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Compact energy surface of a Hamiltonian system

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Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ be points of $\mathbb{R}^n$ and

$H = H(x, y) : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$

a smooth function.

We consider a Hamiltonian system

(H) $\dot{x}_k = H_{y_k}$, $\dot{y}_k = -H_{x_k}$, $k = 1, \ldots, n$.

Along a solution $(x(t), y(t))$ of (H) , $H(x(t), y(t))$ is a constant, so, for fixed $e \in \mathbb{R}$, the set

$H^{-1}(e) = \{ (x, y) ; H(x, y) = e \}$

is an invariant set of the system (H) , called an energy surface.

We assume that

(A) e is a regular value of H , that is, there are no critical points of H on $H^{-1}(e)$.

Then $H^{-1}(e)$ is a smooth submanifold of $\mathbb{R}^{2n}$.

If $H^{-1}(e)$ is not compact, there is not necessarily periodic orbit on it (for example $H = \frac{1}{2}|y|^2 + x_n$) .

Rabinowitz [1] proved that, if $H^{-1}(e)$ is star-shaped, then there exists at least one periodic orbit on it.

Whether " star-shaped " can be replaced by " homeomorphic to the sphere " (or more optimistically " compact ") is not known.

Classically, H is the sum of the kinetic energy T and the potential U , that is,

$$(1) \qquad H = \sum_{i,j=1}^{n} a^{ij}(x)\, y_i\, y_j + U(x) \quad ,$$

where (a^{ij}) is symmetric and positive definite.

We have

Theorem. Assume thet $H = H(x, y)$ is given by (1) . If for some $e \in \mathbb{R}$, H satisfies (A) and $H^{-1}(e)$ is compact, then there exists at least one periodic orbit on $H^{-1}(e)$.

In this case, (H) is equivalent to the Lagrangian system

$$(L) \qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{x}_k} = \frac{\partial}{\partial x_k}\,(\,T - U\,) \quad ,$$

where $T = \sum a_{ij}(x)\,\dot{x}_i\,\dot{x}_j$, $4(a_{ij}) = (a^{ij})^{-1}$.

We consider solutions $x = x(t)$ of (L) with $T(x, \dot{x}) + U(x) = e$. Since $T \geqq 0$, the solution $x(t)$ lies in

$$M = \{\, x \in \mathbb{R}^n \;;\; U(x) \leqq e \,\} \; .$$

M is a compact manifold with boundary $\partial M = \{U = e\}$. In the case $M \approx D^n$, the theorem is proved by H. Seifert [2] .

We prove this theorem by the principle of least action of Maupertuis - Jacobi.

We consider a Riemannian metric

$$(2) \qquad ds^2 = (\,e - U\,)\, a_{ij}\, dx_i\, dx_j \quad ,$$

called Jacobi-metric for e . This is positive on $M - \partial M$ and degenerates on ∂M .

A smooth curve

$\gamma = \gamma(s) : [0, 1] \longrightarrow M$

with $\gamma(0), \gamma(1) \in \partial M$ and $\gamma(s)$, $0 < s < 1$, being a geodesic by the metric (2) on $M - \partial M$, gives a desired periodic solution of (L) after proper time change (see [2]) .

As usual [3], we seek such a geodesic as a critical point of the functional

$$(3) \qquad E(\lambda) = \int_0^1 (e-U(\lambda(t)))\ T(\lambda(t), \dot{\lambda}(t))\ dt \quad .$$

As in [2], for small $\delta > 0$, we define a set $M_\delta \subset M$ as follows.

For $b \in \partial M$, let $x_b(t)$ be the solution of (L) with $x_b(0) = b$ and $\dot{x}_b(0) = 0$.

