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On an Existence Theorem for Quasiperiodic Differntial-Difference Equations

M. Kurihara and M. Hiraiwa
Faculty of Engineering, Yamanashi University

0. Introduction

We study a system of nonlinear quasiperiodic differential-difference equations of the form:

(0.1) $\frac{d}{dt}x(t)=X[t,x(t),x(t-\tau)]$.

Here a function f(t) of a real variable t is called to be quasiperiodic with periods ω_0, $\omega_1,\cdots\cdots,\omega_m$ if it is represented as $f(t)=f_0(t,t,\cdots\cdots,t)$ for some continuous function $f_0(u_0,u_1,\cdots\cdots,u_m)$ periodic in each u_0, $u_1,\cdots\cdots,u_m$ with periods ω_0, $\omega_1,\cdots\cdots,\omega_m$.

To investigate the properties of quasiperiodic functions, M. Urabe[4] defined psuedoperiodic functions. A function f(t,u) $=f(t,u_1,\cdots\cdots,u_m)$ of real variables t and $u=(u_1,\cdots\cdots,u_m)$ is called to be pseudoperiodic with periods ω_0 and $\omega=(\omega_1,\cdots\cdots,\omega_m)$ if it is periodic in each $u_1,\cdots\cdots,u_m$ with periods $\omega_1,\cdots\cdots,\omega_m$ and in addition it satisfies the equality

$$f(t+\omega_0,u_1,\cdots\cdots,u_m)=f(t,u_1+\omega_0,\cdots\cdots,u_m+\omega_0).$$

It is shown [4] that a quasiperiodic function f(t) with periods

$\omega_0, \omega_1, \cdots\cdots, \omega_m$ is represented as $f(t)=\bar{f}(t,0)$ for some continuous pseudoperiodic function $\bar{f}(t,u)$ in t and $u=(u_1,\cdots\cdots, u_m)$ with periods ω_0 and $\omega=(\omega_1,\cdots\cdots,\omega_m)$. In fact, $\bar{f}(t,u)=f_0(t, u_1+t,\cdots\cdots,u_m+t)$. Moreover the pseudoperiodic function $\bar{f}(t,u)$ is shown [5] to be uniquely determined for a quasiperiodic function f(t). In what follows, $\bar{f}(t,u)$ is called to be a continuous pseudoperiodic function corresponding to a given quasiperiodic function f(t).

We associate with the system (0.1) a system of nonlinear pseudoperiodic differential-difference equations of the form:

$$\frac{d}{dt}x(t)=X[t,u,x(t),x(t-\tau)]. \tag{0.2}$$

In this paper we prove two main theorems saying that one can always assure the existence of an exact quasiperiodic solution of the system (0.1) and an exact pseudoperiodic solution of the system (0.2) by checking several conditions on an obtained approximate solution and further they give a method to obtain an error bound of the approximate solution. Finally we give a numerical example of a system of nonlinear quasiperiodic differential-difference equations to apply our main theorems.

The analoguous theorems were originally proved by M. Urabe [5][6][7] for systems of nonlinear quasiperiodic and pseudoperiodic differential equations. After him T. Mitsui [2] completed numerical-analytical methods of M. Urabe's with precise a posteriori estimate and gave some numerical examples for quasiperiodic differential equations.

Throughout the paper we assume without any loss of generality that $\omega_i>0$ for $i=0,1,\cdots\cdots,m$ and that the reciprocals of these periods are rationally linearly independent [5].

In Section 1 we define the regularities of pseudoperiodic differential operators and quasiperiodic ones. Further we give theorems concerning Green functions for these operators. In Section 2 we state our main theorems. In Section 3 and 4 we prove our main theorems by using the facts given in Section 1. In Section 5 we give an example to apply our last theorem.

1. Regularity of Linear Differential Systems

Let $A(t,u)$ be a continuous square matrix pseudoperiodic in t and u with periods ω_0 and ω. Denote by L a pseudoperiodic differential operator of the form:

(1.1) $$Ly=\frac{dy}{dt}-A(t,u)y.$$

M. Urabe [4] called the operator L to be regular if there is a continuous square matrix $P(u)=P(u_1,\cdots\cdots,u_m)$ periodic in each $u_1,\cdots\cdots,u_m$ with periods $\omega_1,\cdots\cdots,\omega_m$ satisfying the conditions as follws:

(1.2) $$P(u)^2=P(u)$$

(1.3) $$\|U(t,u)P(u)\|\leqq K_0e^{-\sigma t} \text{ for } t\geqq 0,$$

(1.4) $$\|U(t,u)[E-P(u)]\|\leqq K_0e^{-\sigma t} \text{ for } t\leqq 0,$$

(1.5) $$P(u+\omega_0)U(\omega_0,u)=U(\omega_0,u)P(u),$$

where E is the unit matrix, $U(t,u)$ is the fundamental matrix of the linear differential system $Ly=0$ satisfying $U(0,u)=E$, K_0 and σ are some positive numbers and $\|\ \|$ denotes any norm.

