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Alexander Polynomials of Two-Bridge Links

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Hartley [H] gave a necessary condition for a polynomial to be the Alexander polynomial of a two-bridge knot or the reduced Alexander polynomial of a two-bridge link. He showed how the coefficients of the polynomial may be read straight from the extended diagram, which is derived from Schubert's normal form of a two-bridge knot or link, and showed the following theorem: If $\Delta(t) = \sum_{i=0}^{n}(-1)^i a_i t^i$, where $a_i > 0$, is the Alexander polynomial of a two-bridge knot or the reduced Alexander polynomial of a two-bridge link, then for some integer s, $a_0 < a_1 < \cdots < a_s = a_{s+1} = \cdots = a_{n-s} > \cdots > a_n$. On the other hand, using surgery techniques, Bailey [B] presented an algorithm for calculating the Alexander polynomial of a two-bridge link from Conway's diagram. As a corollary to this he proved a conjecture of Kidwell about the linking complexity or geometric intersection numbers of a link in the special case of two-bridge links.

The main results of this paper are Theorems 1 and 3, the former provides another algorithm for calculating the Alexander polynomial of a two-bridge link putting every two-bridge link in the special form of Conway's diagram. The latter gives some necessary conditions for a polynomial to be the Alexander polynomial of a two-brdge link. These conditions are analogous to Hartley's theorem above. Theorem 2 and Corollary 1 also give some properties of the Alexander polynomial of a two-bridge link, including the Torres condition [T]. Corollary 2 is a conjecture of Kidwell in the case of two-bridge links.

In Section 2, we show some lemmas for Theorems 1 and 2 using Fox's free differential calculus. In Section 3, we summarize some properties of two-bridge links. In Section 4, we state the above-mentioned results. In Section 5, we prove Theorem 3.

1. Preliminaries

In this paper, a link L will mean a piecewise linear embedding of two oriented circles K_1 and K_2 in the 3-sphere S^3. Two links L and L' are called equivalent, if there is an orientation preserving autohomeomorphism of S^3, which maps L onto L'. The Alexander polynomial $\Delta(x,y)$ of L is an element of the polynomial ring $Z[x,x^{-1},y,y^{-1}] = \Lambda$, and is determined only up to multiplication by a unit $\pm x^i y^j$. Let $G = \pi_1(S^3 - L)$, and let G' be its commutator subgroup. Then $\Lambda = Z[G/G']$; the basis $\{x,y\}$ of G/G' is always taken to be represented by the meridians of K_1 and K_2 respectively. We will calculate the Alexander polynomial of a link by using Fox's free differential calculus, see [F], [T].

Throughout this paper, we will often abbreviate a polynomial $f(x,y)$ in Λ to f and will use the following notation;

$$F_n(x,y) = \begin{cases} \sum_{i=0}^{n-1} (xy)^i & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{i=n}^{-1} (xy)^i & \text{if } n < 0. \end{cases}$$

In the figures of this paper we will use a tangle [C], which is a portion of the link diagram containing two arcs. An integral tangle, which is represented by a circle labeled "i" or "-i", where i is a non-negative integer, is a 2-braid with i or $-i$ crossings, in the manner indicated in Fig. 1.

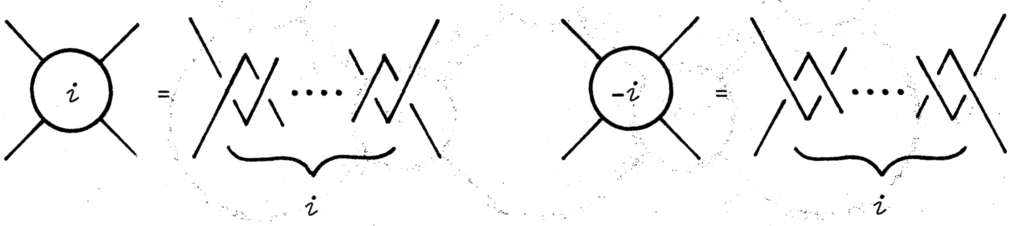

Fig. 1

2. Lemmas

Lemma 1. Let L(q,r,s,t) be a link as shown in Fig. 2, where T is any tangle. Let $\Delta^{(q,r,s,t)}$ be the Alexander polynomial of L(q,r,s,t). If we set $\Delta = \Delta^{(q,r,s,t)}$, $\Delta_0 = \Delta^{(q,r,0,0)}$ and $\Delta_{00} = \Delta^{(0,0,0,0)}$, then

$$(2.1) \qquad \Delta = \{s(x-1)(y-1)F_t + 1\}\Delta_0 + \frac{F_t}{F_r}(xy)^r(\Delta_0 - \Delta_{00}),$$

where $r \neq 0$.

