数理解析研究所講究録
第 430 巻 1981 年 51-67

On the spatial decay of incompressible viscous fluid motion past objects

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1. Introduction

The purpose of this paper is to investigate the decay rate of the solutions of the Navier-Stokes equations (1.1) w.r.t. space variables:

$$(1.1)\quad \begin{cases} \partial/\partial t\, u - \nu\Delta u + (u\nabla)u + \operatorname{grad} p = 0 & \\ \operatorname{div} u = 0 & \text{in } Q_T = \Omega\times(0,T) \\ u(x,t) \to u_\infty \text{ as } |x|\to\infty . & \end{cases}$$

We suppose Ω is an exterior domain in R^3, $\partial\Omega$ is sufficiently smooth, and for convenience, Ω^c contains the origin of the coordinate system. We also suppose u_∞ is a constant vector. With an appropriate change of variables,(1.1) is transformed into

$$(1.2)\quad \begin{cases} \partial/\partial t\, v - \Delta v + (v\nabla)v + \partial/\partial x_1\, v + \operatorname{grad} p = 0 & \\ \operatorname{div} v = 0 & \text{in } Q_T . \\ v(x,t) \to 0 \text{ as } |x|\to\infty . & \end{cases}$$

We consider (1.2) under suitable initial and boundary conditions:

$$(1.3)\quad \begin{cases} v(\ ,0) = v_0 , \\ v|_{\partial\Omega} \text{ is smooth.} \end{cases}$$

On the decay rate of the stationary solutions of (1.2), R.Finn earned many important results. And K.I.Babenko finally obtained the conclusion : every stationary solution that has finite Dirichlet norm $\{\int_{\Omega}|\nabla v|^2 dx\}^{1/2}$ decays satisfying

$$|v(x)| \leqq C'|x|^{-1}(1+s_x)^{-1} \qquad ; \quad s_x = |x| - x_1 ,$$

with some constant C' . We apply his method in this paper .

As to the non-stationary solutions, G.H.Knightly [8][9] and J. Bemelmans [2] considerd the decay problem and constracted the solutions that have certain decay properties.

Here we are concerned mainly in the decay rate of every classical solution that satisfies $\int_{\Omega}|\nabla v(x,t)|^2 dx \leqq C$; $0 \leqq t \leqq T$. We prove the following two theorems.

Theorem 1. Let v be a classical solution of (1.2),(1.3) in Q_T ($T<\infty$) such that

(i) $\nabla v \in L^{\infty}(0,T;\ L^2(\Omega))$,

(ii) $|v_0(x)| \leqq C|x|^{-\lambda}$ for some $\lambda, C>0$.

Then there exists a constant C' depending on T and v such that

$$(1.4) \qquad |v(x,t)| \leqq C'|x|^{-\min(2,\lambda)} \qquad \text{for all } (x,t)\in Q_T .$$

Remark. If we assume the additional condition $\int_{\partial\Omega} v(x,t)\cdot n\ dx = 0$ ($0 \leqq t \leqq T$) , then the exponent of $|x|$ in (1.4) is improved as $-\min(3,\lambda)$.

Remark. We can also prove the same decay of "local mean" of a weak solution v if $v \in L^s(0,T; L^r(\Omega))$ with $r > 3$, $s \leq \infty$, $3/r + 2/s \leq 1$. Spatial regularity and uniqueness theorem are known in this class of solutions. But "mean decay" can be shown independently.

Theorem 2. Let v be a classical solution of (1.2) , (1.3) in Q_T $(T \leq \infty)$. Assume:

(i) $\nabla v \in L^\infty(0,T; L^2(\Omega))$,

(ii) $v_0 = v_s + v_{d0}$; v_s is a stationary solution of (1.2) with finite Dirichlet norm,

; v_{do} satisfies

$$|v_{d0}(x)| \leq C\,|x|^{-2} .$$

In the case $T = \infty$ we further assume:

(iii) $v \in L^\infty(0,\infty; L^r(\Omega))$ for some $r < 3$,

(iv) $\lim_{t,|x| \to \infty} |v(x,t)| = 0$.