M. Urabe [4] proved the following theorem:

Theorem 1. Suppose that the pseudoperiodic differential operator L defined by (1.1) is regular. Then L has a Green function G(t,s,u) with the property

$$\|G(t,s,u)\| \leqq K e^{-\sigma|t-s|} \tag{1.6}$$

for all t, s and u and for some $K>0$ and $\sigma>0$. Here the Green function is given in the form:

$$G(t,s,u)=\begin{cases} U(t,u)P(u)U^{-1}(s,u) & \text{for } t\geqq s \\ -U(t,u)[E-P(u)]U^{-1}(s,u) & \text{for } t\leqq s, \end{cases}$$

where U(t,u) is the fundamental matrix of the linear system $Ly=0$ satisfying $U(0,u)=E$ and P(u) is a matrix in the definition of regularity of L. Moreover for any continuous pseudoperiodic function f(t,u) with periods ω_0 and ω the differential system $Lx=f(t,u)$ has a unique solution $x=x(t,u)$ pseudoperiodic with periods ω_0 and ω and it is given by

$$x(t,u)=\int_{-\infty}^{\infty} G(t,s,u)f(s,u)ds. \tag{1.7}$$

Let A(t) be a square matrix quasiperiodic in t with periods $\omega_0,\omega_1,\cdots\cdots,\omega_m$ and denote by $\bar{A}(t,u)$ the continuous pseudoperiodic matrix corresponding to A(t). The quasiperiodic differential operator L defined by

$$Ly=\frac{dy}{dt}-A(t)y \tag{1.8}$$

is called to be regular if the corresponding pseudoperiodic differential operator $\bar{L}$ defined by

$$\bar{L}y=\frac{dy}{dt}-\bar{A}(t,u)y \tag{1.9}$$

is regular.

Concerning quasiperiodic differential operators, M. Urabe [4] also proved the theorem as follows:

Theorem 2. Suppose that the quasiperiodic differential operator L defined by (1.8) is regular. Then L has a Green function $G(t,s)=\bar{G}(t,s,0)$ with the property

$$\|G(t,s)\| \leqq Ke^{-\sigma|t-s|} \tag{1.10}$$

for all t and s for some $K>0$ and $\sigma>0$, where $\bar{G}(t,s,u)$ is a Green function of the pseudoperiodic differential operator $\bar{L}$ corresponding to L. For any quasiperiodic function f(t) with periods $\omega_0,\omega_1,\cdots\cdots,\omega_m$ and it is given by

$$x(t)=\int_{-\infty}^{\infty} G(t,s)f(s)ds. \tag{1.11}$$

It is noted that a quasiperiodic function is almost periodic [3]. It is seen [7] that a regular quasiperiodic differential operator L in (1.8) is also regular as an almost periodic operator. The notion of regularity of almost periodic operators was introduced by S. Burd, Ju. S. Kolesov and M. A. Krasnosel'skii [1].

2. Main Theorems

We prove the theorem on a system of pseudoperiodic differential-difference equations (0.2) as follows:

Theorem 3. Let D be a bounded domain in N-dimensional Euclidean space R^N with any norm $\| \ \|$. $X[t,u,x,y]$ in (0.2) is assumed to be a continuous function mapping from the space $R\times R^m\times D\times D$ into R^N, pseudoperiodic in t and $u=(u_1,\cdots\cdots,u_m)$ with periods ω_0 and $\omega=(\omega_1,\cdots\cdots,\omega_m)$ and continuously differentiable with respect to (x,y) in $D\times D$. Suppose that the system (0.2) has an approximate solution $x=x_0(t,u)$ pseudoperiodic in t and s with periods ω_0 and ω such that $x_0(t,u)$ in D for all t and u, $dx_0(t,u)/dt$ is continuous of (t,u) in $R\times R^m$ and that

(2.1) $$\left\|\frac{d}{dt}x_0(t,u)-X[t,u,x_0(t,u),x_0(t-\tau,u)]\right\|\leqq r$$

for all t and u. Further suppose that there are a positive number δ, nonnegative number κ and μ and a continuous matrix $A(t,u)$ psuedoperiodic in t and u with periods ω_0 and ω satisfying the conditions that the pseudoperiodic differential operator L defined by (1.1) is regular, D includes D_δ defined by

