infty)}$ and $\inf_{Q_T}\{\rho,\theta\}$, and C_4 depends only on $\|\rho,u,\theta\|_{\mathcal{C}(0,T;V^s)}$ and $\inf_{Q_T}\{\rho,\theta\}$.

Using this proposition, we also have a solution when the initial data belong to V^ℓ.

<u>Corollary 3.2</u> Assume that (ρ,u,θ) and (g^0,g,g^4) are the same as Proposition 3.1 and that $(\eta,v,\zeta)(0) \in V^\ell$. Then there exists a unique solution $(\eta,v,\zeta) \in X_V^\ell(Q_T)$, $D(v,\zeta) \in L^2(0,T;H^\ell)$ of the initial value problem for (3.1), $\varepsilon \in (0,1]$.

Proof. For any $(\eta,v,\zeta)(0) \in V^\ell$, there exists $(\eta',v',\zeta') \in V^{\ell+1} \cap \mathcal{B}^{\ell+1}$ such that

$$(3.5)\qquad (\hat{\eta},\hat{v},\hat{\zeta})(0) \equiv (\eta,v,\zeta)(0) - (\eta',v',\zeta') \in H^\ell .$$

Therefore we seek the solution of (3.1) in the form

$$(3.6)\qquad (\eta,v,\zeta)(t,x) = (\eta',v',\zeta')(x) + (\hat{\eta},\hat{v},\hat{\zeta})(t,x).$$

The equations for $(\hat{\eta},\hat{v},\hat{\zeta})$ are

$$(3.7)\qquad \begin{cases} P^0_{\rho,u}(\hat{\eta};\hat{v}) = \hat{g}^0 , \quad P^i_{\rho,u,\theta}(\hat{v};\hat{\eta},\hat{\zeta}) = \hat{g}^i , \\ \\ P^4_{\rho,u,\theta}(\hat{\zeta};\hat{\eta},\hat{\zeta}) = \hat{g}^4 , \end{cases}$$

where $\hat{g}^i \equiv g^i - K^i_{\rho,u,\theta}(\eta',v',\zeta') + \varepsilon B^i_{\rho,u,\theta}(v';\eta',\zeta')$ etc. Applying Proposition 3.1 to the problem (3.7)-(3.5), we have a unique solution $(\hat{\eta},\hat{v},\hat{\zeta}) \in X^\ell(Q_T)$, $(\hat{v},\hat{\zeta}) \in L^2(0,T;H^{\ell+1})$, and consequently the desired solution of (3.1) by (3.6). This completes the proof of

Corollary 3.2.

Next we show the solution $(\eta,v,\zeta) \in X^{\ell}(Q_T)$ of (3.1), $\varepsilon \in (0,1]$, satisfies the uniform energy estimate with respect to ε.

Proposition 3.3 Let $s\geq 3$, $1\leq \ell\leq s$ and $T>0$. Assume that $(\rho,u,\theta) \in Y_V^s(Q_T)$, $\sqrt{\varepsilon}D(u,\theta) \in L^2(0,T;H^s)$, $\inf_{Q_T}\{\rho,\theta\} > 0$ and $\theta_t \in L^2(0,T;H^{s-1})$ uniformly in $\varepsilon \in (0,1]$ and put

$$(3.8) \qquad \| D(\rho,u,\theta)(t) \|_{s-1}^2 + \varepsilon\int_0^t \| D(u,\theta)(\tau) \|_s^2 \, d\tau \leq E^2 ,$$

$$(3.9) \qquad |\rho_t(t)|_0^2 + \int_0^t |\theta_t(\tau)|_0^2 \, d\tau \leq N^2 .$$

Further assume that $(g^0,g,g^4) \in L^2(0,T;H^{\ell}) \cap L^{\infty}(0,T;H^{\ell-1})$ and $(\eta,v,\zeta)(0) \in H^{\ell}$ uniformly in ε. Then the solution $(\eta,v,\zeta) \in Y^{\ell}(Q_T)$, $(v,\zeta) \in L^2(0,T;H^{\ell+1})$ of (3.1), $\varepsilon \in (0,1]$, satisfies the uniform energy estimate in ε :

$$(3.10) \qquad \| (\eta,v,\zeta)(t) \|_{\ell}^2 + \varepsilon\int_0^t \| (v,\zeta)(\tau) \|_{\ell+1}^2 \, d\tau$$

$$\leq C_6 \, e^{C_7(t+\sqrt{t})} \{ \| (\eta,v,\zeta)(0) \|_{\ell}^2 + \int_0^t \| (g^0,g,g^4)(\tau) \|_{\ell}^2 \, d\tau \}$$

for any $t \in [0,T]$ and constants $C_6>1$ and $C_7>0$ independent of ε. Furthermore the solution has a regularity $(\eta,v,\zeta) \in \mathcal{C}(0,T;H^{\ell})$. Here C_6 depends only on $\| \rho,\theta \|_{L^{\infty}(0,T;L^{\infty})}$ and $\inf_{Q_T}\{\rho,\theta\}$ and C_7 depends only on $\| \rho,u,\theta \|_{L^{\infty}(0,T;L^{\infty})}$, $\inf_{Q_T}\{\rho,\theta\}$, E and N.