Then there exists a constant C' depending on v such that

$$|v(x,t)| \leq C'\,|x|^{-1}\,(1 + s_x)^{-1} \qquad \text{for all } (x,t) \in Q_T.$$

Remark. Both C and C' are used to express various constants. Generally we use C for given or such constants that are determined by some norms of v, while C' expresses constants which can't be explicitly given with such data.

As an application, we show the following

Corollary. The global solution of (1.2),(1.3) which J.Heywood and K.Masuda constructed converges to the stationary solution like

$|x|^{-1} (1+ s_x)^{-1} t^{-1/12 + \varepsilon}$ for every $\varepsilon > 0$ if the initial data decays suitablly.

Remark . See Heywood [6],[7] and Masuda [11]. This solution satisfies the assumptions of Theorem 2, and satisfis

$$(1.5) \quad |v(x,t) - v_s(x)| \leqq C\, t^{-1/4} \qquad \text{for all } (x,t) \in Q_\infty ,$$

$$(1.6) \quad \|\nabla v(\ ,t) - \nabla v_s\|_{L^2(\Omega)} \leqq C\, t^{-1/4} \qquad \text{for all } 0 < t .$$

We depend on these facts.

2. Preliminaries

2.1 Fundamental solution

The fundamental solution tensor E of the linearized system of equations associated with (1.2) i.e.

$$(2.1) \quad \begin{cases} \partial/\partial t\, v - \Delta v + \partial/\partial x_1\, v + \text{grad } p = 0 , \\ \text{div } v = 0 , \end{cases}$$

is given by the following definitions:

$$(2.2) \quad \Gamma(x,t) = (4\pi t)^{-3/2} \exp(-|x-te_1|^2/4t) , \qquad e_1 = (1,0,0).$$

$$(2.3) \quad E_{ij}(x,t) = \delta_{ij}\Gamma(x,t) + 1/4\pi \ \partial^2/\partial x_i \partial x_j \int_{R^3} \Gamma(x-y,t)|y|^{-1}\, dy,$$

$$Q_i(x,t) = - 1/4\pi \ \delta(t)\, \partial/\partial x_i (1/|x|) .$$

To justify this , first we observe

$$(\partial/\partial t - \Delta + \partial/\partial x_1)\,\Gamma(x,t) = \delta(x)\,\delta(t)\ .$$

But then formal calculus shows

$$(\partial/\partial t - \Delta + \partial/\partial x_1) E_{ij}(x,t) + \partial/\partial x_j\, Q_i(x,t) = 0\ ,$$
$$\partial/\partial x_j\, E_{ij}(x,t) = 0\ .$$

(The same index in a term means to take the sum of the terms with that index 1 to 3.)

2.2 Estimates of E

The construction of the fundamental solution tensor E ,and also the following lemma 1 , is owing to Solonnikov [13]. The detailed proof of lemma 2 and 3 is omitted here, but they are elementary.

Lemma 1. The following inequalities hold with some constants C.

(2.4) $$|E_{ij}(x,t)| \leqq C\,(t + |x - te_1|^2)^{-3/2}\ ,$$

(2.5) $$|\nabla E_{ij}(x,t)| \leqq C\,(t + |x - te_1|^2)^{-2}\ .$$

Lemma 2. The following estimates hold independently of t,

(2.6) $$(t + |x - te_1|^2)^{-1} \leqq C \begin{cases} |x|^{-1}(1 + s_x)^{-1} & \text{if } |x| \geqq 1, \\ |x|^{-2} & \text{if } |x| < 1. \end{cases}$$

Lemma 3. For all $p > 1$, there exists a constants C such that

$$(2.7)\qquad \int_0^\infty (\ t + |x - te_1|^2\ )^{-p}\, dt \leqq C \begin{cases} |x|^{-p+1/2}\ (\ 1 + s_x)^{-p+1/2} & \text{if } |x| \geqq 1\ , \\ |x|^{-2p+2} & \text{if } |x| < 1\ . \end{cases}$$