(2.2) $$D_\delta=\{x\,|\,\|x-x_0(t,u)\|<\delta \text{ for some } (t,u) \text{ in } R\times R^m\},$$

(2.3) $$\|\Phi[t,u,x,y]-A(t,u)\|\leqq\kappa/M$$

and

(2.4) $$\|\Psi[t,u,x,y]\|\leqq\mu$$

for any (t,u) in $R\times R^m$ and any (x,y) in $D_\delta\times D_\delta$,

(2.5) $$\kappa+M\mu<1 \text{ and } Mr/(1-\kappa-M\mu)\leqq\delta.$$

Here $\Phi[t,u,x,y]$ and $\Psi[t,u,x,y]$ are the Jacobian matrices of the function $X[t,u,x,y]$ with respect to x and y respectively and $M=2K/\sigma$, where K and σ are positive numbers in (1.6) for a Green function $G(t,s,u)$ of L.

Then the given system (0.2) has a unique solution $x=\hat{x}(t,u)$ pseudoperiodic in t and u with periods ω_0 and ω lying in D_δ for all t and u. Moreover it satisfies the inequality

(2.6) $$\|\hat{x}(t,u)-x_0(t,u)\|\leqq Mr/(1-\kappa-M\mu)$$

for all t and u.

We also prove the theorem on a system of quasiperiodic differential-difference equations (0.1) as follows:

Theorem 4. Let D be a bounded closed domain in R^N with any norm $\| \ \|$. $X[t,x,y]$ in (0.1) is assumed to be a continuous function mapping from $R \times D \times D$ into R^N quasiperiodic in t with periods $\omega_0, \omega_1, \cdots\cdots, \omega_m$ and the pseudoperiodic function $\bar{X}[t,u,x,y]$ corresponding to $X[t,x,y]$ is continuously differentiable with respect to (x,y) in $D \times D$. Suppose that the system (0.1) has an approximate solution $x=x_0(t)$ quasiperiodic in t with periods $\omega_0, \omega_1, \cdots\cdots, \omega_m$ such that the continuous pseudoperiodic function $\bar{x}_0(t,u)$ corresponding to $x_0(t)$ is continuously differentiable with respect to t for all t and u, $x_0(t)$ for all t and

$$(2.7) \qquad \left\| \frac{d}{dt} x_0(t) - X[t, x_0(t), x_0(t-\tau)] \right\| \leqq r$$

for all t. Further suppose that there are a positive number δ, nonnegative numbers κ and μ and a continuous matrix $A(t)$ quasiperiodic in t with periods $\omega_0, \omega_1, \cdots\cdots, \omega_m$ satisfying the conditions that the quasiperiodic differential operator L defined by (1.8) is regular, D includes D_δ defined by

$$D_\delta = \{x \mid \|x - x_0(t)\| < \delta \text{ for some } t \text{ in } R\},$$

$$(2.8) \qquad \|\Phi[t,x,y] - A(t)\| \leqq \kappa/M$$

and

$$(2.9) \qquad \|\Psi[t,x,y]\| \leqq \mu$$

for any t in R and any (x,y) in $D_\delta \times D_\delta$,

$$(2.10) \qquad \kappa + M\mu < 1 \text{ and } Mr/(1-\kappa-M\mu) \leqq \delta.$$

Here $\Phi[t,x,y]$ and $\Psi[t,x,y]$ are the Jacobian matrices of the function $X[t,x,y]$ with respect to x and y respectively and $M=2K/\sigma$, where K and σ are positive numbers such that a Green function $\bar{G}(t,s,u)$ of the pseudoperiodic differential operator $\bar{L}$ in (1.9) corresponding to L satisfies the inequality

(2.11) $\quad \|\bar{G}(t,s,u)\| \leqq Ke^{-\sigma|t-s|}$

for all t, s and u.

Then the given system (0.1) has a unique solution $x=\hat{x}(t)$ quasiperiodic in t with periods $\omega_0,\omega_1,\cdots\cdots,\omega_m$ lying in D_δ for all t. Moreover it satisfies the inequality

(2.12) $\quad \|\hat{x}(t)-x_0(t)\| \leqq Mr/(1-\kappa-M\mu)$

for all t in R.

