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# 不変式論の今昔（その一）

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## 1　不変式論の創始

不変式論は19世紀の中頃に現れ，19世紀後半から今世紀初頭にかけて第一次隆盛期をもった。この時期という点に関しては，群の概念の発生，発展の時期と大いに関係があるのは当然である。

Euclidの幾何学が二千数百年前に立派な体系をもったのに比べて，代数的概念の進展は非常におくれていて，16世紀になって，文字係数の方程式を書いて，方程式を一般的に扱うようになるまでは，ほとんどギリシア数学のレベルにとどまっていたといえるが，それには理由があるように思える。社会環境的理由も，もちろんあるだろうが，それは無視して数学的面だけを考えてみる。

例えば，一般の三角形を考察する場合，不等辺三角形を適

当に書けば，ほとんど特殊性を感ずることなしに一般論をすることができる。しかしながら，代数方程式について考えると，(1) 古い時代には，数を表すことが，まず大変であった。(ヨーロッパの数学が，今の算用数字を用い，十進法に基づいて数を表すことを学んだのがいつであるかはよく知らないが，いわゆるアラビア数字が始まったのが8世紀頃であるから，それより後であることはまちがいない。) (2) どんな方程式でも，数係数でかいてしまうと，特殊性が目立って，一般的とは感じ難いと思われる，ということがいえると思う。(実際には，特殊性を忘却して，一般性を感じとる努力は払われていたようであるが。)

16世紀になって，文字係数の方程式によって，本当に一般的に扱い得るようになって，代数学は急に発展したように思える。3次方程式，4次方程式の解法が見つかったのも，16世紀のことである。5次以上の方程式の解を求めて，見つからないまま19世紀を迎えたのであるが，その間無為に過したわけではない。方程式を解く試みから，根の間の関係を探るようになり，その結果，今の言葉でいうガロア群に着目するようになったのは，大変な進歩であったと思われる。その点で最も有名なのは Galois (1811–1832) であるが，Abel (1802–1829) も大いに貢献している。

その結果，置換群が研究対象にとりあげられた．そして，Cayley（1821－1895）が1854年に抽象的な群の定義を与えた．また，Klein（1849－1925）が群と幾何学との関連をErlangen Programm（1872）において論じたことも有名である．そのような情勢下において，不変式論はCayleyによって創められた．

## 2　19世紀の不変式論

現代の数学者ならば，どの体の上で考えるかは大変気になるところであるが，19世紀の数学者は，そういうことは気にしていなかったように思える。実数体の上で考えるのが中心で，必要に応じて複素数も使うというのが基本であったように思える。

ここでの説明では，標数0の体Kを一つ固定して，Kの上で考えればよいので，そのようにする。

$d$次$n$元形式，すなわち，$n$変数$X_1, \cdots, X_n$についての$d$次斉次式$F(X_1, \cdots, X_n)$は，一般的に次のように表せる。

$$\sum_{|(i)|=d} p_{(i)} c_{(i)} X^{(i)}$$

$\big( (i)=(i_1, \cdots, i_n),\ |(i)| = \sum i_\alpha,\ p_{(i)} = d!/\prod_\alpha i_\alpha!,$

$X^{(i)} = X_1^{i_1} \cdots X_n^{i_n} \big)$

もちろん，多項係数 $p_{(i)}$ で係数を調整しなくてもよい筈ではあるが，不変式の計算の便宜上このような調整を行なっているのである。

一般線型群 $G = GL(n, K)$ の各 $A$ に対し，

$$\sigma_A : (x_1, \cdots, x_n) \to (x_1, \cdots, x_n)A$$

という変換を考えると，

$$\sigma_A F(x_1, \cdots, x_n) = \Sigma_{(i)} p_{(i)} c'_{(i)} X^{(i)}$$

の形にかける。これによって $\{ c_{(i)} \mid |(i)| = d \}$ で生成された加群 $M_d$ の上の一次変換

$$\tau_A : (\cdots, c_{(i)}, \cdots) \to (\cdots, c'_{(i)}, \cdots)$$

がひきおこされる。すなわち，まず，$\{ p_{(i)} X^{(i)} \mid |(i)| = d \}$ で生成された加群 $V_d$ 上に $\sigma_A$ が一次変換

$$\sigma_A : (\cdots, p_{(i)} X^{(i)}, \cdots) \to (\cdots, p_{(i)} X^{(i)}, \cdots) \varphi_d(A)$$