We also need the following elementary

Lemma 4. For all $\lambda > 0$, there exists a constant C such that

$$(2.8)\qquad \int_{\Omega} \Gamma(x-y,t)\ |y|^{-\lambda}\, dy \leqq C\ (\ t + |x - te_1|^2\ )^{-\lambda/2}\ .$$

2.3 Integral representation

The integral representation for smooth solutions of (1.2),(1.3) is given by Knightly [8],[9]. This representation is valid for even weak solutions. As we are here concerned with the classical solution , we don't demonstrate it. The representation formula is :

$$(2.9)\qquad v = L[v] + N[v]\ ,$$

$$(2.10)\qquad L[v] = I_1[v_0] + I_2[v_0] + \Sigma_1[v] + \Sigma_2[v]\ ,$$

$$(2.11)\qquad I_1[v_0](x,t) = \int_{\Omega} \Gamma(x-y,t)\ v_0(y)\ dy\ ,$$

$$(2.12)\qquad I_2[v_0](x,t) = \int_{\partial\Omega} K(x-y,t)\ \ v_0(y)\cdot n_y\ dy\ ,$$

where n_y is the normal vector of $\partial\Omega$ at y ,and

$$(2.13)\qquad K(x,t) = 1/4\pi\ \ \nabla \int_{R^3} \Gamma(x-z,t)\ |z|^{-1}\, dz\ ,$$

$$(2.14)\quad \Sigma_1[v]_i(x,t) = \int_0^t d\tau \int_{\partial\Omega} \Big\{ [\ E_{ij}(x-y,t-\tau)\Big(-p(y,\tau)\delta_{ij} + \partial/\partial y_k\ v_j + \partial/\partial y_j\ v_k\Big) - v_j(y,\tau)\Big(\partial/\partial y_k E_{ij} + \partial/\partial y_j\ E_{ik}\Big)]\ n_{yk} - E_{ij}v_j n_{yl} - E_{ij}v_j v_k n_{yk} \Big\}\ dy\ ,$$

$$(2.15)\quad \Sigma_2[v]_i(x,t) = \int_{\partial\Omega} -1/4\pi\ \partial/\partial x_i\ (|x|^{-1})\ v\ (y,t)\cdot n_y\ dy\ ,$$

$$(2.16)\quad N[v]_i(x,t) = \int_0^t \int_{\Omega} \partial/\partial y_k\ E_{ij}(x-y,t-\tau)\ v_j(y,\tau)\ v_k(y,\tau)\ dy\ d\tau\ .$$

2.4 On N[v]

Proposition 1. The operator $\mathcal{E}(t)$ defined by

$$\mathcal{E}(t)\ :\ f \ \mapsto\ \int_{\Omega} E(x-y,t)\ f(y)\ dy$$

is a bounded operator $L^q(\Omega) \to L^r(\Omega)$, the norm of which is smaller than $C\ (\ t^{-1/2\ -3/2\sigma} + t^{1\ -3/2\sigma}\ )$, where $1/r = 1/q - 1/\sigma$, C is indépendant of t .

Proof. The Fourier transform of E is expressed as

$$\mathcal{F}(\ \partial/\partial x_k\ E_{ij}(x,t))(\xi) = \sqrt{-1}\ \xi_k\ (\ \delta_{ij} - \xi_i\xi_j/|\xi|^2\ )\ \exp\ (-t|\xi|^2 - \sqrt{-1}\ t\ \xi_1\ )\ .$$

Its ξ derivatives can be evaluated by the following inequality with the polynomials p_α and q_α of order $|\alpha|$:

$$|\xi^\alpha \partial_\xi^\alpha \mathcal{F}(\nabla E(\ ,t))(\xi)| \leq C|\xi|(p_\alpha(t|\xi|^2) + q(t|\xi|)) \exp(-t|\xi|^2) .$$

Hence

$$|\xi_1\xi_2\xi_3|^{1/\sigma} |\xi^\alpha \partial_\xi^\alpha \mathcal{F}(\nabla E(\ ,t))(\xi)| \leq C (t^{-1/2\ -3/2\sigma} + t^{-1/2\ -3/2\sigma\ +|\alpha|/2} .$$

By virtue of the theorem of Marcinkiewicz - Mikhlin - Lizorkin, the last inequality implies proposition 1 . ///

In this paper, we only use the proposition 1 in the case $\sigma = \infty$.