3. Proof of Theorem 3

We denote by $PC(R\times R^m,R^N)$ the space of continuous functions $x=x(t,u)$ pseudoperiodic in t and u wirh periods ω_0 and ω mapping from $R\times R^m$ into R^N and define the norm

$$\|x\|_C=\max\{\|x(t,u)\| \,|\, (t,u) \text{ in } R\times R^m\}$$

for any $x=x(t,u)$ in $PC(R\times R^m,R^N)$. Rewrite the given system (0.2) in the form:

(3.1) $\quad \frac{d}{dt}x(t)=A(t,u)x(t)+f(t,u;x)$

where

$$f(t,u;x)=X[t,u,x(t,u),x(t-\tau,u)]-A(t,u)x(t,u).$$

For the given approximate solution $x=x_0(t,u)$ we put

(3.2) $\quad \frac{d}{dt}x_0(t,u)=X[t,u,x_0(t,u),x_0(t-\tau,u)]+h(t,u).$

Note that $h(t,u)$ belongs to $PC(R\times R^m,R^N)$ and that

(3.3) $\quad \|h(t,u)\| \leqq r$

for any (t,u) in $R\times R^m$ by (2.1). The equality (3.2) can be rewritten in the forms of (3.1) as follows:

(3.4) $\frac{d}{dt}x_0(t,u)=A(t,u)x_0(t,u)+f(t,u;x_0)+h(t,u).$

Note that $f(t,u;x_0)$ is in $PC(R\times R^m,R^N)$. Applying Theorem 1 to the system (3.4), we have

(3.5) $x_0(t,u)=\int_{-\infty}^{\infty}G(t,s,u)[f(s,u;x_0)+h(s,u)]ds,$

where $G(t,s,u)$ is a Green function for the pseudoperiodic differential operator L.

To seek an exact pseudoperiodic solution of the system (0.2), we consider the iterative process

(3.6) $x_{n+1}(t,u)=\int_{-\infty}^{\infty}G(t,s,u)f(s,u;x_n)ds$

for $n=0,1,2,\cdots\cdots$. We can prove that the above process can be continued infinitely in the space $PC(R\times R^m,R^N)$ and that

(3.7) $\|x_{n+1}-x_n\|_C\leqq(\kappa+M\mu)^n\|x_1-x_0\|_C$

and

(3.8) $\|x_{n+1}-x_0\|_C\leqq\delta$

for $n=0,1,2,\cdots\cdots$. In fact, for $n=0$ (3.7) is evident. Since

$$x_1(t,u)=\int_{-\infty}^{\infty}G(t,s,u)f(s,u;x_0)ds,$$

it follows from (3.5) that

$$x_1(t,u)-x_0(t,u)=-\int_{-\infty}^{\infty}G(t,s,u)h(s,u)ds.$$

Then by (1.6), (3.3) and (2.5) we have

(3.9) $\|x_1-x_0\|_C\leqq Mr\leqq(1-\kappa-M\mu)\delta<\delta.$

This proves (3.8) for $n=0$. To prove our statement by induction, let us assume that the iterative process (3.6) has been continued up to n-1 and that we have obtained (3.7) and (3.8) up to n-1. We can make $x_{n+1}(t,u)$ by the relation (3.6). Then

$$(3.10)\qquad x_{n+1}(t,u)-x_n(t,u)$$

$$=\int_{-\infty}^{\infty}G(t,s,u)\{X[s,u,x_n(s,u),x_n(s-\tau,u)]$$

$$-X[s,u,x_{n-1}(s,u),x_{n-1}(s-\tau,u)]$$

$$-A(s,u)[x_n(s,u)-x_{n-1}(s,u)]\}ds$$

$$=\int_{-\infty}^{\infty}G(t,s,u)\int_0^1\{\Phi[s,u,x_n^\theta(s,u),x_n^\theta(s-\tau,u)]$$

$$-A(s,u)\}[x_n(s,u)-x_{n-1}(s,u)]d\theta ds$$

$$+\int_{-\infty}^{\infty}G(t,s,u)\int_0^1\Psi[s,u,x_n^\theta(s,u),x_n^\theta(s-\tau,u)]$$

$$\times[x_n(s-\tau,u)-x_{n-1}(s-\tau,u)]d\theta ds,$$

where

$$x_n^\theta(t,u)=x_{n-1}(t,u)+\theta[x_n(t,u)-x_{n-1}(t,u)]$$

for $0\leqq\theta\leqq 1$. It follows from (1.6), (2.3) and (2.4) that

$$(3.11)\qquad \|x_{n+1}-x_n\|_C\leqq\frac{2K}{\sigma}\left(\frac{\kappa}{M}+\mu\right)\|x_n-x_{n-1}\|_C=(\kappa+M\mu)\|x_n-x_{n-1}\|_C.$$

This implies the relation (3.7) by the assumption of induction. Moreover it follows that

$$(3.12)\qquad \|x_{n+1}-x_0\|_C\leqq\sum_{i=0}^{n}(\kappa+M\mu)^n\|x_1-x_0\|_C\leqq[1/(1-\kappa-M\mu)]\|x_1-x_0\|_C$$

$$\leqq Mr/(1-\kappa-M\mu)<\delta.$$

This proves (3.8).

