

A Characterization of QF - rings

Manabu HARADA

Department of Mathematics, Osaka City University

It is well known that T. Nakayama found quasi - Frobenius rings (QF-rings) as a generalization of group algebras [3]. Nowadays we know many characterizations of QF-rings. One of them is that  $R$  is self-injective as a right  $R$ -module. Other one is that every indecomposable and projective right ideal  $e_1R$  contains the unique minimal right ideal  $r_1$  and if  $e_1R \neq e_2R$ ,  $r_1 \neq r_2$  and the above statements are valid for left ideals, where the  $e_i$  is a primitive idempotent.

In this note we shall combine the two weakened conditions. We always assume that  $R$  is a left and right artinian ring with identity and every  $R$ -module is a unitary right  $R$ -module.

Let  $M$  be an  $R$ -module. We consider a diagram for a minimal right ideal  $I$  of  $R$ :

$$\begin{array}{ccccc}
 0 & \longrightarrow & I & \xrightarrow{i} & R \\
 & & \downarrow f & \swarrow h & \\
 & & M & & 
 \end{array}$$

If there exists  $h \in \text{Hom}_R(R, M)$  with  $f = hi$  for any  $f$  and  $i$ , we say  $M$  is right mini-injective. We can define similarly left mini-injective. If  $R$  is right mini-injective

as a right  $R$ -module, we say  $R$  is right self mini-injective. We note that right self mini-injective ring is not left self mini-injective in general.

We shall show

Theorem 1. Let  $R$  be a right and left artinian ring. Then  $R$  is QF-ring if and only if  $R$  is right and left self mini-injective.

Theorem 2. Let  $R$  be an algebra over a field  $K$  of finite dimension. Then  $R$  is a QF-algebra if and only if  $R$  is a right self mini-injective algebra.

Remark. Theorem 2 is not valid for artinian rings.

Proofs of Theorems.

We shall publish the results in this note in [1] and [2] and so we give here a sketch of the proofs.

In this note, we consider only artinian rings and so from now on we understand a ring  $R$  is always right artinian. We denote the Jacobson radical by  $J = J(R)$ .

We put  $\bar{R} = R/J$ . Let

$$R = \sum_{i=1}^n \sum_{j=1}^{\rho(i)} e_{ij} R$$

be the standard decomposition, namely the  $e_{ij}$  is a primitive idempotent and  $e_{ij} R \cong e_{il} R$ ,  $e_{il} R \not\cong e_{jl} R$  if  $i \neq j$ . If the socle  $S(e_{il} R)$  of  $e_{il} R$  is simple for each  $i$ , then we say  $R$  is right QF-2.

Proposition 1. Let  $R$  be as above. If  $R$  is right self mini-injective, then

- 1) If  $e_1R \neq e_2R$ , each minimal submodule in  $e_1R$  is not isomorphic to any minimal one in  $e_2R$ .
- 2)  $S(e_1R) = e_1J^k$  and every minimal submodule in  $e_1R$  is isomorphic each other.
- 3)  $r(J) \supseteq l(J)$ .

where the  $e_i$  is a primitive idempotent,  $r(J) = \{x \in R \mid Jx = 0\}$  and  $l(J) = \{x \in R \mid xJ = 0\}$ .

Proof. 1) is clear from the definition.

2) We take a minimal right ideal  $I$  in  $e_1J^k \neq 0$  ( $e_1J^{k+1} = 0$ ). Using the definition and 1), we can show  $S(e_1R) = e_1J^k$ .

3) Let  $x$  be in a minimal right ideal in  $e_1R$ . Then we can show  $Jx \subseteq \sum_{i=1}^n e_i Jx = 0$  from 1) and 2), where  $1 = \sum_{i=1}^n e_i$ .

Proof of Theorem 2.

First we shall show that  $R$  is right QF-2. Let  $I_1$  be a minimal right ideal in  $e_1R$ . We assume  $I_1 \simeq \overline{e_2'R}$ . Then we can show from the definition and Proposition 1 that there is a monomorphism of  $\text{End}_R(\overline{e_2'R})$  to  $\text{End}_R(\overline{e_1R})$ . Hence,  $[\overline{e_1Re_1} : K] = [\overline{e_2'Re_2'} : K]$ . Next take a minimal right ideal  $I_2$  in  $e_2R$ . Repeating this argument, we have a chain  $\{e_1, e_2', \dots\}$  of primitive idempotents. Then  $e_1 \simeq et'$  for some  $t$  by Proposition 1. Hence,

$[\overline{e_1 R e_1} : K] = [\overline{e_2 R e_2} : K]$  implies that  $I_1$  is a unique minimal right ideal in  $e_1 R$ . Accordingly  $e_1 R$  is uniform and so an injective envelope  $E(e_1 R)$  of  $e_1 R$  is indecomposable. Therefore,  $E(e_i R) \simeq \text{Hom}_K(\text{Re}_{\pi(i)}, K)$ . Since  $S(e_i R) \not\cong S(e_j R)$  if  $e_i R \neq e_j R$  by Proposition 1,  $[R; K] = \sum_{i=1}^n [\text{Re}_{\pi(i)} : K] = \sum_{i=1}^n [E(e_i R) : K] = [E(R) : K]$ .

Proof of Theorem 1.

Let  $xR$  be a minimal right ideal in  $e_1 R$  and  $xR \simeq \overline{e_2 R}$ . Since  $Jx = 0$  by Proposition 1 and  $R$  is left self mini-injective, for any element  $b$  in  $\overline{e_1 R e_1}$ , there exists  $a$  in  $\overline{e_2 R e_2}$  such that  $bx = xa$  as above. Hence,  $xR = S(e_1 R)$  and  $R$  is left and right QF-2.

#### References

- [1] M. Harada, On self mini-injective rings, to appear.
- [2] \_\_\_\_\_, A characterization of QF-rings, to appear.
- [3] T. Nakayama, On Frobenius algebras II, Ann. of Math. 40(1941), 1 - 21.