Proposition 2. If $\nabla v \in L(0,T; L^2(\Omega))$, then $N[v] \in L(0,T; L^3(\Omega))$.

Proof. By virtue of the imbedding theorem of Sobolev, $v \in L(0,T: L^6(\Omega))$. Apply prop.1 with $r = 3$, $\sigma = \infty$. Then the proof is completed . ///

2.5 A theorem of R.Finn.

The following theorem is owing to R.Finn .

Theorem (R.Finn) . Let $v(x)$ satisfy the following inequality with some constant C :

$$|v(x)| \leq C|x|^{-1} (1 + s_x)^{-1} + \bar{N}[v](x) ,$$

where

$$\overline{N}[v](x) = \int_{\Omega} \Phi(x-y)\,|v(y)|^2\,dy ,$$

and Φ satisfies

$$\Phi(x) \leqq C \begin{cases} |x|^{-3/2}(1+s_x)^{-3/2} & \text{if } |x| \geqq 1 , \\ |x|^{-2} & \text{if } |x| < 1 . \end{cases}$$

Then, if v decays as follows:

$$|v(x)| \leqq C|x|^{-\alpha} \qquad \text{for some } C>0,\ \alpha > 1/2 \text{ , and all } x ,$$

v really decays more strongly , that is, satisfying

$$|v(x)| \leq C|x|^{-1}(1+s_x)^{-1} \qquad \text{for some } C>0 \text{ and all } x .$$

We use this theorem for $\Phi(x) = \int_0^\infty |\nabla E(x,t)|\,dt$.

2.6 A basic lemma

The following lemma is based on the argument of Babenko [1] .Both theorem 1 and 2 are proved as applications of this lemma. We give a simpler proof.

Lemma (K.I.Babenko). Let $\varphi(\ )$ be a function such that

(i) $\varphi(R) \geqq 0$ for all sufficiently large R,

(ii) $\varphi(R) \to 0$ as $R \to \infty$,

(iii) there are constants C_1, C_2, $\alpha>0$, $\beta>1$ such that for all sufficiently large R the following inequality holds:

$$(2.17)\qquad \varphi(R) \leqq C_1 R^{-\alpha} + C_2 \{\varphi(R/2)\}^{\beta} .$$

Then, for all $\varepsilon > 0$, there exists R_ε such that if $R > R_\varepsilon$, $\varphi(R) \leqq (C_1 + \varepsilon) R^{-\alpha}$.

Proof. Let $M = (2^{\alpha+1} C_2)^{-1/(\beta-1)}$ and let $\bar{R}_1 = \max \{ \sup\{ R; \varphi(R) \geqq M\} + 1, (2C_1/M)^{1/\alpha} \}$.
Induction shows $\varphi(2^m R_1) \leqq 2^{-m\alpha} M$ for all integer m and all R_1 , $R_1 \in [\bar{R}_1, 2\bar{R}_1)$.
Next we choose $\bar{R}_2$ such that

$$\bar{R}_2^{-\alpha\beta+\alpha} \cdot C_2 K^{\beta} 2^{\alpha\beta} < \varepsilon \quad ; \quad K = M(2R_1)^{\alpha} .$$

Now if $R \geqq \max(2\bar{R}_1, \bar{R}_2)$, then

$$\varphi(R) \leqq C_1 R^{-\alpha} + C_2(K (R/2)^{-\alpha})^{\beta}$$

$$(C_1 + \varepsilon) R^{-\alpha} .$$

///

3. Proof of theorem 1

It is clear that L[v] satisfies

$$(3.1)\qquad L[v](x,t) \leqq C(|x|^{-\lambda} + |x|^{-2})$$

with a constant depending on T and for all $(x,t) \in Q_T$.
To estimate N[v], let us define for fixed x and R,

$$\Omega_1 = \{ y \in \Omega ; |x-y| < R \}, \qquad \Omega_2 = \Omega \setminus \Omega_1$$