をひきおこし，その変換の行列 $\varphi_d(A)$ を使って

$$\tau_A : {}^t(\cdots, c_{(i)}, \cdots) \to \varphi_d(A)\, {}^t(\cdots, c_{(i)}, \cdots)$$

となるのである。

「$d$ 次 $n$ 元形式の不変式」というのは，この係数 $c_{(d,0\cdots0)}$，$\cdots$，$c_{(i)}$，$\cdots$，$c_{(0,\cdots,0,d)}$ の多項式であって，すべての $A \in SL(n, K)$ について不変であるもののことである。$f$ がそのような不変式のとき，一般の $A \in G$ については

$$\tau_A(f) = (\det A)^m f$$

となり，この $m$ は $A$ に依存しない。この $m$ を $f$ の重さという。

19世紀の不変式論は，上記の意味の不変式を探すことが主流であり，いろいろな工夫がなされた。特に $n=2$ のときは，ずい分詳細に調べられた。

上記の変換 $\tau_A$ を，

$$\tau_A : \quad (X_1, \cdots, X_n) \rightarrow (X_1, \cdots, X_n) A^{-1}$$

によって拡張すると，もとの $d$ 次 $n$ 元形式 $F(X_1, \cdots X_n)$ は $G$ 不変になる。このように拡張したときの $\cdots, c_{(i)}, \cdots, X_1, \cdots, X_n$ についての多項式で $SL(n, K)$ 不変元を共変式とよんだ。また，一つの $F(X_1, \cdots X_n)$ だけでなく，複数の $d_j$ 次 $n$ 元形式 $(j = 1, \cdots, s)$ を考え，上と同様に多項係数で調整後の係数全体を代数的独立と考えたときの $SL(n, K)$ 不変元を多重共変式とよんで，それらを見つける方法を考えたのであった。

上記の $\varphi_d$ は，$GL(n, K)$ の既約表現の一つの典型ともいえるものであるから，その意味で，上記の不変式，共変式などは自然な対象といえようが，具体的応用との結びつきが少なかったことと，折角見つけた計算法も，実際に計算しようとすると手間が大変であったことなどから，この方向への発展は19世紀末で止ってしまったように思える。

## 3 Hilbert の第14問題

1890年になってHilbertは上記のような特殊な作用に限定しないで、一般に群が多項式環に作用する場合を考え、また、GL(n, k)がK上の多項式環に作用するときは、不変元全体が有限生成の環になることを証明した。

1900年のParisでのCongressで、Hilbertが23の問題を提出し、その第14番目の問題がつぎのものであった。

LがKと $K(X_1, \cdots, X_n)$ との中間体であるとき、

$$K[X_1, \cdots, X_n] \cap L$$

はK上有限生成の環であるか?

問題の説明文に明示されているように、Hilbertの意識にあったものは、群が $K[X_1, \cdots, X_n]$ に作用(K上ではtrivial)しているとき、不変元全体がK上有限生成であろうか、という問題であった。

この問題についての結果を列記しよう。

(1) R. Weitzenböck は "Über die Invarianten von Linearen Gruppen" Acta Math. 58 (1932) において、不変元の形でのHilbertの問題を肯定的に解いたと称したが、実は帰納法を誤って使っていた。したがって、第一段階だけは正しく、それは次のことを示していた。

標数0の体Kの加法群が多項式環にrationalに作用(K

上trivial）しているときは，その不変元全体なK上有限生成の環になる。

証明は，原論文は難解である。永田のTata Inst. F.R.でのLecture notes p.36-p.40にある，Seshadriのアイディアによる証明がわかりよい。Seshadriの論文（J. Math. Kyoto Univ. vol. 1, No. 3 (1962)）はもう少し一般に扱っている。

(2) 非常に大きい進展をもたらしたのはH. Weylの"The classical groups"（Princeton Univ. Press (1939)）で，$K=R$または$C$の場合，K上有限生成の可換環にcompact Lie群が作用しているとき（連続性はもちろん仮定する），不変元全体がK上有限生成であることを示し，その結果，群がSemi-Simple Lie群でもよいことを示した。