Corollary 3.4 Assume that (ρ,u,θ) and (g^0,g,g^4) are the same as Proposition 3.3 and that $(\eta,v,\zeta)(0) \in V^{\ell}$ uniformly in $\varepsilon \in (0,1]$. Then the solution $(\eta,v,\zeta) \in Y_V^{\ell}(Q_T)$, $D(v,\zeta) \in L^2(0,T;H^{\ell})$ of (3.1),

$\varepsilon \in (0,1]$, satisfies the uniform estimate in ε :

$$(3.11) \qquad \| D(\eta,v,\zeta)(t) \|_{\ell-1}^2 + \varepsilon\int_0^t \| D(v,\zeta)(\tau) \|_\ell^2 \, d\tau$$

$$\le C_6 \, e^{C_7(t+\sqrt{t})} \{ \| D(\eta,v,\zeta)(0) \|_{\ell-1}^2 + \int_0^t \| D(g^0,g,g^4)(\tau) \|_{\ell-1}^2 \, d\tau \}$$

for any $t \in [0,T]$, where C_6 and C_7 are constants in (3.10). Furthermore we have a regularity $(\eta,v,\zeta) \in \mathcal{C}(0,T;V^\ell)$ for $\ell \ge 2$.

Proof of Proposition 3.3. First we show (3.10) under the assumptions that $(\rho,u,\theta) \in Y_V^{s+1}(Q_T)$, $(g^0,g,g^4) \in L^\infty(0,T;H^\ell)$ uniformly in ε and $(\eta,v,\zeta) \in Y^{\ell+2}(Q_T)$. Then the estimate (3.10) is also valid under the assumptions of Proposition 3.3 by use of the Friedrichs' mollifier.

Now we apply D^k $(0 \le k \le \ell)$ to (3.1) :

$$(3.12) \qquad \begin{cases} P^0_{\rho,u}(D^k\eta; D^k v) = G^{0,k}(\eta,v) , \\ P^i_{\rho,u,\theta}(D^k v; D^k\eta, D^k\zeta) = G^{i,k}(\eta,v,\zeta) , \\ P^4_{\rho,u,\theta}(D^k\zeta; D^k\eta, D^k v) = G^{4,k}(\eta,v,\zeta) , \end{cases}$$

where $G^{i,k}(\eta,v,\zeta) \equiv D^k g^i - [D^k, P^i_{\rho,u,\theta}](\eta,v,\zeta)$ and $[D^k, P^i_{\rho,u,\theta}](\eta,v,\zeta) \equiv D^k P^i_{\rho,u,\theta}(v;\eta,\zeta) - P^i_{\rho,u,\theta}(D^k v; D^k\eta, D^k\zeta)$ etc. Noting that (3.12), $\varepsilon=0$, is a symmetrizable hyperbolic system in $D^k(\eta,v,\zeta)$, we multiply the equations of $D^k\eta$, $D^k v^i$ and $D^k\zeta$ by $\frac{P_\rho}{\rho} D^k\eta$, $\rho D^k v^i$ and $\frac{\rho c_v}{\theta} D^k\zeta$ respectively and integrate them with respect to $x \in \mathbb{R}^3$:

$$(3.13) \qquad \sum_{k=0}^{\ell} \int \frac{P_\rho}{\rho} D^k\eta \cdot P^0_{\rho,u}(D^k\eta; D^k v) + \rho D^k v^i \cdot P^i_{\rho,u,\theta}(D^k v; D^k\eta, D^k\zeta)$$

$$+ \frac{\rho c_v}{\theta} D^k\zeta \cdot P^4_{\rho,u,\theta}(D^k\zeta; D^k\eta, D^k v)dx = \sum_{k=0}^{\ell} \int \frac{p_\rho}{\rho} D^k\eta \cdot G^{0,k}(\eta,v)$$

$$+ \rho D^k v^i \cdot G^{i,k}(\eta,v,\zeta) + \frac{\rho c_v}{\theta} D^k\zeta \cdot G^{4,k}(\eta,v,\zeta)dx .$$

Integrating by parts, we obtain

(3.14) $$\text{the left member of (3.13)} \geq \frac{1}{2}\frac{\partial}{\partial t} E_\ell(t) + \varepsilon\nu_1 \| v,\zeta \|^2_{\ell+1}$$

$$- C(1 + |\rho_t, \theta_t|_0) \| \eta,v,\zeta \|^2_\ell - \varepsilon C \| \eta,v,\zeta \|_\ell \| v,\zeta \|_{\ell+1}$$

for any $t \in [0,T]$ and constants $\nu_1 > 0$ and $C > 0$ independent of ε, where we defined the energy norm by

(3.15) $$E_\ell(t) = \sum_{k=0}^{\ell} \int E_{\rho,\theta}(D^k\eta, D^k v, D^k\zeta)(t)dx$$

with $E_{\rho,\theta}(\eta,v,\zeta) = \frac{p_\rho}{\rho} |\eta|^2 + \rho|v|^2 + \frac{\rho c_v}{\theta} |\zeta|^2$. The function $E_\ell(t)$ is equivalent to the norm $\| (\eta,v,\zeta)(t) \|^2_\ell$:

(3.16) $$\alpha_1 \| \eta,v,\zeta \|^2_\ell \leq E_\ell \leq \alpha_2 \| \eta,v,\zeta \|^2_\ell$$

for constants $\alpha_1 < \alpha_2$ independent of ε. Here ν_1, α_1 and α_2 depend only on $\| \rho,\theta \|_{L^\infty(0,T;L^\infty)}$ and $\inf_{Q_T} \{\rho,\theta\}$, and C depends only on $\| \rho, u,\theta \|_{L^\infty(0,T;V^s)}$ and $\inf_{Q_T} \{\rho,\theta\}$.