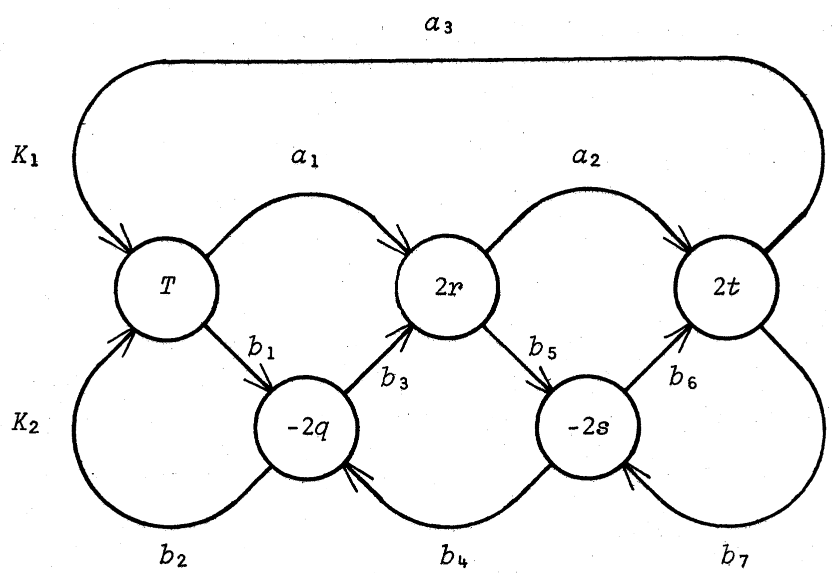

Fig. 2

Proof. We have a presentation of $\pi_1(S^3 - L(q,r,s,t))$ as follows:

generators; $a_1, a_2, a_3, b_1, b_2, b_3, b_4, b_5, b_6, b_7,$

$c_i \quad (7 \leqq i \leqq n+1),$

relations; (i) $b_3 = (b_1^{-1}b_2)^q b_1 (b_2^{-1}b_1)^q,$

(ii) $b_4 = (b_1^{-1}b_2)^q b_2 (b_2^{-1}b_1)^q,$

(iii) $a_2 = (b_3a_1)^r a_1 (a_1^{-1}b_3^{-1})^r,$

(iv) $b_5 = (b_3a_1)^r b_3 (a_1^{-1}b_3^{-1})^r,$

(v) $b_6 = (b_5^{-1}b_4)^s b_5 (b_4^{-1}b_5)^s,$

(vi) $b_7 = (b_5^{-1}b_4)^s b_4 (b_4^{-1}b_5)^s,$

(vii) $a_3 = (b_6a_2)^t a_2 (a_2^{-1}b_6^{-1})^t,$

(viii) $b_7 = (b_6a_2)^t b_6 (a_2^{-1}b_6^{-1})^t,$

(ix_j) $S_j = 1 \quad (5 \leqq j \leqq n),$

where c_i and $S_j = 1$ are obtained in the tangle T, so S_j is a word in $a_1, a_2, b_1, b_2, c_7, \ldots, c_{n+1}$.

(i) and (ii), (iii) and (iv), (v) and (vi), (vii) and (viii) imply

(ii') $b_4 = b_3b_1^{-1}b_2,$

(iv') $b_5 = b_3a_1a_2^{-1},$

(vi') $b_7 = b_6b_5^{-1}b_4,$

(viii') $b_7 = b_6a_2a_3^{-1},$

respectively, and we eliminate (ii), (iv), (vi) and (viii).

Using (vi') and (viii'), we have

(x) $b_5^{-1}b_4 = a_2a_3^{-1}$

and we eliminate b_7. Substituting (iv') and (x) in (v), we have

$$(v') \quad b_6 = (a_2a_3^{-1})^s b_3 a_1 a_2^{-1} (a_3a_2^{-1})^s$$

and we eliminate (v). Substituting (v') in (vii), we have $R_1 = 1$, where

$$R_1 = \{(a_2a_3^{-1})^s b_3 a_1 (a_2^{-1}a_3)^s\}^t a_2 \{(a_3^{-1}a_2)^s a_1^{-1} b_3^{-1} (a_3a_2^{-1})^s\}^t a_3^{-1}$$

and we eliminate b_6. Substituting (ii') and (iv') in (x), we have $R_3 = 1$, where

$$R_3 = a_3 a_1^{-1} b_1^{-1} b_2$$

and we eliminate b_4 and b_5.

Hence $\pi_1(S^3 - L(q,r,s,t))$ is presented by

generators; $a_1, a_2, a_3, b_1, b_2, b_3, c_7, \ldots, c_{n+1}$,

relators; $R_1, R_2, R_3, R_4, S_5, \ldots, S_n$,

where

$$R_2 = (b_3a_1)^r a_1 (a_1^{-1}b_3^{-1})^r a_2^{-1}$$

and

$$R_4 = (b_1^{-1}b_2)^q b_1 (b_2^{-1}b_1)^q b_3^{-1},$$

which come from (iii)and (i) respectively.

From this presentation we have the Alexander matrix of $L(q,r,s,t)$ as follows:

$$
\begin{array}{c}
 \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ S_5 \\ \vdots \\ S_n
\end{array}
\begin{array}{c}
\begin{array}{cccccc|ccc}
b_3 & a_2 & a_3 & a_1 & b_2 & b_1 & c_7 & \cdots & c_{n+1}
\end{array} \\
\left[\begin{array}{cccccc|ccc}
(1-x)F_t & f & g & (1-x)yF_t & 0 & 0 & & & \\
(1-x)F_r & -1 & 0 & (y-1)F_r+1 & 0 & 0 & & O & \\
0 & 0 & 1 & -1 & y^{-1} & y^{-1} & & & \\
-1 & 0 & 0 & 0 & q(y^{-1}-1) & -q(y^{-1}-1)+1 & & & \\
\hline
0 & 0 & \alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & \beta_{57} & \cdots & \beta_{5,n+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \alpha_{n4} & \beta_{n7} & \cdots & \beta_{n,n+1}
\end{array}\right]
\end{array}
$$

where $f = s(x-1)(y-1)F_t + (xy)^t$ and $g = -s(x-1)(y-1)F_t - 1$. Add the third column to the second column, the first column multiplied by $-y$ to the fourth column, and the fifth column to the sixth column, and delete the last column. Then we have an $n \times n$ matrix as follows:

