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On some properties of a minimal flow

— A topological characterization of the strict ergodicity —

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1. Introduction.

A flow ϕ_t $(t \in \mathbb{R})$ on a compact metric space M is said to be minimal, if M is equal to the orbit closure of each of its points. In the following, we always assume that our flow ϕ_t is a minimal flow on M. ϕ_t is called strictly ergodic, if it admits a unique normalized invariant Borel measure. In [1], Oxtoby has given some nice characterizations of the strict ergodicity (see Theorem A below).

It is known that ϕ_t is Lyapunov-stable if and only if ϕ_t has a unique normalized invariant measure μ and the unitary operator U_t in $L_2(M,\mu)$ defined by $U_t f = f \circ \phi_t$ has pure point spectrum. As is implied by this example, some metrical property can be characterized by a topological property. The Oxtoby's characterization of the strict ergodicity seems not to be purely topological. And so, in what follows, we shall try to give a purely topological characterization of the strict ergodicity.

2. Known results.

By $C(M)$ we denote the set of real valued continuous functions on M . And for $f \in C(M)$, we define $M(f,m,T)$ and $M(f,m)$ $(m \in M,\ T \in \mathbb{R})$ as follows:

$$M(f,m,T) = \frac{1}{T} \int_0^T f(\phi_t(m))dt , \quad M(f,m) = \lim_{T\to\infty} M(f,m,T) .$$

A point p M is called <u>quasi-regular</u>, if $M(f,p)$ is defined for every $f \in C(M)$. And by Q we denote the set of all quasi-regular points. Oxtoby's characterization is the following.

<u>THEOREM A</u>. ([1]) For a minimal flow (M, ϕ_t) , the following three conditions are equivalent.

(i) ϕ_t is strictly ergodic.

(ii) $Q = M$.

(iii) $M(f,m,T)$ $(T \to \infty)$ converges uniformly on M to a constant for each $f \in C(M)$.

It is known (see [1]) that for each $p \in Q$ there is a unique normalized invariant measure μ_p such that

$$M(f,p) = \int_M f \, d\mu_p$$

for any $f \in C(M)$. The following theorem of Schwartzman gives a relation between this invariant measure μ_p and the "asymptotic cycles".

<u>THEOREM B</u>. ([2]) For any $p \in Q$ and any $g \in C(M,S^1)$,

$$A_p(g) = \int_M A_m(g) \, d\mu_p(m) ,$$

where $C(M,S^1)$ ($S^1 = \{z \in \mathbb{C} ; |z| = 1\}$) is the set of S^1-valued continuous functions, and $A_m : C(M,S^1) \to R$ $(m \in M)$ is defined by

$$A_m(g) = \lim_{T\to\infty} \frac{1}{2\pi T} (\arg(g(\phi_t(m)) - \arg g(m)) .$$

From these two theorems, it follows that if ϕ_t is strictly ergodic, then $A_m(g)$ does not depend on m .

3. Topological characterization of the strict ergodicity.

In this section, we assume that M is an n-dimensional smooth manifold and ϕ_t is a smooth minimal flow on M . An open submanifold Σ with codimension one is said to be a local section if it does not contact with the flow everywhere. Moreover we call Σ a δ-local section $(\delta > 0)$ if the mapping h defined by $h(m,t) = \phi_t(m)$ is a homeomorphism from $\overline{\Sigma} \times (-\delta, \delta)$ onto $\bigcup_{-\delta<t<\delta} \phi_t(\overline{\Sigma})$. For two δ-local sections Σ_1 and Σ_2 , we define a subset $K(\Sigma_1, \Sigma_2)$ of Σ_1 to be

$$K(\Sigma_1, \Sigma_2) = \{m \in \Sigma_1 ; \int_0^t (\chi_1(\phi_t(m)) - \chi_2(\phi_t(m)))dt > c > -\infty \text{ for any positive } t \} ,$$

where χ_j is the characteristic function of $\bigcup_{0<t<\delta} \phi_t(\Sigma_j)$. It can be shown that if $K(\Sigma_1, \Sigma_2)$ is neither empty nor Σ_1 , then it is a countable union of nowhere dense closed subset of Σ_1 . With this definition we can give a topological characterization of the strict ergodicity as the following.

THEOREM C. ϕ_t is strictly ergodic if and only if the following condition (*) is satisfied:

(*) whenever $K(\Sigma_1, \Sigma_2)$ is neither empty nor Σ_1, for any compact subset K of $K(\Sigma_1, \Sigma_2)$ and for any local section Σ, there is a bounded function $F : K \to R$ such that $\hat{F}(K) \subset \Sigma$ and $\hat{F}$ is injective ($\hat{F}(m) = \phi_{F(m)}(m)$) .