In order to estimate the right hand side of (3.13) we need the estimates for $[D^k, P^j_{\rho,u,\theta}](\eta,v,\zeta)$:

(3.17) $$\begin{cases} \sum_{j=0}^{4} \sum_{k=1}^{\ell} \| [D^k, K^j_{\rho,u,\theta}](\eta,v,\zeta) \| \leq C \| D(\eta,v,\zeta) \|_{\ell-1} , \\ \sum_{j=1}^{4} \sum_{k=1}^{\ell} \| [D^k, B^j_{\rho,u,\theta}](\eta,v,\zeta) \| \leq C \| D(v,\zeta) \|_\ell \\ \qquad + C(1 + \| D(u,\theta) \|_s) \| D(\eta,v,\zeta) \|_{\ell-1} . \end{cases}$$

Taking into account the definitions of $G^{i,k}(\eta,v,\zeta)$ and $P^{i}_{\rho,u,\theta}(v;\eta,\zeta)$ etc., we obtain by (3.14) and (3.17) :

$$\text{(3.18)}\qquad \frac{\partial}{\partial t}E_{\ell}(t) + \varepsilon\nu_1\|v,\zeta\|^2_{\ell+1} \le \|g^0,g,g^4\|^2_{\ell}$$

$$+ C(1+\varepsilon\|D(u,\theta)\|_s + |\rho_t,\theta_t|_0)\|\eta,v,\zeta\|^2_{\ell}$$

for any $t \in [0,T]$ and a constant $C=C(\nu_1)$ independent of ε. The estimate (3.10) is obtained by the integration of (3.18) with respect to $t \in [0,T]$, where we use (3.16) and the conditions (3.8) and (3.9).

Finally we show $(\eta,v,\zeta) \in \mathcal{C}(0,T;H^{\ell})$. Apply the Friedrichs' mollifier $\phi_{\delta}*$ to (3.1). Then $(\eta_{\delta},v_{\delta},\zeta_{\delta}) \equiv \phi_{\delta}*(\eta,v,\zeta)$ belongs to $\mathcal{C}(0,T;H^{\ell})$ and satisfies (3.1) with g^j replaced by $g^{j,\delta}$, where $g^{i,\delta} \equiv \phi_{\delta}*g^i-[\phi_{\delta}*,P^i_{\rho,u,\theta}](\eta,v,\zeta)$ and $[\phi_{\delta}*,P^i_{\rho,u,\theta}](\eta,v,\zeta) \equiv \phi_{\delta}*P^i_{\rho,u,\theta}(v;\eta,\zeta)-P^i_{\rho,u,\theta}(v_{\delta};\eta_{\delta},\zeta_{\delta})$ etc. Therefore the difference $(\hat{\eta},\hat{v},\hat{\zeta})_{\delta,\delta'} \equiv (\eta_{\delta}-\eta_{\delta'},v_{\delta}-v_{\delta'},\zeta_{\delta}-\zeta_{\delta'})$ satisfies (3.1) with g^j replaced by $g^{j,\delta}-g^{j,\delta'}$. Apply the estimate (3.10) to the system for $(\hat{\eta},\hat{v},\hat{\zeta})_{\delta,\delta'}$:

$$\text{(3.19)}\qquad \|(\hat{\eta},\hat{v},\hat{\zeta})_{\delta,\delta'}(t)\|^2_{\ell} + \varepsilon\int_0^t \|(\hat{v},\hat{\zeta})_{\delta,\delta'}(\tau)\|^2_{\ell+1}\,d\tau$$

$$\le C(T)\{\|(\hat{\eta},\hat{v},\hat{\zeta})_{\delta,\delta'}(0)\|^2_{\ell} + \sum_{j=0}^{4}\int_0^t \|(g^{j,\delta}-g^{j,\delta'})(\tau)\|^2_{\ell}\,d\tau\} .$$

Since $g^{j,\delta}-g^{j,\delta'} \to 0$ strongly in $L^2(0,T;H^{\ell})$ as $\delta,\delta' \to 0$, we have $(\hat{\eta},\hat{v},\hat{\zeta})_{\delta,\delta'} \to 0$ strongly in $\mathcal{C}(0,T;H^{\ell})$ as $\delta,\delta' \to 0$, which implies $(\eta_{\delta},v_{\delta},\zeta_{\delta}) \to (\eta,v,\zeta)$ strongly in $\mathcal{C}(0,T;H^{\ell})$ as $\delta \to 0$. Thus we have $(\eta,v,\zeta) \in \mathcal{C}(0,T;H^{\ell})$. This completes the proof of Proposition 3.3.