Now let

$$N_k[v](x,t) = \int_0^t \int_{\Omega_k} \nabla E(x-y,t-\tau)\ v(y,\tau)\ v(y,\tau)\ dy\, d\tau \ ; \ k = 1,2$$

so that $N[v] = N_1[v] + N_2[v]$. It is easy to show

$$(3.2) \quad |N_1[v](x,t)| \leqq C\ (1+t)^3 \sup_{|x-y|<R,\ 0\le\tau\le t} |v(y,\tau)|^2 ,$$

$$(3.3) \quad |N_2[v](x,t)| \leqq C\ (1+t)^3 \sup_{0\le\tau\le t} \{\| v(\ ,\tau)\|_{L^3(\Omega)}\}^2 \cdot R^{-3} .$$

By proposition 2, (3.3) implies

$$(3.4) \quad |N_2[v](x,t)| \leqq C\ (1+t)^3\ R^{-3} .$$

Let $R = |x|/2$ in (3.1), (3.2), (3.4). Togethet with (2.9),those inequalities imply

$$(3.5) \quad |v(x,t)| \leqq C_1 (|x|^{-\lambda} + |x|^{-2} + |x|^{-3}) + \sup_{|y|\ge|x|/2,\ 0\le\tau\le t} \{ |v(y,\tau)| \}^2 .$$

Take sup of both sides of (3.5) over $\{ (x,t); |x|\ge R,\ 0\le t\le T\} = S_{R,T}$, obtaining

$$(3.6) \quad \sup_{S_{R,T}} |v(x,t)| \leqq C_1 |x|^{-\min(2,\lambda)} + C_2 \{ \sup_{S_{R/2,T}} |v(x,t)| \}^2 .$$

Hence we can apply Babenko's lemma to $\sup_{S_{R,T}} |v(x,t)| = \varphi(R)$, concluding

$$\sup_{S_{R,T}} |v(x,t)| \leqq (C_1 + \varepsilon) R^{-\min(2,\lambda)}$$

for large R and therefore, for all x, t,

$$|v(x,t)| \leqq C' |x|^{-\min(2,\lambda)} . \qquad ///$$

4 Proof of theorem 2

It is easy to show under assumptions of theorem 2 that

$$(4.1) \qquad |L[v](x,t)| \leqq C|x|^{-1} (1 + s_x)^{-1} .$$

Let us define this time

$$\Omega_1 = \Omega \cap \{ x ; (x_1^2 + x_2^2)^{1/2} > R \} ,$$

$$\Omega_2 = \Omega \cap \{ x ; |x_1| > R \} \setminus \Omega_1 ,$$

$$\Omega_3 = \Omega \setminus (\Omega_1 \cup \Omega_2) .$$

Let $N_k[v]$ be defined in the same way as in the preceding section, so that $N[v] = N_1[v] + N_2[v] + N_3[v]$ this time. Calculation shows

$$(4.2) \qquad |N_1[v](x,t)| \leqq \int_0^\infty \left\{ \int_{\Omega_1} |\nabla E(x-y,t-\tau)|^p \, dy \right\}^{1/p} d\tau$$

$$\sup_{0 \le \tau \le t} \left\{ \| v(\cdot ,\tau)\|_{L^{2q}(\Omega)} \right\}^2$$

$$\leqq C R^{-2+3/p} \sup_{0\le\tau\le t}\{\| v(\ ,\tau)\|_{L^{2q}(\Omega)}\}^2 ,$$

(4.3) $$|N_2[v](x,t)| \leq C R^{-1+3/2p} \sup_{0\le\tau\le t}\{\| v(\ ,\tau)\|_{L^{2q}(\Omega)}\}^2 ,$$