$$
(2.2) \qquad
\left[\begin{array}{cccccc|c}
(1-x)F_t & (xy-1)F_t & g & 0 & 0 & 0 & \\
(1-x)F_r & -1 & 0 & (xy)^r & 0 & 0 & \\
0 & 1 & 1 & -1 & y^{-1} & 0 & O \\
-1 & 0 & 0 & y & q(y^{-1}-1) & 1 & \\
\hline
\mathbf{0} & A_1 & A_1 & A_2 & A_3 & A_3+A_4 & B
\end{array}\right]
$$

where $\mathbf{0}$ is a zero column vector of dimnsion $n-4$, A_i is the column vector $\begin{bmatrix} \alpha_{5i} \\ \vdots \\ \alpha_{ni} \end{bmatrix}$ and B is the matrix $\begin{bmatrix} \beta_{57} & \cdots & \beta_{5n} \\ \vdots & & \vdots \\ \beta_{n7} & \cdots & \beta_{nn} \end{bmatrix}$.

Let $\Phi^{(q,r,s,t)}$ be the determinant of (2.2). Then

$$(2.3) \qquad \Phi^{(q,r,s,t)} = (z-1)\Delta^{(q,r,s,t)},$$

where z is x or y according as c_{n+1} is represented by the meridian of K_1 or K_2 [T, p.61]. Let C be the $(n-2) \times (n-2)$ matrix obtained from (2.2) by deleting the first and second row, and first and second column. The expansion of (2.2) according to the first row gives us

$$(2.4) \quad \Phi^{(q,r,s,t)} = \{s(x-1)(y-1)F_t + 1\}\Phi^{(q,r,0,0)} + \varphi^{(q,r,s,t)}.$$

Here

$$\varphi^{(q,r,s,t)} = (1-x)F_t \begin{vmatrix} -1 & 0 & (xy)^r & \mathbf{0}^T \\ 1 & & & \\ 0 & & C & \\ A_1 & & & \end{vmatrix} - (xy-1)F_t \begin{vmatrix} (1-x)F_r & 0 & (xy)^r & \mathbf{0}^T \\ 0 & & & \\ -1 & & C & \\ \mathbf{0} & & & \end{vmatrix},$$

where $\mathbf{0}^T$ is a zero row vector of dimension $n-4$. Since

$$\begin{vmatrix} -1 & 0 & (xy)^r & \mathbf{0}^T \\ 1 & & & \\ 0 & & C & \\ A_1 & & & \end{vmatrix} = \begin{vmatrix} -1 & 0 & (xy)^r & \mathbf{0}^T \\ 0 & & & \\ 0 & & C & \\ \mathbf{0} & & & \end{vmatrix},$$

we have

$$\varphi^{(q,r,s,t)} = (xy)^r F_t \begin{vmatrix} x-1 & 0 & 1 & \mathbf{0}^T \\ 0 & & & \\ xy-1 & & C & \\ \mathbf{0} & & & \end{vmatrix}.$$

On the other hand $\Phi^{(q,r,0,0)} = -\begin{vmatrix} (1-x)F_r & -1 & (xy)^r & \mathbf{0}^T \\ 0 & & & \\ -1 & & C & \\ \mathbf{0} & & & \end{vmatrix}$.

Since $L(q,0,0,0)$ is equivalent to $L(0,0,0,0)$, $\Phi^{(q,0,0,0)} = \Phi^{(0,0,0,0)}$ by (2.3), so we have

$$\Phi^{(q,r,0,0)} - \Phi^{(0,0,0,0)} = F_r \begin{vmatrix} x-1 & 0 & 1-xy & \mathbf{0}^T \\ 0 & & & \\ -1 & & C & \\ \mathbf{0} & & & \end{vmatrix}.$$

Since it is easily seen that

$$\begin{vmatrix} x-1 & 0 & 1 & \mathbf{0}^T \\ 0 & & & \\ xy-1 & & C & \\ \mathbf{0} & & & \end{vmatrix} = \begin{vmatrix} x-1 & 0 & 1-xy & \mathbf{0}^T \\ 0 & & & \\ -1 & & C & \\ \mathbf{0} & & & \end{vmatrix}$$

we have $\varphi^{(q,r,s,t)} = (xy)^r \frac{F_t}{F_r} (\Phi^{(q,r,0,0)} - \Phi^{(0,0,0,0)})$ if $r \neq 0$.

Thus it follows from (2.4) and (2.3), we obtain (2.1). □

<u>Lemma 2.</u> Besides the notation in Lemma 1, let $\Delta_0' = \Delta^{(q,r,0,t)}$ and $\Delta^{(t_0)} = \Delta^{(q,r,s,t_0)}$. Then

(2.5) $\Delta = s(x-1)(y-1)F_t\Delta_0 + \Delta_0'$;

(2.6) $\Delta^{(t)} = F_t\Delta^{(1)} - xyF_{t-1}\Delta_0$;

(2.7) $\Delta^{(t)} + xy\Delta^{(t-2)} = (1+xy)\Delta^{(t-1)}$.

Remarks. (1) In the above notation $\Delta^{(t)} = \Delta$ and $\Delta^{(0)} = \Delta_0$.

(2) (2.7) is a special case of Conway's result [C, p.338], see also [K, p.462].

Proof. Putting $s = 0$ in (2.1) we have

$$\Delta_0' = \Delta_0 + \frac{F_t}{F_r}(xy)^r(\Delta_0 - \Delta_{00}).$$

Combining this formula with (2.1) we obtain (2.5).