§ 4. Invariant set under iterations

We consider the linear system of (1.1) :

$$
(4.1)\quad \begin{cases} P^0_{\rho,u}(\eta;v) = 0\ , \quad P^i_{\rho,u,\theta}(v;\eta,\zeta) = f^i, \quad i=1,2,3, \\ P^4_{\rho,u,\theta}(\zeta;\eta,v) = 0\ , \end{cases}
$$

with the initial data

$$
(4.2)\quad (\eta,v,\zeta)(0) = (\rho,u,\theta)(0).
$$

Let (η,v,ζ) be a solution of the problem (4.1), (4.2). We show that the mapping $(\rho,u,\theta) \to (\eta,v,\zeta)$ has an invariant set in the following sense. We introduce the set $Z^s_T = Z^s_T(M,m,E,N)$: $(\rho,u,\theta) \in Z^s_T$ means that $(\rho,u,\theta) \in X^s_V(Q_T)$, $\sqrt{\varepsilon}D(u,\theta) \in L^2(0,T;H^s)$, $\inf\{\rho,\theta\} > 0$ and $\theta_t \in L^2(0,T;H^{s-1})$ and that

$$
(4.3)\quad \begin{cases} \| (\rho,u,\theta)(t) \|_{L^\infty} \le M\ , \quad \inf_x \{\rho(t),\theta(t)\} \ge m > 0\ , \\ \| D(\rho,u,\theta)(t) \|^2_{s-1} + \varepsilon\int_0^t \| D(u,\theta)(\tau) \|^2_s d\tau \le E^2\ , \\ |\rho_t(t)|^2_0 + \int_0^t |\theta_t(\tau)|^2_0\, d\tau \le N^2 \qquad \text{for} \quad t \in [0,T]. \end{cases}
$$

First we note that if $(\rho,u,\theta) \in Z^s_T(M,m,E,N)$ and if $(\eta,v,\zeta) \in X^s_V(Q_T)$, $\sqrt{\varepsilon}D(v,\zeta) \in L^2(0,T;H^s)$ is a solution of (4.1), $\varepsilon \in (0,1]$, satisfying for any $t \in [0,T]$

$$
\| D(\eta,v,\zeta)(t) \|^2_{s-1} + \varepsilon\int_0^t \| D(v,\zeta)(\tau) \|^2_s d\tau \le \hat{E}^2\ ,
$$

then $\zeta_t \in L^2(0,T;H^{s-1})$ and the estimate

$$(4.4)\qquad |\eta_t(t)|_0^2 + \int_0^t |\zeta_t(\tau)|_0^2 \, d\tau \le C_8(1+T)\hat{E}^2$$

holds for any $t \in [0,T]$, where $C_8=C_8(M,m,E)$ is a constant independent of ε and N.

Now assume the conditions (2.5) and (2.6) in Theorem 1 and set

$$(4.5)\qquad \begin{cases} M = 2\|(\rho,u,\theta)(0)\|_{L^\infty}, \quad m = \frac{1}{2}\inf_x \{\rho(0),\theta(0)\}, \\ E_\tau^2 = 2C_6(\|D(\rho,u,\theta)(0)\|_{s-1}^2 + \|f\|_{L^2(0,\tau;H^s)}^2), \\ N_\tau^2 = C_8(1+\tau)E_\tau^2, \end{cases}$$

where $C_6=C_6(M,m)$ and $C_8=C_8(M,m,E_\tau)$ are constants in Proposition 3.3 and (4.4) respectively.

Proposition 4.1 Let $s\ge 3$ and $T>0$. Assume that the initial data (4.2) and the external force satisfy the conditions (2.5) and (2.6) respectively and set the constants M, m, E_τ and N_τ by (4.5). Then there exists a positive constant T_0 ($\le T$) independent of $\varepsilon \in (0,1]$ such that if $(\rho,u,\theta) \in Z_{T_0}^s = Z_{T_0}^s(M,m,E_0,N_0)$, then the problem (4.1), (4.2) has a unique solution $(\eta,v,\zeta)(t,x)$ for any $\varepsilon \in (0,1]$ which belongs to the same $Z_{T_0}^s$, where $E_0=E_{T_0}$ and $N_0=N_{T_0}$.

Proof. By Corollary 3.2 ($\ell=s$) we have a unique solution $(\eta,v,\zeta) \in X_V^s(Q_{T_0})$, $D(v,\zeta) \in L^2(0,T_0;H^s)$. And by Corollary 3.4 ($\ell=s$) the solution satisfies

$$(4.6)\qquad \|D(\eta,v,\zeta)(t)\|_{s-1}^2 + \varepsilon\int_0^t \|D(v,\zeta)(\tau)\|_s^2 \, d\tau$$

$$\le C_6 e^{C_7(t+\sqrt{t})}(\|D(\rho,u,\theta)(0)\|_{s-1}^2 + \|f\|_{L^2(0,t;H^s)}^2)$$

$$\leq \frac{1}{2} e^{C_7(T_0+\sqrt{T_0})} E_0^2 \leq E_0^2$$

for any $t \in [0,T_0]$ provided $e^{C_7(T_0+\sqrt{T_0})} \leq 2$, where $C_7=C_7(M,m,E_0,N_0)$ is a constant in Proposition 3.3. Next integrating (4.1) with respect to $t \in [0,T_0]$, we have a constant $C_9=C_9(M,m,E_0)$ independent of ε such that

$$\|(\eta,v,\zeta)(t)\|_{L^\infty} \leq \|(\rho,u,\theta)(0)\|_{L^\infty}$$

$$+ C_9(E_0 + \|f\|_{\mathcal{C}(0,T_0;H^2)})(T_0 + \sqrt{T_0}) \leq M ,$$

$$\inf_x \{\eta(t),\zeta(t)\} \geq \inf_x \{\rho(0),\theta(0)\} - C_9E_0(T_0 + \sqrt{T_0}) \geq m$$