Then we obtain an infinite sequence $\{x_n(t,u)\}$ in the space $PC(R\times R^m,R^N)$ by the iterative process (3.6). It is easily seen from (3.7) and (2.5) that the sequence $\{x_n(t,u)\}$ is uniformly convergent to a function $\hat{x}(t,u)$ in $PC(R\times R^m,R^N)$. Further letting n to infinity in (3.12) and (3.6), we have (2.6) and

$$\hat{x}(t,u)=\int_{-\infty}^{\infty} G(t,s,u)f(s,u;\hat{x})ds,$$

which implies that $d\hat{x}(t,u)/dt=A(t,u)\hat{x}(t,u)+f(t,u;\hat{x})=X[t,u,\hat{x}(t,u),\hat{x}(t-\tau,u)]$. Thus $x=\hat{x}(t,u)$ is our desired solution in $PC(R\times R^m,R^N)$ of the system (0.2) lying in D_δ for any (t,u) in $R\times R^m$ and satisfying (2.6).

In order to prove the uniqueness of pseudoperiodic solutions of the system (0.2), we consider another solution $x=\hat{x}'(t,u)$ in the space $PC(R\times R^m,R^N)$ of the system (0.2) lying in D_δ for any (t,u) in $R\times R^m$. Then we have $d\hat{x}'(t,u)/dt=X[t,u,\hat{x}'(t,u),\hat{x}'(t-\tau,u)]=A(t,u)\hat{x}'(t,u)+f(t,u;\hat{x}')$, which implies

$$\hat{x}'(t,u)=\int_{-\infty}^{\infty} G(t,s,u)f(s,u,\hat{x}')ds.$$

Using the relations (1.6), (2.3) and (2.4), we have

(3.13) $\qquad \|\hat{x}-\hat{x}'\|_C\leqq(\kappa+M\mu)\|\hat{x}-\hat{x}'\|_C$

by the same arguments as those in proceeding from (3.10) to (3.11). It follows from (2.5) and (3.13) that $\|\hat{x}-\hat{x}'\|_C=0$. This proves the uniqueness of pseudoperiodic solutions of the system (0.2) lying in D_δ for all t.

4. Proof of Theorem 4

At first we use for the proof of Theorem 4 the following lemma proved by M. Urabe [5].

Lemma. Let $f_i(t)$ $(i=1,2,\cdots\cdots,n)$ be arbitrary functions quasiperiodic in t with periods $\omega_0,\omega_1,\cdots\cdots,\omega_m$ and $\bar{f}_i(t,u)=f_i(t,u_1,u_2,\cdots\cdots,u_m)$ $(i=1,2,\cdots\cdots,n)$ be continuous pseudoperiodic functions corresponding to $f_i(t)$ respectively. Then for arbitrary t and u to any positive number ε corresponds a number ξ such that $\|\bar{f}_i(t,u)-f_i(\xi)\|<\varepsilon$ $(i=1,2,\cdots\cdots,n)$.

Now we are in a position to prove Theorem 4. Let $\bar{x}_0(t,u)$ be the continuous pseudoperiodic function corresponding to $x_0(t)$. Then by Lemma for arbitrary t and u to any positive number ε corresponds a number ξ such that

$$(4.1) \qquad \|\bar{x}_0(t,u)-x_0(\xi)\|<\varepsilon.$$

Since $x_0(t)$ is in D for all t, $\bar{x}_0(t,u)$ is in $\bar{D}=D$ for all t and u, where $\bar{D}$ is the closure of D. Let $\bar{X}[t,u,x,y]$ be the continuous function pseudoperiodic in t with periods ω_0 and $\omega=(\omega_1,\cdots\cdots,\omega_m)$ corresponding to $X[t,x,y]$. The function $d\bar{x}_0(t,u)/dt-\bar{X}[t,u,\bar{x}_0(t,u),\bar{x}_0(t-\tau,u)]$ is continuous and pseudoperiodic in t and u with periods ω_0 and $\omega=(\omega_1,\cdots\cdots,\omega_m)$. Therefore by Lemma for arbitrary t and u to any positive number ε corresponds a number ξ such that

$$(4.2) \qquad \Big\|\{\frac{d}{dt}\bar{x}_0(t,u)-\bar{X}[t,u,\bar{x}_0(t,u),\bar{x}_0(t-\tau,u)]\} - \{\frac{d}{dt}x_0(t)-X[t,x_0(t),x_0(t-\tau)]\}_{t=\xi}\Big\|<\varepsilon.$$