Moreover we can prove that

THEOREM D. If ϕ_t is not strictly ergodic, then for any normalized invariant Borel measure μ there are δ-local sections Σ_1 and Σ_2 such that $K(\Sigma_1, \Sigma_2) \neq \phi$, Σ_1 and $\mu(\bigcup_{0<t<\delta} \phi_t(K))$ is positive for some compact set $K \subset K(\Sigma_1, \Sigma_2)$.

In the following, we shall give an outline of the proof of Theorem C . First we state a lemma.

LEMMA. For any local section Σ we can construct a minimal flow $\tilde{\phi}_t$ on a compact metric space $\tilde{M}$ which has the following properties:

(i) $\tilde{\phi}_t$ is an extension of ϕ_t , namely there is a continuous map $p : \tilde{M} \to M$ such that $p \circ \tilde{\phi}_t = \phi_t \circ p$,

(ii) $\tilde{\Sigma} = \overline{p^{-1}(\Sigma)}$ is a cross-section of $\tilde{\phi}_t$,

(iii) to each invariant measure $\tilde{\mu}$ of $\tilde{\phi}_t$, there corresponds an invariant measure μ of ϕ_t such that $\mu = \tilde{\mu} \circ p^{-1}$ and this correspondence is one-to-one.

For the method for constructing such a minimal flow, one should refer to [3].

Outline of the proof of THEOREM A.

(Sufficiency) By the lemma, it is sufficient to show that the Poincare-map T induced by $\tilde{\phi}_t$ on the cross-section Σ has a unique normalized invariant Borel measure. The condition (*) implies that T has an invariant measure ν which satisfies that $\nu(p^{-1}(\Sigma_1)) = p^{-1}(\Sigma_2))$ (Σ_1 and Σ_2 are open subsets of Σ) if and only if $K(\Sigma_1, \Sigma_2) \neq \phi$, Σ_1 or $K(\Sigma_1, \Sigma_2) - \Sigma_1 = K(\Sigma_2, \Sigma_1) - \Sigma_2 = \phi$. And moreover we can see that this invariant measure ν is the unique one.

(Necessity) Suppose that $K(\Sigma_1, \Sigma_2)$ is neither empth nor Σ_1 and for some its compact subset there is a local section Σ for which there is no bounded function F such that $\hat{F}$ is injective. We can construct a minimal flow $\overline{(\tilde{M}, \tilde{\phi}_t)}$ satisfying (i),(ii),(iii) of the lemma so that $\tilde{\Sigma}_j = p^{-1}(\Sigma_j)$ $(j = 1, 2)$ are both cross-sections of $\tilde{\phi}_t$. Here we define functions τ_j and T_j $(j = 1, 2)$ on M as follows:

$$\tau_j(\tilde{m}) = \sup\{ t \leqq 0 \mid \tilde{\phi}_t(\tilde{m}) \in \tilde{\Sigma}_j \}$$

$$T_j(\tilde{m}) = \inf\{ t > 0 \mid \tilde{\phi}_t(\tilde{m}) \in \tilde{\Sigma}_j \} ,$$

and define $f_j : \tilde{M} \to S^1$ by

$$f_j(\tilde{m}) = \exp(2\pi\sqrt{-1}\tau_j(\tilde{m})/T_j(\hat{\tau}_j(\tilde{m}))) , \quad (\hat{\tau}_j(\tilde{m}) = \tilde{\phi}_{\tau_j(\tilde{m})}(\tilde{m})).$$

Because $\tilde{\Sigma}_j$ is a cross-section, f_j is continuous. Then, according to the assumption on Σ_j , we can see that for $f = f_1/f_2 \in C(\tilde{M}, S^1)$, $A_{\tilde{m}}(f)$ cannot be independent of $\tilde{m}$. Hence, by Theorems A and B, $\tilde{\phi}_t$ is not strictly ergodic, and so ϕ_t is also not strictly ergodic.

REFERENCES

[1] Oxtoby, J.C., Ergodic sets, Bull. Amer. Math. Soc. 58 (1952), 116-136.

[2] Schartzman, S., Asymptotic cycles, Ann. of Math. 66 (1957), 270-284.

[3] Ishii, I., On the first cohomology group of a minimal set, Tokyo J. of Math. 1 (1978), 41-56.