We put $F_b = \{ x_b(t_1) \in M : t_1 \gneqq 0$ and the length of the curve $x_b(t)$, $0 \leqq t \leqq t_1$, by the metric (2) is less than $\delta \}$ and define

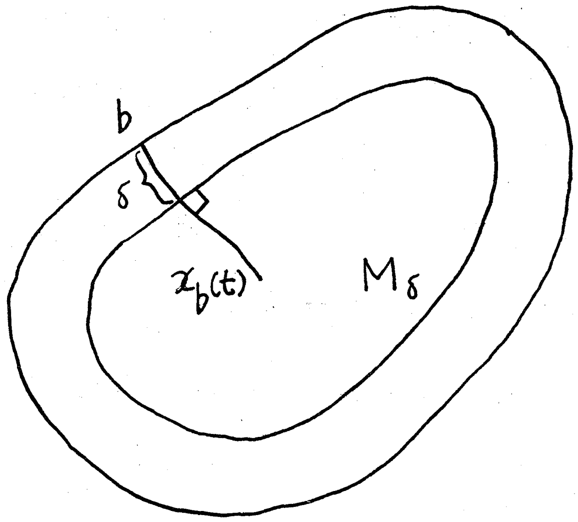
$$M_\delta = M - \bigcup_{b \in \partial M} F_b$$

Put $B_\delta = \partial M_\delta$. For small $\delta > 0$, $M_\delta \approx M$ and

$\dot{x}_b(t) \perp T_{x_b(t)} B_\delta$

if $x_b(t) \in B_\delta$. (t : small)

So, a geodesic $\gamma = \gamma(s) : [0, 1] \longrightarrow M_\delta$ with $\gamma(i) \in B_\delta$ and $\dot{\gamma}(i) \perp T_{\gamma(i)} B_\delta$ ($i = 0, 1$) gives a desired solution.



In general, let $\Omega(X; A, B)$ be the set of continuous curves $\omega = \omega(t) : [0, 1] \longrightarrow X$ with $\omega(0) \in A$ and $\omega(1) \in B$, endowed with the compact open topology.

Consider a compact connected smooth manifold M with boundary $\partial M = B$, and put $Y = \Omega(M; B, B)$. We identify $b \in B$ with the constant curve $\tilde{b}$ whose image is b, so $B \subset Y$.

Then we have

Lemma 1. $H_0(Y, B) \neq 0$ or $\pi_k(Y, B) \neq 0$ for some $k \geqq 1$.

(proof) It is easily proved that, if B is not arcwise connected then $H_0(Y, B) \neq 0$; moreover, if B is arcwise connected and Y is not arcwise connected, then $H_0(Y, B) \neq 0$.

So we assume that B and Y are arcwise connected and $\pi_k(Y, B) = 0$ for all $k \geqq 1$.

We put

$$Y_0 = \Omega(B; B, B) \quad , \quad B \subset Y_0 \overset{j}{\subset} Y .$$

Since $B \simeq Y_0$, we have

$$\pi_k(Y, B) \cong \pi_k(Y, Y_0) = 0 \qquad \text{for } k \geqq 1 .$$

Let $\pi : Y \longrightarrow B \times B$ be the fibration.

$$\omega \longmapsto (\omega(0), \omega(1))$$

We put $F = \pi^{-1}(*) = \Omega M$; the loop space, $\pi_0 = \pi_{|Y_0} : Y_0 \longrightarrow B \times B$ and $F_0 = \pi_0^{-1}(*) = \Omega B$.

Then we have a commutative diagram of fibrations

$$\begin{array}{ccccc} \Omega B & \longrightarrow & Y_0 & \xrightarrow{\pi_0} & B \times B \\ \cap\ \Omega i & & \cap\ j & & \| \\ \Omega M & \longrightarrow & Y & \xrightarrow{\pi} & B \times B \end{array} .$$

This derives the following commutative diagram of long exact sequence of homotopy groups of fibrations

$$\begin{array}{ccccccccc} \pi_k(Y_0) & \longrightarrow & \pi_k(B\times B) & \longrightarrow & \pi_{k-1}(\Omega B) & \longrightarrow & \pi_{k-1}(Y_0) & \longrightarrow & \pi_{k-1}(B\times B) \\ \downarrow j_* & & \| & & \downarrow (\Omega i)_* & & \downarrow j_* & & \| \\ \pi_k(Y) & \longrightarrow & \pi_k(B\times B) & \longrightarrow & \pi_{k-1}(\Omega M) & \longrightarrow & \pi_{k-1}(Y) & \longrightarrow & \pi_{k-1}(B\times B) \end{array} .$$