Then by the assumption (2.7) it follows that

$$(4.3) \qquad \Big\|\frac{d}{dt}\bar{x}_0(t,u)-\bar{X}[t,u,\bar{x}_0(t,u),\bar{x}_0(t-\tau,u)]\Big\|<r+\varepsilon.$$

Since ε is arbitrary, letting ε to +0 in (4.3), we have

$$(4.4) \qquad \Big\|\frac{d}{dt}\bar{x}_0(t,u)-\bar{X}[t,u,\bar{x}_0(t,u),\bar{x}_0(t-\tau,u)]\Big\|\leqq r$$

for any (t,u) in $R\times R^m$.

Consider x such that

(4.5) $\|x-\bar{x}_0(t,u)\|<\delta$.

Then there is a positive number $\delta_1<\sigma$ such that

(4.6) $\|x-\bar{x}_0(t,u)\|<\delta_1<\delta$.

Let ε be an arbitrary positive number such that

(4.7) $0<\varepsilon<\delta-\delta_1$.

Since $\bar{x}_0(t,u)$ is continuous and pseudoperiodic in t and u with periods ω_0 and ω, by Lemmma, there is a number ξ such that $\|\bar{x}_0(t,u)-x_0(\xi)\|<\varepsilon$. Then by (4.6) and (4.7) we have $\|x-x_0(\xi)\|<\delta$. It follows from the assumptions of Theorem 4 that

(4.8) $D_\delta=\{x:\|x-\bar{x}_0(t,u)\|<\delta$ for some t and u$\}$ in D.

Let $\bar{\Phi}[t,u,x,y]$ and $\bar{\Psi}[t,u,x,y]$ be the Jacobian matrices with respect to x and y respectively. Moreover let $\bar{A}(t,u)$ be the continuous pseudoperiodic matrix corresponding to A(t). By Lemma for any positive number ε satisfyinf (4.7) there exists a number ξ such that $\|\bar{x}_0(t,u)-x_0(\xi)\|<\varepsilon$, $\|\{\bar{\Phi}[t,u,x,y]-\bar{A}(t,u)\}-\{\Phi[\xi,x,y]-A(\xi)\}\|<\varepsilon$ and $\|\bar{\Psi}[t,u,x,y]-\Psi[\xi,x,y]\|<\varepsilon$. Then for any x satisfying (4.6) we have $\|x-x_0(\xi)\|<\delta$. By the assumptions (2.8) and (2.9) we obtain $\|\bar{\Phi}[t,u,x,y]-\bar{A}(t,u)\|<(\kappa/M)+\varepsilon$ and $\|\bar{\Psi}[t,u,x,y]\|<\mu+\varepsilon$. Since ε is an arbitrary positive number satisfying (4.7), letting ε to +0, we have

(4.9) $\|\bar{\Phi}[t,u,x,y]-\bar{A}(t,u)\|\leqq\kappa/M$

(4.10) $\|\bar{\Psi}[t,u,x,y]\|\leqq\mu$.

This proves that (4.9) and (4.10) are valid for any (t,u) in $R\times R^m$ and (x,y) in $D_\delta\times D_\delta$.

By (4.4), (4.8), (4.9) and (4.10) we see that all the conditions of Theorem 3 are fulfilled for the pseudoperiodic differential system

(4.11) $\frac{d}{dt}x(t)=\bar{X}[t,u,x(t),x(t-\tau)]$.

Hence by Theorem 3 it is seen that the system (4.11) has a solution $x=\bar{x}(t,u)$ pseudoperiodic in t and u with periods ω_0 and ω such that

(4.12) $\|\bar{x}-\bar{x}_0\|_C \leqq Mr/(1-\kappa-M\mu)<\delta$.

Put $\hat{x}(t)=\bar{x}(t,0)$. Then $x=\hat{x}(t)$ is a solution of the system (0.1) quasiperiodic in t with periods $\omega_0,\omega_1,\cdots\cdots,\omega_m$. It follows from (4.12) that $\|\hat{x}-x_0\|_C \leqq Mr/(1-\kappa-M\mu)$, which implies (2.12).