for $t \in [0,T_0]$ provided that $C_9(E_0+\|f\|_{\mathcal{C}(0,T_0;H^{s-1})})(T_0+\sqrt{T_0}) \leq \|(\rho,u,\theta)(0)\|_{L^\infty}$ and $2C_9E_0(T_0+\sqrt{T_0}) \leq \inf_x\{\rho(0),\theta(0)\}$, where we use (4.6). Finally from (4.4) and (4.6) we have

$$|\eta_t(0)|_0^2 + \int_0^t |\zeta_t(\tau)|_0^2 \, d\tau \leq C_8(1 + T_0)E_0^2 = N_0^2$$

for $t \in [0,T_0]$. This completes the proof of Proposition 4.1.

§ 5. Proof of Theorem 1

We consider the system (1.1), $\varepsilon \in (0,1]$, of a viscous compressible fluid :

$$(5.1) \quad \begin{cases} P^0_{\rho,u}(\rho;u) = 0 , \quad P^i_{\rho,u,\theta}(u;\rho,\theta) = f^i , \quad i=1,2,3, \\ P^4_{\rho,u,\theta}(\theta;\rho,u) = 0 , \end{cases}$$

with the initial data $(\rho,u,\theta)(0)$ satisfying (2.5). Let us introduce the successive approximate sequence $\{(\rho,u,\theta)_n(t,x)\}_{n=0}^{\infty}$ as follows :

$$(5.2)_0 \qquad (\rho,u,\theta)_0(t,x) \equiv (\overline{\rho},0,\overline{\theta}) \ ,$$

where $\overline{\rho}$ and $\overline{\theta}$ are constants satisfying $\inf_x \rho(0) \le \overline{\rho} \le \|\rho(0)\|_{L^\infty}$, $\inf_x \theta(0) \le \overline{\theta} \le \|\theta(0)\|_{L^\infty}$.

$$(5.2)_n \quad \begin{cases} P^0_{\rho_{n-1},u_{n-1}}(\rho_n;u_n) = 0 \ , \quad P^i_{\rho_{n-1},u_{n-1},\theta_{n-1}}(u_n;\rho_n,\theta_n) = f^i \ , \\ P^4_{\rho_{n-1},u_{n-1},\theta_{n-1}}(\theta_n;\rho_n,u_n) = 0 \ , \\ (\rho,u,\theta)_n(0) = (\rho,u,\theta)(0), \quad n=1,2,\cdots \ . \end{cases}$$

By Proposition 4.1 the sequence $(\rho,u,\theta)_n$ is uniformly bounded with respect to $n \ge 0$ and $\varepsilon \in (0,1]$:

$$(5.3) \qquad (\rho,u,\theta)_n \in Z^s_{T_0} \qquad \text{for all} \quad n \ge 0 \ .$$

We show the convergence of the sequence $(\rho,u,\theta)_n$. Subtract the system $(5.2)_n$ from $(5.2)_{n+1}$. The difference $(\hat{\rho},\hat{u},\hat{\theta})_n \equiv (\rho_{n+1}-\rho_n, u_{n+1}-u_n, \theta_{n+1}-\theta_n)$ satisfies

$$(5.4) \quad \begin{cases} P^0_{\rho_n,u_n}(\hat{\rho}_n;\hat{u}_n) = g^0_n \ , \quad P^i_{\rho_n,u_n,\theta_n}(\hat{u}_n,\hat{\rho}_n,\hat{\theta}_n) = g^i_n \ , \\ P^4_{\rho_n,u_n,\theta_n}(\hat{\theta}_n;\hat{\rho}_n,\hat{u}_n) = g^4_n \ , \\ (\hat{\rho},\hat{u},\hat{\theta})_n(0) = 0 \ , \qquad n=1,2,\cdots \ , \end{cases}$$

where $g^i_n \equiv -\{P^i_{\rho_n,u_n,\theta_n}(u_n;\rho_n,\theta_n) - P^i_{\rho_{n-1},u_{n-1},\theta_{n-1}}(u_n;\rho_n,\theta_n)\}$ etc.

Apply Proposition 3.1 (ℓ=s-1) to (5.4). We have $(\hat\rho,\hat u,\hat\theta)_n \in \mathcal{C}(0,T;H^{s-1})$ and

$$(5.5)\quad \|(\hat\rho,\hat u,\hat\theta)_n(t)\|_{s-1}^2 + \varepsilon\nu\int_0^t \|(\hat u,\hat\theta)_n(\tau)\|_s^2\,d\tau$$

$$\le e^{C_4 t/\varepsilon}\{\int_0^t \|g_n^0(\tau)\|_{s-1}^2\,d\tau + \frac{C_5}{\varepsilon}\sum_{j=1}^4 \int_0^t \|g_n^j(\tau)\|_{s-2}^2\,d\tau\}\ .$$

By the estimates for composite functions we know a constant C independent of n≥0 such that

$$(5.6)\quad \|g_n^0\|_{s-1},\ \sum_{j=1}^4 \|g_n^j\|_{s-2} \le C\|(\hat\rho,\hat u,\hat\theta)_{n-1}\|_{V^{s-1}}\ .$$