Since $\pi_k(Y, Y_0) = 0$, we have $j_* : \pi_k(Y_0) \cong \pi_k(Y)$. Hence by the 5 lemma, we have

$$\begin{array}{cccc} (\Omega i)_* : & \pi_{k-1}(\Omega B) & \cong & \pi_{k-1}(\Omega M) \\ & \wr\| & \circlearrowleft & \wr\| \\ & \pi_k(B) & \xrightarrow{i_*} & \pi_k(M) \end{array}$$

Therefore $i_* : \pi_k(B) \cong \pi_k(M)$ for $k \geqq 1$.

B and M are arcwise connected and CW complexes, hence

$i : B \subset M$

is homotopy equivalence.

But, on the other hand, $H_m(M, B; Z_2) \neq 0$ ($m = \dim M$).

This is a contradiction. Q.E.D.

Now we define

$\Lambda_\delta = \{ \lambda : [0, 1] \longrightarrow M_\delta$; piecewise smooth with $\lambda(0), \lambda(1) \in B_\delta \}$

with a distance as §16 in [4]. Then, as Theorem 17.1 in [4], we can prove

$$\Lambda_\delta \simeq \Omega(M_\delta;\ B_\delta,\ B_\delta) \approx \Omega(M;\ B,\ B)\ .$$

From Lemma 1, we have $H_0(\Lambda_\delta, B_\delta) \neq 0$ or $\pi_k(\Lambda_\delta, B_\delta) \neq 0$.

For example, let $\alpha \in \pi_k(\Lambda_\delta, B_\delta)$ be the nontrivial element.

A representative $f \in \alpha$ is a continuous function $D^k \longrightarrow \Lambda_\delta$ with $f(S^{k-1}) \subset B_\delta$.

We define

(4) $$c_\delta = \inf_{f\in\alpha} \operatorname{Max} E(\ \mathrm{Im} f\ )$$

For the case of homology, take a component A with $A \cap B_\delta = \phi$ and define $c_\delta = \inf_{a\in A} E(a)$.

The following lemma is easily proved.

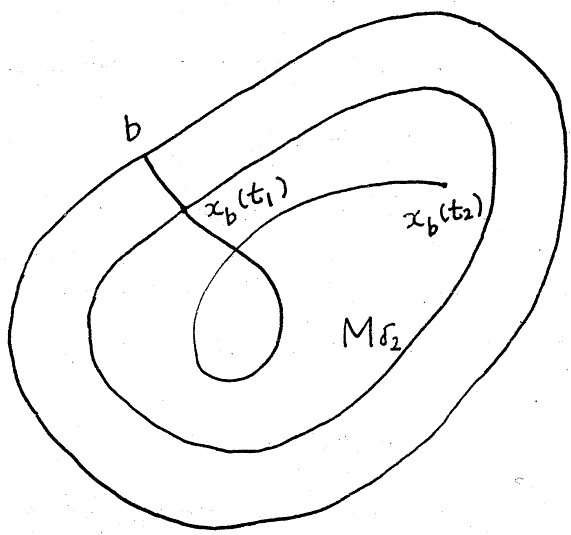
Lemma 2. *There exist* $\delta_1 > 0$ *and* $K \geqq 1$ *such that*

$$c_\delta + 1 \leqq K \quad \text{if} \quad 0 < \delta \leqq \delta_1\ .$$

(proof of Theorem)

Assume that there are no periodic orbit on $H^{-1}(e)$. Then any solution $x_b(t)$, $b \in \partial M$, of (L) does not reach at the boundary.

Hence we can choose δ_2 , $0 < \delta_2 < \delta_1$ in Lemma 2, such that any solution $x_b(t)$ lies in M_{δ_2} for $t_1 \leqq t \leqq t_2$, where the length of $x_b(t)$, $0 \leqq t \leqq t_1$, by the metric



(2) is δ_2 and the length of $x_b(t)$, $t_1 \leqq t \leqq t_2$, by (2) is $K^{1/2}$.