Weylはcompact群の場合には積分を利用し，次に，$G_1$が$G$のdense subgroupならば，「$G_1$不変」⇔「$G$不変」ということを利用したのである。

Weylが意識したかどうかはよくわからないが，$K=R$またはCという仮定はあまり強いわけではない。というのは，

(イ) 標数0ならば，$K \subseteq C$（または$R$）の場合にreduceすることは易しい。

(ロ) 一般に$GL(n, k)$の部分群$G$がK上有限生成の環にrationalに作用しているとき，$K'$がKの拡大体であれ

ば，$G$ の $GL(n, K')$ における Zariski 位相による閉包 $\bar{G}$ をとると，$R$ における $G$ 不変元全体 $R^G$ と，$R \otimes_K K'$ における $\bar{G}$ 不変元との間には

$$(R \otimes_K K')^{\bar{G}} = R^G \otimes_K K'$$

という関係がある。したがって，

$R^G$ が $K$ 上有限生成 $\Leftrightarrow$ $(R \otimes_K K')^{\bar{G}}$ が $K'$ 上有限生成

ということがわかるのである。

(3) 永田は標数 0 を外すために，積分を避けて，表現の完全可約性を利用して Weyl の上記の結果の証明をした。すなわち，$GL(n, k)$ の部分群 $G$ の rational な表現がすべて完全可約であるならば，$R^G$ は有限生成である，という形にしたのである（前出，Tata Inst. の Lecture notes 参照）。

しかしながら，残念なことに，標数 $p \neq 0$ の場合，この完全可約性をもつ代数群は，単位元の連結成分が torus 群で，連結成分の数が $p$ と素な群ということで特徴づけられるので，割合弱いといわざるを得ない。（Nagata, J. Math. Kyoto. Univ. vol 1. No. 1. (1961) 参照）標数 0 のときの代数群ならば，よく知られているように，根基が torus 群であることが必要充分条件である。

その頃 D. Mumford が，いわゆる Mumford 予想を立てた。その一つの formulation は次のように述べられる。

代数群G（⊆ GL(n, K)）の根基がtorus群であれば，Gのrational な表現 $\rho: G \to GL(m, K)$ によりひきおこされる m 変数多項式環 $K[X_1, \dots, X_m]$ への作用

$$^t(X_1, \dots, X_m) \to \rho(g)\,^t(X_1, \dots, X_m)$$

について，つぎのことが成り立つ：

$N = \sum_{i \geq 2} X_i K$ がG認容で，$X_1 \pmod N$ がG不変であれば，$X_1$ についての monic な多項式

$$F = X_1^s + F_1(X)X_1^{s-1} + \cdots + F_s(X) \qquad (s \geq 1, \; F_i(X) \in K[X_2, \dots, X_m])$$

でG不変なものが存在する。

（この予想は W. J. Haboush によって1974年に解かれた。Ann. of Math (2) 102 (1975)）

rational な表現がすべて完全可約であれば，$s = 1$ で F がとれるのである。

永田は，完全可約の場合の証明に工夫を加えて，Mumford 予想の結論の成り立つような群の不変元について，いろいろな結果を得た。（J. Math. Kyoto Univ. Vol. 3. No. 3. (1964) 参照）Mumford 予想の解けている現在では，その主要結果は次のように述べられる。

定理．体K上の代数群Gの根基がtorus群であるとき，GがK上有限生成な可換環Rに rational に作用すれば，

(i) G不変元全体 $R^G$ は K上有限生成である。

(ii) I, J が G認容な R のイデアルであって，$I+J=R$ であれば，$R^G$ の元 f で $f\in I$, $1-f\in J$ となるものがある。

この (ii) は，幾何学的にいえば，上のようなGがアフィン多様体V に rational に作用しているとき，$R^G$ の定めるアフィン多様体（RはVの座標環）は closed orbit 全体を集めたものに対応する．一般の orbit で考えるならば，二つの orbits PG, QG （P, Q　V）に対し，それらの closures $\overline{PG}$, $\overline{QG}$ に共通点があれば，$R^G$ の各元は PG と QG とで同一の値をとり，共通点がなければ，PG で 0, QG で 1 となる $R^G$ の元が存在することを示している。

