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On elements contained in the radical of a group algebra

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What kind of nilpotent element in a ring is contained in the (Jacobson) radical ? Answers to this problem are little known in ring theory. It seem to be difficult to say much about this problem. In commutative rings or in the ring of upper triangular matrices of a finite degree over a field, every nilpotent element is contained in the radical. On the other hand, an Artinian simple ring contains many nilpotent elements. In this note, we shall present a lemma for this problem and some applications of this lemma to a group algebra over a field, especially a group algebra of a finite group with a (B, N)-pair.

Throughtout this note we shall use the following notations: Let p be a fixed prime number, KG a group algebra of a group G over a field K of characteristic p, $J(KG)$ the radical of KG, and for a finite group G, $t(G)$ the nilpotency index of $J(KG)$. Further, for a finite subset U of G, U denotes the sum of all elements of U.

The following lemma is useful and plays an important role in this note.

Lemma 1. Let A be a ring. Let B, I and J be subsets of A satisfying the following conditions:

(1) $IAI = IBI$, (2) $IJ = JI$, (3) $BJ \subseteq JB$.

Then $(JIA)^n \subseteq J^nIA$. Moreover, if $J^n = 0$, then JIA is contained in the radical of A.

The following proposition shall have many applications.

Proposition 2. Suppose that a group G has finite subgroups H and U such that $G = UN_G(H)U$ and $H \subseteq N_G(U)$. Then we have $J(KH)\hat{U} \subseteq J(KG)$.

The justifications for Lemma 1 and Proposition 2 are given in the following examples.

Example 3. Let $q = p^e$, F the finite field of q^p elements, F_q the finite field of q elements, H the Galois group of F over F_q, b a generator of the multiplicative group of F and $T = \langle b^{q-1} \rangle$. We consider permutation groups on F as follows: $V = \{v_a : x \to x + a \mid a \in F\}$, $U = \{u_t : x \to tx \mid t \in T\}$, and $G = \langle H, U, V \rangle$. Since $C_G(H)$ contains $\{v_a \mid a \in F_q\}$, $F = TF_q$ because of $(q - 1, (q^p - 1)/(q - 1)) = 1$, and $u_t v_a u_t^{-1} = v_{ta}$, we can see that $G = UC_G(H)U$. Noting that $hu_t h^{-1} = u_{h(t)}$ for all $h \in H$, we obtain $J(KH)\hat{U} \subseteq J(KG)$.

The above example played a fundamental role in [3]. The following shall show that Proposition 2 is applicable to finite groups of Lie type (See [1]).

Proposition 4. Suppose that a finite group G has a split (B, N)-pair of characteristic r such that B is a semi-direct product of a normal r-subgroup U of B and an abelian r'-subgroup $H = B \cap N$ (See [1]). Then $J(KH)\hat{U} \subseteq J(KG)$.

The following proposition follows from Dade's theorem [2] and Proposition 4.

Proposition 5. Let r be a prime number, $q = r^e$, p an odd prime divisor of $q - 1$, and $G = SL(2, q)$. Then $t(G)$ is p^m the p-part of $q - 1$.

The similar consideration to that given in Proposition 5 is also valid for finite groups of Lie type. Let N be a normal p'-subgroup of a finite group G. K. Morita [4] had given a theorem such that the structure of KG can be obtained from that of KN. The following is a partial modification of it by Lemma 1.

Proposition 6. Let N be a normal subgroup of a finite group G, e a centrally primitive idempotent of KN and $H = \{x \in G \mid xex^{-1} = e\}$. Then $J(KHe) = J(KH)e \subseteq J(KG)$.

References

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