Substituting (5.6) into (5.5), we have for $t \in [0,T_0]$,

$$\|(\hat\rho,\hat u,\hat\theta)_n(t)\|_{V^{s-1}}^2 \le C_\varepsilon(T_0)\int_0^t \|(\hat\rho,\hat u,\hat\theta)_{n-1}(\tau)\|_{V^{s-1}}^2\,d\tau\ .$$

Therefore as $n \to \infty$ the limit exists such as $(\rho,u,\theta)_n \to (\rho,u,\theta)$ strongly in $\mathcal{C}(0,T_0;V^{s-1})$ for every fixed $\varepsilon \in (0,1]$. On the other hand it follows from (5.3) that $D(u,\theta)_{n'} \to D(u,\theta)$ weakly in $L^2(0,T_0;H^s)$ and that $D(\rho,u,\theta)_{n''} \to D(\rho,u,\theta)$ weakly in H^{s-1} for every fixed $t \in [0,T_0]$, where n' and n"=n"(t) are subsequences of n and n' respectively. Thus the limit (ρ,u,θ) is a solution of (5.1), $\varepsilon \in (0,1]$, satisfying $(\rho,u,\theta) \in Y_V^s(Q_{T_0})$ and $D(u,\theta) \in L^2(0,T_0;H^s)$. Then by Corollary 3.4 we have $(\rho,u,\theta) \in \mathcal{C}(0,T_0;V^s)$. The estimates (2.7), (2.8) and (2.9) are consequences of (5.3), where we take $C_1 \equiv 2C_6$. This completes the proof of Theorem 1.

§ 6. Approximation of the viscous fluids by the perfect fluids

Let us assume that $(\rho_0,u_0,\theta_0)(t,x)$ is a solution of (1.1), $\varepsilon=0$, satisfying (2.10). We seek the solution of (1.1), $\varepsilon \in (0,1]$, in the form

$$(6.1)\qquad (\rho,u,\theta)(t,x) = (\rho_0,u_0,\theta_0)(t,x) + \varepsilon(\eta,v,\zeta)(t,x).$$

Substituting (6.1) into (1.1), then we have the system for (η,v,ζ) :

$$(6.2)\qquad \begin{cases} \hat{P}^0_{\rho,u}(\eta;v) \equiv P^0_{\rho,u}(\eta;v) + M^0(\eta,v) = g^0 \equiv 0\ , \\ \hat{P}^i_{\rho,u,\theta}(v;\eta,\zeta) \equiv P^i_{\rho,u,\theta}(v;\eta,v) + M^i(\eta,v,\zeta) - \varepsilon L^i_{\rho,\theta}(v) \\ \qquad = g^i(\rho,\theta), \quad i=1,2,3, \\ \hat{P}^4_{\rho,u,\theta}(\zeta;\eta,v) \equiv P^4_{\rho,u,\theta}(\zeta;\eta,v) + M^4(\eta,v,\zeta) - \varepsilon L^4_{\rho,\theta}(v,\zeta) \\ \qquad = g^4(\rho,\theta), \end{cases}$$

where (ρ,u,θ) and $P^j_{\rho,u,\theta}$ are given by (6.1) and (3.1) respectively, and

$$(6.3)\qquad M^i(\eta,v,\zeta) \equiv v^j u^i_{0,x_j} + \eta\{(\frac{\overline{p_\rho}}{\rho})_\rho\, \rho_{0,x_i} + (\frac{\overline{p_\theta}}{\rho})_\rho\, \theta_{0,x_i}\}$$
$$+ \zeta\{(\frac{\overline{p_\rho}}{\rho})_\theta\, \rho_{0,x_i} + (\frac{\overline{p_\theta}}{\rho})_\theta\, \theta_{0,x_i}\} \quad \text{etc.},$$

$$(6.4)\qquad L^i_{\rho,\theta}(v) \equiv \frac{1}{\rho}\{(\mu_\rho \rho_{0,x_j} + \mu_\theta \theta_{0,x_j})(v^i_{x_j} + v^j_{x_i})$$
$$+ (\mu'_\rho \rho_{0,x_i} + \mu'_\theta \theta_{0,x_i}) v^j_{x_j}\} \quad \text{etc.},$$

(6.5) $g^i(\rho,\theta) \equiv \frac{1}{\rho}\{\mu u^i_{0,x_j x_j} + (\mu+\mu')u^j_{0,x_j x_i} + (\mu_\rho \rho_{0,x_j} + \mu_\theta \theta_{0,x_j})\cdot$

$\cdot(u^i_{0,x_j} + u^j_{0,x_i}) + (\mu'_\rho \rho_{0,x_i} + \mu'_\theta \theta_{0,x_i})u^j_{0,x_j}\}$ etc.,

where $\bar{a} \equiv \int_0^1 a(\rho_0+\varepsilon\tau\eta, \theta_0+\varepsilon\tau\zeta)d\tau$.

We write M^j and $L^j_{\rho,\theta}$ as follows :

$$(6.6)\quad \begin{cases} M^j(\eta,v,\zeta) = M^j_0\eta + M^j_i v^i + M^j_4\zeta, & j=0,1,\cdots,4, \\ L^j_{\rho,\theta}(v,\zeta) = L^j_{ik}v^i_{x_k} + L^j_{4k}\zeta_{x_k}, & j=1,\cdots,4. \end{cases}$$

If (ρ,u,θ), M^j_i, L^j_{ik} and g^j are given functions, then (6.2) is a linear system in (η,v,ζ) whose principal part is identical with that of (3.1). Therefore the arguments in §3 are applicable to the linear system (6.2) with slight modification.