It now remains to prove the uniqueness of quasiperiodic solutions. Let $x=\hat{x}(t)$ be an arbitrary solution of the system (0.1) quasiperiodic in t with periods $\omega_0,\omega_1,\cdots\cdots,\omega_m$ lying in D_δ for all t. Then we have $d\hat{x}(t)/dt=X[t,\hat{x}(t),\hat{x}(t-\tau)]=A(t)\hat{x}(t)+\{X[t,\hat{x}(t),\hat{x}(t-\tau)]-A(t)\hat{x}(t)\}$. By Theorem 2 for the regular operator L we have

(4.13) $$\hat{x}(t)=\int_{-\infty}^{\infty}G(t,s)\{X[s,\hat{x}(s),\hat{x}(s-\tau)]-A(s)\hat{x}(s)\}ds.$$

Suppose that the system (0.1) has two solutions $x=\hat{x}(t)$ and $x=\hat{x}'(t)$ quasiperiodic in t with periods $\omega_0,\omega_1,\cdots\cdots,\omega_m$ lying in D_δ for all t. Then we also obtain

(4.14) $$\hat{x}'(t)=\int_{-\infty}^{\infty}G(t,s)\{X[s,\hat{x}'(s),\hat{x}'(s-\tau)]-A(s)\hat{x}'(s)\}ds.$$

Making use of the same arguments as those in proceeding from (3.10) to (3.11) for the relations (4.13) and (4.14) and noting the assumptions (2.11), (2.7) and (2.8), we have

(4.15) $\|\hat{x}-\hat{x}'\|_C \leqq (2K/\sigma)(\kappa/M+\mu)\|\hat{x}-\hat{x}'\|_C \leqq (\kappa+M\mu)\|x-x'\|_C$.

It follows from (2.10) and (4.15) that $\|\hat{x}-\hat{x}'\|_C=0$. This proves the uniqueness of quasiperiodic solutions and thus completes the proof of Theorem 4.

5. An Example

In order to apply Theorem 4, we give an example introduced and computed by M. Hiraiwa [8]. Consider a system of differential-difference equations of the form:

$$(5.1)\qquad \begin{cases} \frac{d}{dt}x_1(t)=x_2(t) \\ \frac{d}{dt}x_2(t)=\frac{1}{4}\cos t+\frac{1}{2}\cos\sqrt{2}t-2x_1(t)-\frac{1}{8}x_2(t)+\frac{1}{16}x_1(t-1)^2. \end{cases}$$

The right member of (5.1) denotes X[t,x(t),x(t-1)] for $x(t)=(x_1(t),x_2(t))$. The function X[t,x,y] is smooth of (t,x,y) in $R\times R^2\times R^2$ and quasiperiodic in t with periods 2π and $2\pi/\sqrt{2}$. Note that the function

$$(5.2)\qquad \bar{X}[t,u,x,y]=\begin{cases} x_2 \\ \frac{1}{4}\cos t+\frac{1}{2}\cos\sqrt{2}(u+t)-2x_1-\frac{1}{8}x_2+\frac{1}{16}y_1{}^2 \end{cases}$$

is the pseudoperiodic one corresponding to X[t,x,y].

By using the Galerkin's procedure and carrying out by a digital computer, we can obtain an approximate solution of the system (5.1) of the form $x_0(t)=(x_1^0(t),dx_1^0(t)/dt)$. Here $x_1^0(t)=T_1^0(t,t)$ and

$$(5.3)\qquad T_1^0(u_1,u_2)=\sum_{|p|\leq 4}\{\alpha_p\cos(p,\nu,u)+\beta_p\sin(p,\nu,u)\},$$

where $p=(p_1,p_2)$ is a pair of integers with $|p|=|p_1|+|p_2|$, $(p,\nu,u)=p_1\nu_1u_1+p_2\nu_2u_2$ for $\nu_1=1$ and $\nu_2=\sqrt{2}$. The coefficients α_p and β_p in (5.3) are computed as indicated in Table. It is noted that the function $x_1^0(t)$ is quasiperiodic with periods 2π and

Table. The coefficients of $T_1^0(u_1,u_2)$

p	p	p
(0, 0)	.3795834E+00	
(1, 0)	-.1574205E-01	-.6167170E-01
(0, 1)	.5606771E-02	-.1574205E-01
(2, 0)	.4749099E-04	.2953473E-01
(1, 1)	.1342754E-04	.5606771E-02
(0, 2)	.1948652E-05	.3961999E-04
(-1, 1)	.2778561E-03	.4749099E-04
(3, 0)	.4026646E-07	.5281227E-05
(2, 1)	.1739084E-07	.1342254E-04
(1, 2)	.2720843E-07	.1967783E-08
(0, 3)	.4795622E-07	.1948652E-05
(-1, 2)	-.7669865E-07	-.3352412E-04
(-2, 1)	-.2413022E-06	.2778561E-03
(4, 0)	.9457390E-09	-.1037135E-07
(3, 1)	.1770202E-09	.4026646E-07
(2, 2)	.2757679E-09	-.9750039E-08
(1, 3)	.1413319E-07	.1739084E-07
(0, 4)	.3151296E-08	.1704890E-08
(-1, 3)	-.2259874E-07	.2720843E-07
(-2, 2)	.6152877E-08	-.4391073E-07
(-3, 1)	.4286024E-07	.4795622E-07