(4.4) $$|N_3[v](x,t)| \leqq \int_0^{t-1}\left\{\int_{\Omega_3} |\nabla E(x-y,t-\tau)|^p\, dy\right\}^{1/p} d\tau$$

$$\cdot \sup_{0\le\tau\le t-1}\{\| v(\ ,\tau)\|_{L^{2q}(\Omega)}\}^2$$

$$+ \int_{t-1}^{t}\int_{\Omega_3} |E(x-y,t-\tau)|\, dy\, d\tau \cdot \sup_{y\in\Omega_3,\ t-1\le\tau\le t} |v(y,\tau)|^2$$

$$\leqq C \sup_{0\le\tau\le t}\{\| v(\ ,\tau)\|_{L^{q(1-\varepsilon)}(\Omega)}\}^{1-\varepsilon} \sup_{y\in\Omega_3,\ 0\le\tau\le t} |v(y,\tau)|^{1+\varepsilon}$$

for every $\varepsilon > 0$ in (4.4) and every p ; $p > 3/2$ in (4.2) $\sim$ (4.4). Let $q = r/2$ in (4.2) and (4.3) , $(1-\varepsilon)\, q = r$ in (4.4), so that

(4.5) $$|N[v](x,t)| \leqq C\,(R^{1-6/r} + R^{1/2-3/r})$$

$$+ C \sup_{y\in\Omega_3,\ 0\le\tau\le t} |v(y,\tau)|^{1+\varepsilon} .$$

Now let $R = |x|/2\sqrt{2}$ so that $y \in \Omega_3$ implies $|x-y| \leqq |x|/2$. Then, from (4.1) and (4.5),

(4.6) $$|v(x,t)| \leqq C_1 |x|^{-\alpha} + C_2 \sup_{|y|\ge|x|/2,\ 0\le\tau\le t} |v(y,\tau)|^{1+\varepsilon} .$$

where $-\alpha = 1/2 - 3/r < -1/2$.

By virtue of assumption (iv) and theorem 1, $|v(x,t)| \to 0$ as $|x| \to \infty$ uniformly in $0 \leqq t$. Hence essentially the same argument as at the end of the preceding section lead us to

$$(4.7) \quad |v(x,t)| \leqq C'|x|^{-\alpha} .$$

By Finn's theorem, (4.7) implies

$$|v(x,t)| \leqq C'|x|^{-1} (1 + s_x)^{-1} .$$ ///

5 Proof of corollary

Let $v_d = v - v_s$. (See the remark after corollary.) We define

$$N[f,g](x,t) = \int_0^t \int_\Omega \nabla E(x-y,t-\tau)\, f(y,\tau)\, g(y,\tau)\, dy\, d\tau .$$

Then v_d is represented as follows:

$$(5.1) \quad v_d = L[v] + N[v_d,v_d] + N[v_d,v_s] + N[v_s,v_d] .$$

We have already seen

$$(5.2) \quad |v_d(x,t)| \leqq C' \min\{|x|^{-1} (1 + s_x)^{-1} , t^{-1/4}\},$$

$$(5.3) \quad |v_s(x,t)| \leqq C'|x|^{-1} (1+ s_x)^{-1} .$$

(5.2) can be replaced by

(5.4) $|v_d(x,t)| \leqq C' \, t^{-1/8} |x|^{-1/2} (1+ s_x)^{-1/2}$.

We state a lemma to go further, but don't prove it.

Lemma. For every p ; $p > 1$, and α ; $\alpha > 0$, there exists a constant C such that

(5.5) $$I_{\alpha,p}(x,t) = \int_0^t \tau^{-\alpha} \{ t-\tau + |x-(t-\tau)e_1|^2 \}^{-p} \, d\tau$$

$$\leqq C \begin{cases} t^{-\alpha/2} |x|^{-p+1/2} (1+ s_x)^{-p+1/2} & \text{if } |x| \geqq 1, \\ t^{-\alpha} |x|^{-2p+2} & \text{if } |x| < 1. \end{cases}$$

Then, from (5.2), (5.3), (5.4), we get

(5.6) $|v_d(x,t)| \leqq C' \, t^{-1/16} |x|^{-1} (1+ s_x)^{-1}$,

and by this and (5.2),

(5.7) $|v_d(x,t)| \leqq C' \, t^{-1/8 -1/32} |x|^{-1/2} (1+ s_x)^{-1/2}$.

This improves (5.4) . Repeated argument proves

(5.8) $|v_d(x,t)| \leqq C' \, t^{-1/12 +\varepsilon} |x|^{-1} (1+ s_x)^{-1}$,

instead of (5.6) , which ends the proof . ///

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