Next, from (2.1) we have

$$\Delta^{(t)} - \Delta_0 = F_t\{s(x-1)(y-1)\Delta_0 + (xy)^r\frac{1}{F_r}(\Delta_0 - \Delta_{00})\}$$
$$= F_t(\Delta^{(1)} - \Delta_0).$$

Since $1 - F_t = -xyF_{t-1}$, we obtain (2.6).

Finally, using (2.6) we have

$$\Delta^{(t)} - \Delta^{(t-1)} = (F_t - F_{t-1})\Delta^{(1)} - xy(F_{t-1} - F_{t-2})\Delta_0$$
$$= (xy)^{t-1}(\Delta^{(1)} - \Delta_0).$$

Hence

$$\Delta^{(t)} - \Delta^{(t-1)} = xy(\Delta^{(t-1)} - \Delta^{(t-2)}),$$

and (2.7) follows. □

3. Two-bridge links

According to Conway's presentation [C], every two-bridge link can be put in the form as shown in Fig. 3. It will be denoted by $C(a_1, a_2, \ldots, a_n)$ including the indicated orientation of each component. The diagram is slightly different in the cases $n = 2k$ and $n = 2k+1$, as indicated in Fig. 3. From this projection we can see that a two-bridge link is a link with two components each of which is a trivial knot. Moreover a two-bridge link is interchangeable, that is, there is an isotopy of S^3 which interchanges the two components. This follows immediately from Schubert's normal form [Sc], or Bailey [B, p.48] also proves this using Conway's diagram.

Let $\alpha (> 0)$ and β be the coprime integers computed from the continued fraction:

$$\frac{\alpha}{\beta} = a_1 + \frac{1}{a_2} + \ldots + \frac{1}{a_n}$$

Then α is even and $0 < |\beta| < \alpha$. The two-fold cover of S^3 branched over this link is the lens space $L(\alpha, \beta)$, see [C], [S]. This link is equivalent to the link with Schubert's normal form (α, β), denoted by $S(\alpha, \beta)$, endowed with suitable orientations. According to Schubert [Sc, p.144], $S(\alpha, \beta)$ and $S(\alpha', \beta')$ are equivalent if and only if $\alpha = \alpha'$ and $\beta^{\pm 1} \equiv \beta'$ (mod 2α). Furthermore, if $\beta' \equiv \beta + \alpha$ (mod 2α) or $\beta\beta' \equiv \alpha + 1$ (mod 2α), then $S(\alpha, \beta)$ differs from $S(\alpha, \beta')$ only by the orientation of one of the components (cf. [S, p.7]).