<u>Proposition 6.1</u> Let $s\geq 3$ and $T>0$.

(i) Let $2\leq \ell\leq s$. In addition to the hypotheses of Proposition 3.1 we assume that M^j_i and $L^j_{ik} \in \mathcal{C}(0,T;V^s)$. Then the initial value problem for the linear system (6.2), $\varepsilon \in (0,1]$, has a unique solution $(\eta,v,\zeta) \in X^\ell(Q_T)$, $(v,\zeta) \in L^2(0,T;H^{\ell+1})$ which satisfies the energy estimate (3.4) for C_4 replaced by a constant C'_4 depending only on C_4 and $\|M^j_i, L^j_{ik}\|_{\mathcal{C}(0,T;V^s)}$.

(ii) Let $1\leq \ell\leq s$. In addition to the hypotheses of Proposition 3.3 we assume that $M^j_i, L^j_{ik} \in L^\infty(0,T;V^s)$ uniformly in ε. Then the solution $(\eta,v,\zeta) \in Y^\ell(Q_T)$, $(v,\zeta) \in L^2(0,T;H^{\ell+1})$ of the linear system (6.2), $\varepsilon \in (0,1]$, satisfies the uniform energy estimate in ε :

(6.7) $\|(\eta,v,\zeta)(t)\|^2_\ell + \varepsilon\int_0^t \|(v,\zeta)(\tau)\|^2_{\ell+1}d\tau$

$$\leq C_6 e^{C_7'(t+\sqrt{t})} \{ \| (\eta,v,\zeta)(0) \|_\ell^2 + \int_0^t \| (g^0,g,g^4)(\tau) \|_\ell^2 d\tau \} ,$$

for $t \in [0,T]$, where C_7' is a constant depends only on C_7 and $\| M_i^j, L_{ik}^j \|_{L^\infty(0,T;V^s)}$, and C_6 and C_7 are given in Proposition 3.3. Furthermore we have a regularity $(\eta,v,\zeta) \in \mathcal{C}(0,T;H^\ell)$.

Now let us assume that (ρ,u,θ) in (6.2) is given in the form

$$(6.8) \qquad (\rho,u,\theta) = (\rho_0,u_0,\theta_0) + \varepsilon(\eta',v',\zeta') ,$$

and let (η,v,ζ) be a solution of the linear system (6.2). Then we show that the mapping $(\eta',v',\zeta') \to (\eta,v,\zeta)$ has an invariant set $W_T^s(R,K)$ for some R and K : $(\eta,v,\zeta) \in W_T^s(R,K)$ means that $(\eta,v,\zeta) \in X^s(Q_T)$, $\sqrt{\varepsilon}(v,\zeta) \in L^2(0,T;H^{s+1})$ and $\zeta_t \in L^2(0,T;H^{s-1})$ and that

$$(6.9) \qquad \begin{cases} \| (\eta,v,\zeta)(t) \|_s^2 + \varepsilon \int_0^t \| (v,\zeta)(\tau) \|_{s+1}^2 d\tau \leq R^2 , \\ |\eta_t(t)|_0^2 + \int_0^t |\zeta_t(\tau)|_0^2 d\tau \leq K^2 \qquad \text{for } t \in [0,T]. \end{cases}$$

We note the following. Let (ρ_0,u_0,θ_0) satisfy (2.10) and $(\eta', v',\zeta') \in W_T^s(R,K)$. And we choose a constant $\varepsilon_0 \in (0,1]$ as

$$(6.10) \qquad \begin{cases} \varepsilon_0 C_0 R \leq \| \rho_0,u_0,\theta_0 \|_{\mathcal{C}(0,T;L^\infty)} , \quad 2\varepsilon_0 C_0 R \leq \inf_{Q_T} \{\rho_0,\theta_0\} , \\ 6\varepsilon_0^2 R^2 \leq (1+T) \| D(\rho_0,u_0,\theta_0) \|_{\mathcal{C}(0,T;H^s)}^2 , \\ 6\varepsilon_0^2 K^2 \leq (1+T) \| \rho_{0,t},\theta_{0,t} \|_{\mathcal{C}(0,T;\mathcal{B}^0)}^2 , \end{cases}$$

where C_0 is a constant in (2.2). Then (ρ,u,θ) given by (6.8) satisfies

$$(6.11) \qquad (\rho,u,\theta) \in Z_T^s(M,m,E,N) \qquad \text{for } \varepsilon \in (0,\varepsilon_0] ,$$

where we take the constants as follows :

$$(6.12)\quad \begin{cases} M = 2\| \rho_0,u_0,\theta_0\|_{\mathcal{C}(0,T;L^\infty)} , \quad m = \frac{1}{2}\inf_{Q_T} \{\rho_0,\theta_0\} , \\ E^2 = 2(1+T)\| D(\rho_0,u_0,\theta_0)\|^2_{\mathcal{C}(0,T;H^s)} , \\ N^2 = 2(1+T)\| \rho_{0,t},\theta_{0,t}\|^2_{\mathcal{C}(0,T;\mathcal{B}^0)} . \end{cases}$$