Then we change the metric ds to $d\tilde{s}$ so that

(5) $ds = d\tilde{s}$ on M_{δ_2} ,

(6) $ds \geqq d\tilde{s}$ on $M_{\delta_3} - M_{\delta_2}$ for some $0 < \delta_3 < \delta_2$,

(7) M_{δ_3} is geodesically convex w.r.t. $d\tilde{s}$.

This is done as in [2]. The condition (6) is fulfilled if we modify the function λ used in [2] so as to $\lambda \leqq 1$ but $|\lambda'(\delta)|$; sufficiently large.

Remark that then $x_b(t)$ $0 \leqq t \leqq t_1$ is also a geodesic w.r.t. $d\tilde{s}$ after a time change, because $d\tilde{s}$ is a conformal trnsformation of ds by the function λ depending only on y_n in [2] .

Now let $d(\ ,\ )$ be the Riemannian distance on $M - \partial M$ w.r.t. $d\tilde{s}$.

We choose $\eta > 0$ so that

(8) two points $x, y \in M_{\delta_3}$, with $d(x, y) \leqq \eta$, is uniquely combined by the shortest geodesic in M_{δ_3} ,

(9) for $x \in M_{\delta_3}$ with $d(x, B_{\delta_3}) \leqq \eta$, there is the unique $r(x) \in B_{\delta_3}$ such that $d(x, r(x)) = d(x, B_{\delta_3})$.

We put $N = (K/\eta)^2$.

Then for $\lambda \in \Lambda_{\delta_3}$ with $\tilde{E}(\lambda) \leqq K$, where $\tilde{E}$ is defined by (3) replacing ds with $d\tilde{s}$, we have

$$d(\lambda(t_1), \lambda(t_2)) \leqq \int_{t_1}^{t_2} |\dot{\lambda}(t)|_{d\tilde{s}}\, dt$$

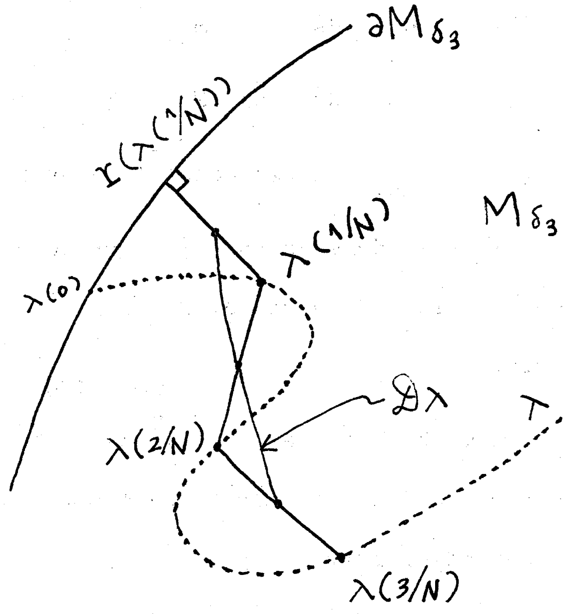
$$\leqq (t_2-t_1)^{1/2}\, \tilde{E}(\lambda) \leqq \eta ,$$

if $0 \leqq t_2 - t_1 \leqq 1/N$.

We put $\widetilde{\Lambda} = \{ \lambda \in \Lambda_{\delta_3} ; \widetilde{E}(\lambda) \leqq K \}$.

For $\lambda \in \widetilde{\Lambda}$, we join $r(\lambda(1/N))$, $\lambda(1/N)$, $\lambda(2/N)$, ..., $\lambda(1-1/N)$, $r(\lambda(1-1/N))$ by the shortest geodesics, mark the centers of the geodesics and join them by another geodesics (see [2]) .