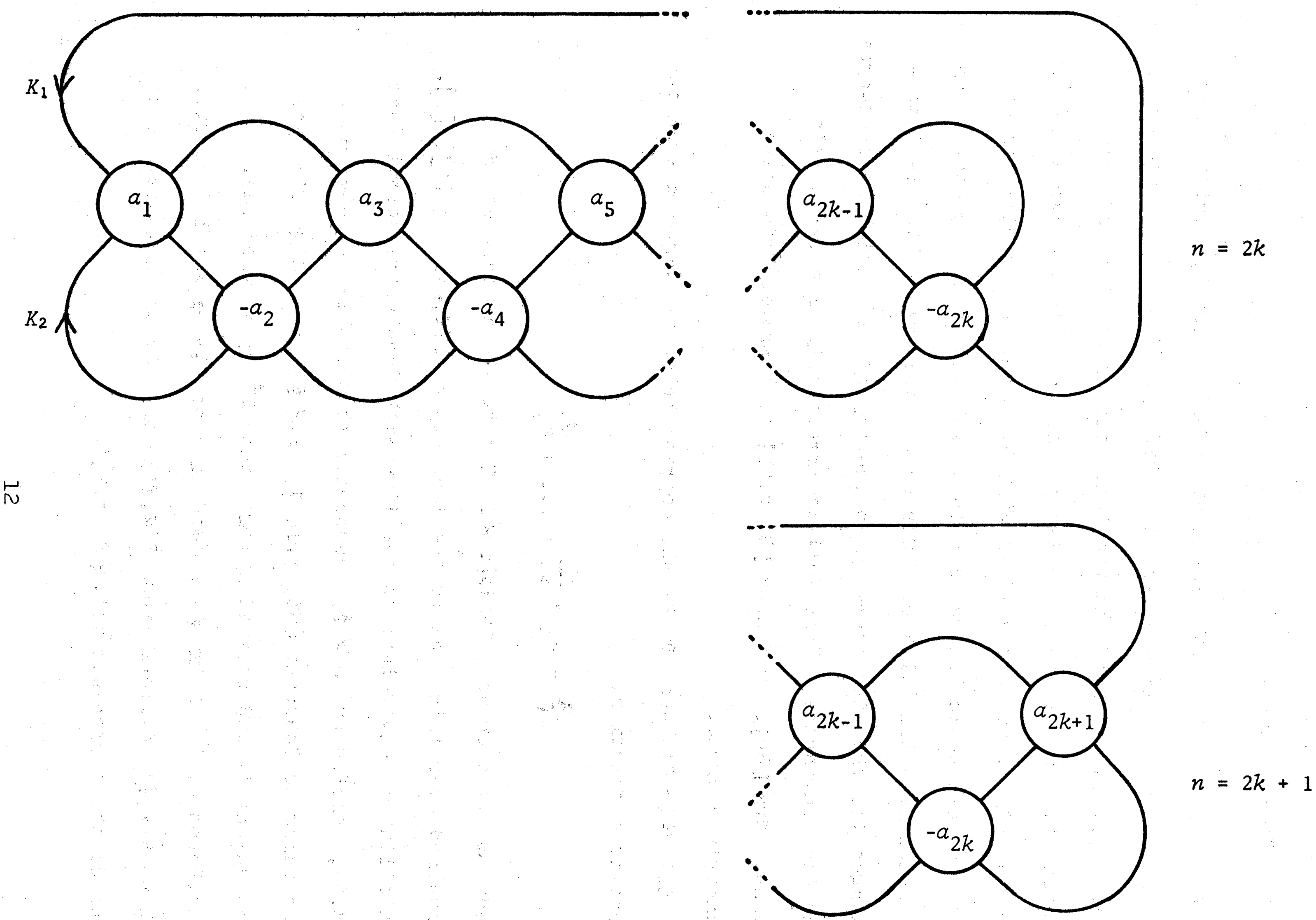

Fig. 3

We can obtain easily another continued fraction:

$$\frac{\alpha}{\beta} = 2b_1 + \frac{1}{2b_2} + \dots + \frac{1}{2b_m},$$

where m is odd. $C(2b_1, 2b_2, \dots, 2b_m)$ is then equivalent to $C(a_1,a_2,\dots,a_n)$ and will be denoted by $D(b_1,b_2,\dots,b_m)$. In the following we will consider every two-bridge link putting in this form (cf. [S, p.13]).

4. Main theorems

From Lemma 1, we have

Theorem 1. Let $L_0 = D(0)$ and for $n \geq 1$ let $L_n = D(p_1,q_1, p_2,q_2,\dots,p_{n-1},q_{n-1},p_n)$, where $\prod_{i=1}^{n} p_i \prod_{j=1}^{n-1} q_j \neq 0$. Let $\Delta_n(x,y)$ be the polynomial inductively defined as follows:

$$\Delta_0 = 0;$$

(4.1) $$\Delta_1 = F_{p_1};$$

$$\Delta_n = \{q_{n-1}(x-1)(y-1)F_{p_n} + 1\}\Delta_{n-1} + (xy)^{p_{n-1}}\frac{F_{p_n}}{F_{p_{n-1}}}(\Delta_{n-1} - \Delta_{n-2}),$$

for $n \geq 2$.

Then $\Delta_n(x,y)$ is the Alexander polynomial of L_n.

In the following, by the Alexander polynomial of a two-bridge link we mean the polynomial defined in Theorem 1 and we will use the following notation besides that in Theorem 1. Let $\Delta_n^{(p)}$ be the Alexander polynomial of $D(p_1,q_1,p_2,q_2,\dots,p_{n-1},q_{n-1},p)$; thus $\Delta_n^{(p_n)} = \Delta_n$ and $\Delta_n^{(0)} = \Delta_{n-1}$. Let $\ell_n = \sum_{i=1}^{n} p_i$, that is the linking number of L_n. Let $\tilde{\ell}_n = \sum_{i=1}^{n} |p_i|$.

From Lemma 2, we have

<u>Theorem 2.</u>

(4.2) $\Delta_n = q_{n-1}(x-1)(y-1)F_{p_n}\Delta_{n-1} + \Delta_{n-1}^{(p_{n-1}+p_n)}$;

(4.3) $\Delta_n^{(p)} = F_p\Delta_n^{(1)} - xyF_{p-1}\Delta_{n-1}$;

(4.4) $\Delta_n^{(p)} + xy\Delta_n^{(p-2)} = (1+xy)\Delta_n^{(p-1)}$.

Using (4.4) or Theorem 1 we can show easily the following each formula by induction on n.

<u>Corollary 1.</u>

(4.5) $\Delta_n(x,y) = \Delta_n(y,x)$;

(4.6) $\Delta_n(x,y) \equiv F_{\ell_n}(x,y) \mod (x-1)(y-1)$:

(4.7) $\Delta_n(x,y) = (xy)^{\ell_n - 1}\Delta_n(x^{-1},y^{-1})$.

The fact that a two-bridge link is interchangeable assures us of (4.5). From (4.6), we have immediately

(4.8) $\Delta_n(x,1) = F_{\ell_n}(x,1)$.

(4.7) and (4.8) constitute the Torres conditions [T] for two-bridge links.

<u>Definition 1.</u> Let $f(x,y)$ be a polynomial in Λ. If $f(x,y) \neq 0$, then $\deg_x f$ = (maximum x-power of any term of f) minus (minimum x-power of any term of f). If $f(x,y) = 0$, then

$\deg_x f = -1$. We define $\deg_y f$ in the same way.

Definition 2. $\Lambda^{+1}(r,s)$ denotes the set of all polynomials $f(x,y) = \sum_{r \leq i,j \leq s} a_{ij}x^i y^j$ in Λ satisfying the following conditions.

(i) $\deg_x f = \deg_y f = s-r$.

(ii) Let $M(f) = \begin{bmatrix} a_{sr} & \cdots & a_{ss} \\ \vdots & & \vdots \\ a_{rr} & \cdots & a_{rs} \end{bmatrix}$ and $W(f) = \begin{bmatrix} a_{rr} & \cdots & a_{rs} \\ \vdots & & \vdots \\ a_{sr} & \cdots & a_{ss} \end{bmatrix}$.

Then both $M(f)$ and $W(f)$ are symmetric matrices.

(iii) $a_{ij} \geqq 0$ if $i+j$ is even and $a_{ij} \leqq 0$ if $i+j$ is odd.

