数理解析研究所講究録
第 438 巻 1981 年 6-8

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REALIZATIONS OF LIE ALGEBRAS

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This is an expository talk on realizations of Lie algebras.

1. The classical invariant theory

Let us choose a generic polynomial

$$f(\xi|z) = \sum_{\ell=0}^{n} \binom{n}{\ell} \xi^{(\ell)} z^{(\ell)},$$

on which $SL_2(K)$ acts as follows

$$(1) \qquad f(\begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix} \xi | z) = (\gamma z+\delta)^m f(\xi | \frac{\alpha z+\beta}{\gamma z+\delta}),$$

i.e.,

$$(2) \qquad (\begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix} \xi)^{(\ell)} = \sum_{p,q} \binom{\ell}{p} \binom{n-\ell}{q} \xi^{(\ell-p+q)} \alpha^{\ell-p} \beta^q \gamma^p \delta^{n-\ell-q}.$$

The corresponding realization of $sl_2(K)$ is given by

$$\begin{cases} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \mathcal{D} = \sum \ell \xi^{(\ell-1)} \dfrac{\partial}{\partial \xi^{(\ell)}} , \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \Delta = \sum (n-\ell) \xi^{(\ell+1)} \dfrac{\partial}{\partial \xi^{(\ell)}} , \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \mathcal{H} = \sum (n-2\ell) \xi^{(\ell)} \dfrac{\partial}{\partial \xi^{(\ell)}} . \end{cases}$$

Definition 1.

$\rho^{[m]}$ = {covariants of index m}

$$= \left\{ F(\xi,z) = \sum \binom{m}{\ell} c_\ell(\xi) z^\ell \;\middle|\; \begin{array}{l} c_\ell(z) \in K[\xi], \\ F\left(\begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix}\xi, z\right) = (\gamma z+\delta)^m F\left(\xi, \frac{\alpha z+\beta}{\gamma z+\delta}\right) \end{array} \right\}.$$

Definition 2.

$$\mathfrak{S}^{[m]} = \{\text{semi-invariants of index } m\}$$
$$= \{\varphi \in K[\xi] \,|\, \mathcal{D}\varphi = 0,\ \mathcal{H}\varphi = m\varphi\}.$$

Problem. To seek all covariants of index m.

Solution. (Robert's theorem)

$$\rho^{[m]} = \{\exp(z\Delta)\varphi(\xi) \,|\, \varphi(\xi) \in \mathfrak{S}^{[m]}\}.$$

Remark. This solution $\exp(z\Delta)\varphi(\xi)$ is a typical explicit solution of mathematical problems.

2. Automorphic forms

Let us choose a formal power series

$$f(\xi|z) = \sum_{\ell=0}^{\infty} \frac{(-2k)_\ell}{\ell!} \xi^{(\ell)} z^\ell$$

with variable coefficients, where

$$(-2k)_\ell = (-2k)(-2k-1)\cdots(-2k-\ell+1).$$

Denoting

$$\begin{cases} \mathcal{D} = \sum \ell \xi^{(\ell-1)} \dfrac{\partial}{\partial \xi^{(\ell)}}, \\ \Delta = \sum (-2k-\ell) \xi^{(\ell+1)} \dfrac{\partial}{\partial \xi^{(\ell)}}, \\ \mathcal{H} = \sum (-2k-2\ell) \xi^{(\ell)} \dfrac{\partial}{\partial \xi^{(\ell)}}, \end{cases}$$

we have a realization of $sl_2(\mathbb{C})$. Denote

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$$\mathfrak{G}^{[-2m]} = \{\varphi \in K[\xi] \mid \mathcal{D}\varphi = 0,\ \mathcal{H}\varphi = -2m\varphi\}.$$

Problem. Let $h(z)$ be an automorphic form of dimension $-2k$. To seek all automorphic forms of dimension $-2m$ which are differential polynomials of $h(z)$.

Solution. Assume that the Zariski closure of the automorphic group coincides with $PSL_2(\mathbb{C})$. And denote

$$h(z) = \sum_{\ell=0}^{\infty} \frac{(-2k)_\ell}{\ell!} \alpha^{(\ell)} z^\ell.$$

Then

$$\{\exp(z\Delta)\varphi(\xi)\big|_{\xi=\alpha} \mid \varphi(\xi) \in \mathfrak{G}^{[-2m]}\}$$

$$= \left\{\begin{array}{l}\text{automorphic forms of dimension } -2m \\ \text{which are differential polynomials of } h(z)\end{array}\right\}.$$

Reference

Hisasi Morikawa, Invariant theory, Kinokuniya.