Thus we deform λ to the new curve $\mathscr{D}\lambda$.



$$\mathscr{D} : \widetilde{\Lambda} \longrightarrow \widetilde{\Lambda}$$

is continuous and

(10) $\mathscr{D} \simeq i_d$,

(11) $\mathscr{D}$: E - decreasing.

Let c be defined by (4) putting $\delta = \delta_3$ and replacing E with $\widetilde{E}$. We have

(12) $c > 0$.

Because α is nontrivial in the relative sense (if $c = 0$, Im f is deformed into B_{δ_3}) .

Then $c \leqq c_{\delta_3} \leqq K - 1$ by (6) and Lemma 2.

Now for a natural number j , we choose $f \in \alpha$ with

$$c \leqq \text{Max}\ \widetilde{E}(\ \text{Im}\ f\ ) \leqq c + 1/j \ .$$

By (10) and (11) , $\mathscr{D}\circ f \in \alpha$ and $\text{Max}\ E(\ \text{Im}\ \mathscr{D}\circ f\ ) \leqq c + 1/j$. So we have $\lambda_j \in \widetilde{\Lambda}$ with

(13) $c \leqq \tilde{E}(\mathcal{D}\lambda_j) \leqq \tilde{E}(\lambda_j) \leqq c + 1/j$.

For this sequence $\{ \lambda_j \}_{j = 1, 2, \ldots}$, we can assume

$$\lambda_j(k/N) \longrightarrow p_k \quad ; \quad k = 0, 1, \ldots, N .$$

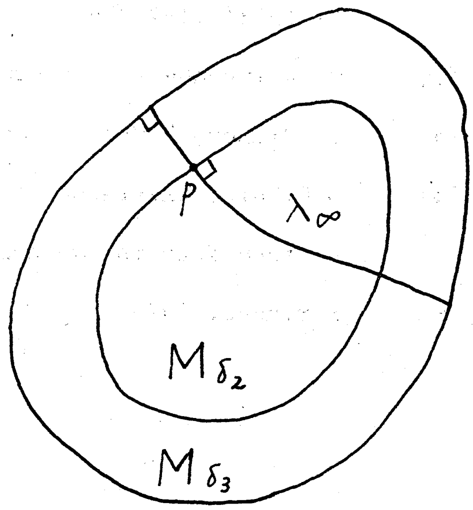
Consider the curve λ_∞ given by combining $p_0, p_1, \ldots, p_N$ by the shortest geodesic. Then we can prove that λ_∞ is a smooth geodesic w.r.t. $d\tilde{s}$ with

$$\tilde{E}(\lambda_\infty) = c \quad \text{and} \quad \dot{\lambda}_\infty(i) \perp T_{\lambda_\infty(i)}B_{\delta_3} \quad (i = 0, 1) .$$

This corresponds to the condition (C) of Palais-Smale [3] .

Consider the point $p \in B_{\delta_2}$, at which λ_∞ encounter M_{δ_2} for the first time. ; $p = \lambda_\infty(s_1)$. Then

$$\dot{\lambda}_\infty(s_1) \perp T_pB_{\delta_2} .$$



But, by the construction of δ_2 , the geodesic $\lambda_\infty(s)$; $s_1 \leqq s \leqq 1$, is contained in M_{δ_2} , because the length of the curve $\lambda_\infty(s)$; $s_1 \leqq s \leqq 1$, w.r.t. ds ($= d\tilde{s}$ as long as $\lambda_\infty(s) \in M_{\delta_2}$ by (5)) is less than $K^{1/2}$.
($\tilde{E}(\lambda_\infty) = c \leqq K$ implies that the length of λ_∞ w.r.t. $d\tilde{s} \leqq K^{1/2}$.)

This is a contradiction, proving the theorem. Q.E.D.

[2] treats the analytic system, but it is not essential for our argument. In [5] , Seifert's result is proved for C^3 - Finsler systems.

For the case $M \approx D^n$, Seifert conjectured that there may be at least n periodic orbits.

For counting the number of critical points, we use the homology group (pairwise subordinated homology classes [3]) instead of the homotopy group

But I don't know whether Lemma 1 is valid, replacing π with H .

References

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