(iv) Let $b_{ij} = a_{i+r,j+r}$. Then

$$|b_{k,0}| \leqq |b_{k-1,1}| \leqq \cdots \leqq |b_{k-u,u}|,$$

and $$|b_{k,0}| \leqq |b_{k+1,1}| \leqq \cdots \leqq |b_{k+v,v}|$$

for $0 \leqq k \leqq s-r$, where $u = \left[\frac{k}{2}\right]$ *) and $v = \left[\frac{k-s+r}{2}\right] + 1$.

Furthermore $\Lambda^{-1}(r,s)$ denotes the set of all polynomials $f(x,y)$ in Λ such that $-f(x,y) \in \Lambda^{+1}(r,s)$.

Theorem 3. For $n \geqq 1$, $\Delta_n \in \Lambda^{\varepsilon_n}(r_n, s_n)$, where $\varepsilon_n = \prod_{i=1}^{n} \frac{p_i}{|p_i|} \prod_{j=1}^{n-1} \frac{q_j}{|q_j|}$, $r_n = \frac{\ell_n - \tilde{\ell}_n}{2}$ and $s_n = \frac{\ell_n + \tilde{\ell}_n}{2} - 1$.

Note that $r_n \leqq 0 \leqq s_n$, $r_n - r_{n-1} = \frac{p_n - |p_n|}{2}$ and $s_n - s_{n-1} = \frac{p_n + |p_n|}{2}$. The proof of Theorem 3 will be given in Section 5.

*) [ ] denotes the Gaussian symbol.

Concerning the reduced Alexander polynomial of a two-bridge link, Theorem 3 yields the weaker result than that of Hartley stated in the beginning of this paper.

For the sake of Corollary 2 below, we need some preliminaries.

Definition 3. Let $L = K_1 \cup K_2$ be a link and S be a Seifert surface for K_1 with S and K_2 in general position. If γ_S = 2(genus of S) plus (the number of times K_2 intersects S), then $\gamma_1 = \min_S \gamma_S$ is the linking complexity of K_2 with K_1. We define γ_2 in the same way. We call the ordered pair (γ_1, γ_2) the linking complexity of the link L.

This definition follows Bailey [B, p.45], see also [K].

Proposition 1. (Kidwell) If $\Delta(x,y)$ is the Alexander polynomial of a link L with linking complexity (γ_1, γ_2), then $\gamma_1 - 1 \geq \deg_x \Delta(x,y)$.

Proof. See [B, p.46]. □

Corollary 2. Let (γ_1, γ_2) be the linking complexity of L_n. Then

(4.9) $\gamma_1 = \gamma_2$;

(4.10) $\deg_x \Delta_n(x,y) + 1 = \gamma_1 = \tilde{\ell}_n$.

Remark. The first equality of (4.10) is Proposition 6 of [B, p.57].

Proof. (4.9) follows from interchangeability of a two-bridge link or (4.10). For (4.10), from the diagram of L_n, we see that $\gamma_1 \leqq \tilde{\ell}_n$. By Theorem 3, $\deg_x \Delta_n + 1 = \tilde{\ell}_n$ and by Proposition 1, $\gamma_1 \geqq \deg_x \Delta_n + 1$. □

5. Proof of Theorem 3

In this section we use the following trivial lemma without mention.

<u>Lemma 3.</u> Let $f \in \Lambda^{\varepsilon}(r,s)$ and $g \in \Lambda^{\varepsilon}(r-k,s+k)$ $(k \geqq 0)$. Then $f + g \in \Lambda^{\varepsilon}(r-k,s+k)$.

<u>Lemma 4.</u> Let $f \in \Lambda^{\varepsilon}(r,s)$. Then

$$F_n f \in \begin{cases} \Lambda^{\varepsilon}(r,\ s+n-1) & \text{if } n > 0, \\ \Lambda^{-\varepsilon}(r+n,\ s-1) & \text{if } n < 0, \end{cases}$$

$$G_n f \in \Lambda^{(-1)^{n-1}\varepsilon}(r,\ s+n-1) \quad \text{if } n > 0,$$

where $G_n(x,y) = x^{n-1}F_n(x^{-1},y)$.

Proof. We show that $f \in \Lambda^{+1}(r,s)$ implies $F_n f \in \Lambda^{+1}(r,\ s+n-1)$ if $n > 0$. The other case can be proved similarly.

It is clear that $F_n f$ satisfies the conditions (i), (ii), (iii) and the first inequality of (iv) in Definition 2. The second inequality of (iv) can be reduced to Sublemma below. □

Sublemma. Let $f(x) = \sum_{i=0}^{n} a_i x^i$, where $a_i = a_{n-i}$ and $0 < a_0 \leq a_1 \leq \cdots \leq a_{[n/2]}$. Let $(\sum_{j=0}^{m} x^j) f(x) = \sum_{k=0}^{m+n} b_k x^k$. Then $b_k = b_{m+n-k}$ and $0 < b_0 \leq b_1 \leq \cdots \leq b_{[(m+n)/2]}$.