R= .6693911E-06 W= .4736636E+00

$2\pi/\sqrt{2}$ and $\bar{x}_1^0(t,u)=T_1^0(t,t+u)$ is the pseudoperiodic function corresponding to $x_1^0(t)$.

Denote that $\|x(t)\|=\max\{|x_1(t)|,|x_2(t)|\}$ for any function $x(t)=(x_1(t),x_2(t))$. For the obtained approximation $x_0(t)$ the positive number r in (2.6) and w satisfying

(5.4) $\max\{\|x_0(t)\|:t \text{ in } R\}\leqq w$

are also computed as indicated in Table.

The Jacobian matrices of X[t,x,y] with respect to x and y are derived in the forms:

(5.5) $\Phi[t,x,y]=\begin{pmatrix} 0 & 1 \\ -2 & -1/8 \end{pmatrix}$

and

$$(5.6)\qquad \Psi[t,x,y]=\begin{bmatrix} 0 & 0 \\ y_1/8 & 0 \end{bmatrix}.$$

We choose a matrix $A(t)$ in the assumptions of Theorem 4 as follows:

$$(5.7)\qquad A(t)=\bar{A}(t,u)=\begin{bmatrix} 0 & 0 \\ -2 & -1/8 \end{bmatrix},$$

which is quasiperiodic and pseudoperiodic. The fundamental matrix $U(t,u)$ with $U(0,u)=E$ (the unit matrix) of the linear system

$$(5.8)\qquad \bar{L}y=\frac{dy}{dt}-\bar{A}(t,u)y=0$$

is given in the form:

$$U(t,u)=e^{-(1/8)t}\begin{bmatrix} \cos\lambda t+(1/8\lambda)\sin\lambda t & (1/\lambda)\sin\lambda t \\ -(2/\lambda)\sin\lambda t & \cos\lambda t-(1/8\lambda)\sin\lambda t \end{bmatrix},$$

where $\lambda=\sqrt{2-(1/64)}$. If we choose $P(u)=E$, the axioms (1.2)-(1.5) are fulfilled. Hence, the pseudoperiodic differential operator $\bar{L}$ in (5.8) is regular. Therefore the quasiperiodic differential operator L defined by $Ly=dy/dt-A(t)y$ is regular. A Green function $\bar{G}(t,s,u)$ of the operator $\bar{L}$ is given as follows:

$$\bar{G}(t,s,u)=\begin{cases} U(t-s) & \text{for } t\geqq s \\ 0 & \text{for } t\leqq s \end{cases}$$

and hence $\|\bar{G}(t,s,u)\|\leqq K\exp[-(1/8)|t-s|]$, where $K=\sqrt{2}(\sqrt{2}+1)/\lambda$.

For any x in D_δ defined in the assumptions of Theorem 4 we have $\|x\|\leqq\delta+\|x_0(t)\|\leqq\delta+w$ by (5.4) and thus $\|\Psi[t,x,y]\|=(1/8)|y_1|\leqq(\delta+w)/8$. The inequality (2.8) automatically holds. In order for all assumptions of Theorem 4 concerning the system (5.1) to be fulfilled, it is sufficient that we choose $\kappa=0$, $\mu=(\delta+w)/8$ and a positive number δ satisfying the relations (2.10), that are, (5.9) $M(\delta+w)/8<1$ and $Mr/[1-M(\delta+w)/8]\leq\delta$,

where

(5.10) $M=2K/(1/8)=16\sqrt{2}(\sqrt{2}+1)/\lambda$.

Actually we can choose $\delta=4.176844\times10^{-6}$ so as to satisfy the relations (5.9) and (5.10).

Hence it is concluded by Theorem 4 that there exists uniquely an exact solution $x=\hat{x}(t)=(\hat{x}_1(t),\hat{x}_2(t))$ of the system (5.1) quasiperiodic with periods 2π and $2\pi/\sqrt{2}$ lying in D_δ for all t and it satisfies the error estimation of the form:

$$\|\hat{x}(t)-x_0(t)\|\leq\delta=4.176844\times10^{-6}.$$

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