Then the coefficients of (6.2) have the estimates :

$$(6.13)\qquad \| M^j_i,L^j_{ik}\|_{\mathcal{C}(0,T;V^s)} \le C_{10} , \quad \sum_j \| g^j\|_{\mathcal{C}(0,T;H^s)} \le C_{11}$$

for $\varepsilon \in (0,\varepsilon_0]$ and constants C_{10} and C_{11} independent of ε. Here C_{10} and C_{11} depend on $\| \rho_0,u_0,\theta_0\|_{\mathcal{C}(0,T;V^{s+1})}$ and $\| \rho_0,u_0,\theta_0\|_{\mathcal{C}(0,T;V^{s+2})}$ respectively. Furthermore if $(\eta,v,\zeta) \in X^s(Q_T)$, $\sqrt{\varepsilon}(v,\zeta) \in L^2(0,T;H^{s+1})$ is a solution of the linear system (6.2), $\varepsilon \in (0,\varepsilon_0]$, satisfying

$$\| (\eta,v,\zeta)(t)\|^2_s + \varepsilon\int_0^t \| (v,\zeta)(\tau)\|^2_{s+1}\, d\tau \le \hat{R}^2 ,$$

then we have $\zeta_t \in L^2(0,T;H^{s-1})$ and

$$(6.14)\qquad |\zeta_t(t)|^2_0 + \int_0^t |\zeta_t(\tau)|^2_0\, d\tau \le C'_8(1+T)(\hat{R}^2+C^2_{11})$$

for a constant $C'_8=C'_8(C_8,C_{10})$ independent of ε, where C_8 is a constant in (4.4).

Now we determine the constants R and K :

$$(6.15)\quad \begin{cases} R^2 = C_6\, e^{C'_7(T+\sqrt{T})}(\| (\eta,v,\zeta)(0)\|^2_s + TC^2_{11}) , \\ K^2 = C'_8(1+T)(R^2+C^2_{11}) , \end{cases}$$

where C_6 and C_7' are constants in Proposition 6.1 (ii). Then by (6.11), (6.13), Proposition 6.1 (i), (ii) and (6.14) the set $W_T^s(R,K)$ is invariant under the mapping $(\eta',v',\zeta') \to (\eta,v,\zeta)$:

<u>Proposition 6.2</u> Let $s\geq 3$ and $T>0$. Assume that (ρ_0,u_0,θ_0) and $(\eta,v,\zeta)(0)$ satisfy (2.10) and (2.11) respectively and set the constants R and K by (6.15). Also assume that (ρ,u,θ) is given by (6.8) with $(\eta',v',\zeta') \in W_T^s(R,K)$. Then there exists a constant $\varepsilon_0 \in (0,1]$ depending only on (ρ_0,u_0,θ_0), $\|(\eta,v,\zeta)(0)\|_s$ and T (cf. (6.10) and (6.15)), such that the initial value problem for the linear system (6.2) has a unique solution $(\eta,v,\zeta)(t,x)$ for any $\varepsilon \in (0,\varepsilon_0]$, which belongs to the same $W_T^s(R,K)$.

Finally we solve the nonlinear system (6.2) by the successive approximation scheme. We introduce the sequences $\{(\eta,v,\zeta)_n(t,x)\}_{n=0}^{\infty}$ and $\{(\rho,u,\theta)_n(t,x)\}_{n=0}^{\infty}$ as follows :

$$(6.16)_0 \qquad (\eta,v,\zeta)_0(t,x) = 0 ,$$

$$(6.17)_0 \qquad (\rho,u,\theta)_0(t,x) = (\rho_0,u_0,\theta_0)(t,x),$$

$$(6.16)_n \qquad \begin{cases} \hat{P}^0_{\rho_{n-1},u_{n-1}}(\eta_n;v_n) = 0 , \\ \hat{P}^i_{\rho_{n-1},u_{n-1},\theta_{n-1}}(v_n;\eta_n,\zeta_n) = g^i(\rho_{n-1},\theta_{n-1}), \\ \hat{P}^4_{\rho_{n-1},u_{n-1},\theta_{n-1}}(\zeta_n;\eta_n,v_n) = g^4(\rho_{n-1},\theta_{n-1}), \\ (\eta,v,\zeta)_n(0) = (\eta,v,\zeta)(0), \end{cases}$$

$$(6.17)_n \qquad (\rho,u,\theta)_n(t,x) = (\rho_0,u_0,\theta_0)(t,x) + \varepsilon(\eta,v,\zeta)_n(t,x), \qquad n=1,2,\cdots .$$

By Proposition 6.2 and (6.11) the sequences $(\eta,v,\zeta)_n$ and $(\rho,u,\theta)_n$ are uniformly bounded with respect to $n\geq 0$ and $\varepsilon \in (0,\varepsilon_0]$:

(6.18) $\quad (\eta,v,\zeta)_n \in W_T^s(R,K), \quad (\rho,u,\theta)_n \in Z_T^s(M,m,E,N)$

for constants in (6.15) and (6.12). The convergence of the sequence $(\eta,v,\zeta)_n$ is shown in the same way as §5, by using Proposition 6.1 (i) and (6.18). Thus as a limit of $(\eta,v,\zeta)_n$ we obtain a solution (η,v,ζ) of (6.2), and consequently a solution (ρ,u,θ) of (1.1) for any $\varepsilon \in (0,\varepsilon_0]$. This completes the proof of Theorem 2.

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