Proof. We proceed by induction on n. For $n = 0, 1$, the sublemma is trivial. Assume the sublemma proved for polynomials of degree $< n$. Write $f(x) = a_0 \sum_{i=0}^{n} x^i + xg(x)$, where $g(x) = \sum_{j=0}^{n-2} a_{j+1} x^j$. Let $(\sum_{j=0}^{m} x^j) g(x) = \sum_{i=0}^{m+n-2} c_i x^i$. Then $c_i = c_{m+n-2-i}$ and $0 < c_0 \leq c_1 \leq \cdots \leq c_{[(m+n-2)/2]}$ by inductive hypothesis. Thus if $(\sum_{j=0}^{m} x^j) f(x) = \sum_{k=0}^{m+n} b_k x^k$, it is easy to see that $b_k = b_{m+n-k}$ and $0 < b_0 \leq b_1 \leq \cdots \leq b_{[(m+n)/2]}$. □

Lemma 5. If $\Delta_{n-1} \in \Lambda^{-\varepsilon}(r, s-1)$ and $\Delta_n^{(1)} \in \Lambda^{\varepsilon}(r,s)$, then

$$\Delta_n^{(p)} \in \begin{cases} \Lambda^{\varepsilon}(r, s+p-1) & \text{if } p > 0, \\ \Lambda^{-\varepsilon}(r+p, s-1) & \text{if } p < 0. \end{cases}$$

Proof. (4.2) in Theorem 2 states that $\Delta_n^{(p)} = F_p \Delta_n^{(1)} - xyF_{p-1} \Delta_{n-1}$. The case $p = 1$ is the hypothesis. If $p \geq 2$, then using Lemma 4, $F_p \Delta_n^{(1)} \in \Lambda^{\varepsilon}(r, s+p-1)$ and $-xyF_{p-1}\Delta_{n-1} \in \Lambda^{\varepsilon}(r+1, s+p-2)$. Thus $\Delta_n^{(p)} \in \Lambda^{\varepsilon}(r, s+p-1)$. If $p \leq -1$, then $F_p \Delta_n^{(1)}$, $-xyF_{p-1}\Delta_{n-1} \in \Lambda^{-\varepsilon}(r+p, s-1)$, so $\Delta_n^{(p)} \in \Lambda^{-\varepsilon}(r+p, s-1)$. □

Lemma 6. Let $\Delta_n^{\langle m\rangle}$ be the Alexander polynomial of $D(p_1, q_1, \ldots, p_{n-m}, q_{n-m}, 1, q_{n-m+1}, 1, \ldots, q_{n-1}, 1)$. Then we have

$$(5.1)\quad \Delta_n^{\langle m\rangle} = G_{m+1}\Delta_{n-m} - xyG_m\Delta_{n-m}^{(p_{n-m}-1)} + (x-1)(y-1)\sum_{k=1}^{m}(q_{n-k}+1)G_k\Delta_{n-k},$$

where the last term denotes zero if $m = 0$.

Proof. We prove (5.1) by induction on m. For $m = 0$, it is clear that $\Delta_n^{\langle 0\rangle} = \Delta_n$. Assume that (5.1) proved for $m-1$. Substituting $p_{n-m+1} = 1$ in $\Delta_n^{\langle m-1\rangle}$ we have

$$\Delta_n^{\langle m\rangle} = G_m\Delta_{n-m+1}^{(1)} - xyG_{m-1}\Delta_{n-m+1}^{(0)} + (x-1)(y-1)\sum_{k=1}^{m-1}(q_{n-k}+1)G_k\Delta_{n-k}.$$

By (4.2), $\Delta_{n-m+1}^{(1)} = q_{n-m}(x-1)(y-1)\Delta_{n-m} + \Delta_{n-m}^{(p_{n-m}+1)}$. Thus we have

$$\Delta_n^{\langle m\rangle} = G_m\{-(x-1)(y-1)\Delta_{n-m} + \Delta_{n-m}^{(p_{n-m}+1)}\} - xyG_{m-1}\Delta_{n-m}$$
$$+ (x-1)(y-1)\sum_{k=1}^{m}(q_{n-k}+1)G_k\Delta_{n-k}.$$

By (4.4), $\Delta_{n-m}^{(p_{n-m}+1)} = (xy+1)\Delta_{n-m} - xy\Delta_{n-m}^{(p_{n-m}-1)}$. Thus we have

$$\Delta_n^{\langle m\rangle} = \{(x+y)G_m - xyG_{m-1}\}\Delta_{n-m} - xyG_m\Delta_{n-m}^{(p_{n-m}-1)}$$
$$+ (x-1)(y-1)\sum_{k=1}^{m}(q_{n-k} + 1)G_k\Delta_{n-k}.$$

Since $(x+y)G_m - xyG_{m-1} = G_{m+1}$, we have (5.1). □

Now we are in position to prove Theorem 3. We use induction on n. For $n = 1$, the theorem is clear. Assume the theorem proved for Δ_k, where $1 \leqq k \leqq n-1$. Without loss of generality we may suppose that $q_{n-1} < 0$. By Lemma 5 we have only to prove for the case $p_n = 1$. Then there exists an integer m such that:

(I) $1 \leq m \leq n-1$, $p_{n-m+1} = p_{n-m+2} = \dots = p_{n-1} = 1$, $p_{n-m} \neq 1$ and $q_{n-m}, q_{n-m+1}, \dots, q_{n-1} < 0$,

(II) $1 \leq m \leq n-2$, $p_{n-m} = p_{n-m+1} = p_{n-m+2} = \dots = p_{n-1} = 1$, $q_{n-m}, q_{n-m+1}, \dots, q_{n-1} < 0$ and $q_{n-m-1} > 0$, or

(III) $m = n-1$, $p_1 = p_2 = \dots = p_{n-1} = 1$, $q_1, q_2, \dots, q_{n-1} < 0$.

To prove Theorem 3, it suffices to prove that $\Delta_{n-m} \in \Lambda^{\varepsilon}(r,s)$ implies $\Delta_n \in \Lambda^{(-1)^m\varepsilon}(r, s+m)$, where by Lemma 6

$$(5.2)\quad \Delta_n = G_{m+1}\Delta_{n-m} - xyG_m\Delta_{n-m}^{(p_{n-m}-1)} + (x-1)(y-1)\sum_{k=1}^{m}(q_{n-k}+1)G_k\Delta_{n-k}.$$

By Lemma 4, we have

$$(5.3)\quad G_{m+1}\Delta_{n-m} \in \Lambda^{(-1)^m\varepsilon}(r, s+m).$$

By inductive hypothesis, $\Delta_{n-k} \in \Lambda^{(-1)^{m-k}\varepsilon}(r, s+m-k)$ for $1 \leq k \leq m$. Then by Lemma 4, $G_k\Delta_{n-k} \in \Lambda^{(-1)^{m-1}\varepsilon}(r, s+m-1)$; hence we obtain

$$(5.4)\quad (x-1)(y-1)\sum_{k=1}^{m}(q_{n-k}+1)G_k\Delta_{n-k} \begin{cases} = 0 & \text{if } q_{n-k} = -1 \text{ for any } k, \\ \in \Lambda^{(-1)^m\varepsilon}(r, s+m) & \text{otherwise.} \end{cases}$$

Case (I). If $p_{n-m} \neq 1$, then by inductive hypothesis,

$$\Delta_{n-m}^{(p_{n-m}-1)} \in \begin{cases} \Lambda^{\varepsilon}(r, s-1) & \text{if } p_{n-m} \geq 2, \\ \Lambda^{\varepsilon}(r-1, s) & \text{if } p_{n-m} \leq -1. \end{cases}$$

Thus, using Lemma 4, we have

(5.5) $$-xyG_m\Delta_{n-m}^{(p_{n-m}-1)} \in \begin{cases} \Lambda^{(-1)^m\varepsilon}(r+1,\ s+m-1) & \text{if } p_{n-m} \geqq 2, \\ \Lambda^{(-1)^m\varepsilon}(r,\ s+m) & \text{if } p_{n-m} \leqq -1. \end{cases}$$

Case (II). If $p_{n-m} = 1$ and $q_{n-m-1} > 0$, then by inductive hypothesis,

$$\Delta_{n-m}^{(p_{n-m}-1)} = \Delta_{n-m-1} \in \Lambda^{\varepsilon}(r,\ s-1).$$

Thus, using Lemma 4, we have

(5.6) $$-xyG_m\Delta_{n-m}^{(p_{n-m}-1)} \in \Lambda^{(-1)^m\varepsilon}(r+1,\ s+m-1).$$

Case (III). Since $m = n-1$ and $p_1 = 1$, we have

(5.7) $$-xyG_m\Delta_{n-m}^{(p_{n-m}-1)} = 0.$$

From (5.2) ~ (5.7), we have $\Delta_n \in \Lambda^{(-1)^m\varepsilon}(r,\ s+m)$. This completes the proof of Theorem 3.

APPENDIX

Alexander Polynomials of Two-Bridge Links of 10 Crossings

For every two-bridge link of 10 crossings in the table of Conway [C, p.353], we list the Alexander polynomial. Two-bridge links are presented by Conway's notation; $p_1p_2\cdots p_n$ denotes a two-bridge link with the notation $C(p_1,p_2,\ldots,p_n)$ in this paper. The Alexander polynomial is abbreviated in the same manner as Rolfsen's table [R, Appendix C].

60

10

				1
			1	
		1		
	1			
1				

73

			1	
		1	-1	1
	1	-1	1	
1	-1	1		
	1			

622

		-1	2
	-1	3	-1
-1	3	-1	
2	-1		

55

		1		
	1	-1	1	
1	-1	1	-1	1
	1	-1	1	
		1		

523

			1	
		2	-2	1
	2	-3	2	
1	-2	2		
	1			

5122

		-2	1
	-2	3	-2
-2	3	-2	
1	-2		

442

	-2	3
-2	5	-2
3	-2	

424

		-1	3
	-2	4	-1
-1	4	-2	
3	-1		

4213

	-1	2	
-1	3	-3	2
2	-3	3	-1
	2	-1	

41212

	1	-2	1
1	-3	3	-2
-2	3	-3	1
1	-2	1	

411112

	1	-2	1
1	-3	5	-2
-2	5	-3	1
1	-2	1	

343

		1		
	2	-2	1	
1	-2	3	-2	1
	1	-2	2	
		1		

3322

	2	-1	
2	-4	4	-1
-1	4	-4	2
	-1	2	

3223

		1		
	2	-3	2	
1	-3	5	-3	1
	2	-3	2	
		1		

32122

	1	-2	
1	-5	5	-2
-2	5	-5	1
	-2	1	

3142

	-3	2
-3	5	-3
2	-3	

31213

		-2	1
	-4	5	-2
-2	5	-4	
1	-2		

311212

	1	-2	1
1	-5	5	-2
-2	5	-5	1
1	-2	1	

3111112

	1	-2	1
1	-5	7	-2
-2	7	-5	1
1	-2	1	

262

-3	4
4	-3

23212

2	-4	2
-4	7	-4
2	-4	2

231112

2	-4	2
-4	9	-4
2	-4	2

22222

1	-4	4
-4	9	-4
4	-4	1

221122

2	-5	2
-5	9	-5
2	-5	2

21412

-1	2	-1	
2	-3	2	-1
-1	2	-3	2
	-1	2	-1

213112

-1	2	-1	
2	-5	4	-1
-1	4	-5	2
	-1	2	-1

2112112

	1	-2	1
1	-6	7	-2
-2	7	-6	1
1	-2	1	

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