この事実は orbit space を考える上で重要なことである。

(4) 以上，肯定的方向だけを述べてきたが，Hilbert の第14問題自身は永田によって否定的結論が得られた。( Nagata, Amer. J. Math. 81 (1959) ) その一つの例は GL (32, K) の部分群 G で，つぎのもので与えられる

対角線に沿って $B_i=\begin{pmatrix}1 & b_i\\ 0 & 1\end{pmatrix}$ $(i=1, 2, \cdots, 16)$ が並び，他は0という行列で $(b_1, \cdots, b_{16})$ について，

素体上代数的独立な元 $c_{ij}$ $(i=1,2,3;\ j=1, 2, \cdots 16)$ により，$\sum_j c_{ij}b_j=0$ $(i=1, 2, 3)$ をみたすもの全部

を動かす」ことによって得られた，13次元のベクトル空間と同型な群である。

上記の（1），（2），（4）については，「数学」12巻4号（1960）にも，永田による紹介がある．

# Geometric Invariant Theory
## 不変式論の今昔 (その二)

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The aim of this note is to give a brief introduction to geometric invariant theory and its application to moduli problems.

## § I. Geometrically reductive groups

For simplicity, we assume that:

k : an algebraically closed field with char $k = p \geq 0$

G : a connected linear algebraic group defined over k

X, Y, ... : algebraic schemes over k (only schemes sometimes)

V, W, ... : vector spaces over k (rational G-modules).

Remark 0. Almost all results in this note are obtained for more general cases, e.g., the case where k is a Noetherian Japanese domain A and G is a reductive group scheme $G_A$ over A which splits over A; the assumption on the splitting of $G_A$ is important because Borel subgroups, root systems, maximal tori, etc. play essential roles in this theory.

(1.0) Linearly reductive algebraic groups:

Let V be a finite-dimensional rational G-module, i.e., there is a homomorphism $\rho : G \longrightarrow GL(V)$ as algebraic groups. If V decomposes into irreducible rational G-modules, say $V = \oplus V_i$, $V_i$ an irreducible G-module, then $\rho$ is called a completely reducible representation of G, i.e.,

$$\rho = \begin{pmatrix} \rho_1 & & & \\ & \rho_2 & & \\ & & \ddots & \\ & & & \rho_r \end{pmatrix} , \quad \rho_i : G \longrightarrow GL(V_i) .$$

<u>Definition</u> 1. If every representation of G is completely reducible then G is called a linearly reductive group.

<u>Remark</u> 1. G is linearly reductive if and only if $H^i(G,V) = 0$ $(i \geq 1)$ for any finite-dimensional rational G-module, where $H^i(G,V)$ is a Hochschild cohomology group.

As for linearly reductive groups, it is known that:

<u>Theorem</u> 1. (1) (H. Weyl [15]). If char $k = 0$ then G is linearly reductive if and only if G is a reductive group, i.e., the radical R of G is a torus group.
(2) (M. Nagata [62]). If char $k = p > 0$ then G is linearly reductive if and only if G is a torus group.

<u>Remark</u> 2. When char $k = p > 0$, non-abelian reductive groups are not linearly reductive. A simple example is:

<u>Example</u> 1. $G = SL(2)$, char $k = 2$ and a representation is given by

$$\rho : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & ac & bd \\ 0 & a^2 & b^2 \\ 0 & c^2 & d^2 \end{pmatrix} ,$$

where $\rho$ is not completely reducible, i.e., $H^1(G,V) \neq 0$.

(1.1) Geometrically reductive groups:

Linearly reductive groups have many nice algebro-geometric properties. Analyzing those properties, D. Mumford derived the following concept of geometrically reductive groups and posed a conjecture, called, since then, the Mumford conjecture (D. Mumford [116]):

Definition 2. Let $G$ be a connected linear algebraic group defined over $k$ with char $k = p > 0$. If $G$ satisfies the following condition, $G$ is said to be a geometrically reductive group:

Let $V$ be a finite-dimensional rational $G$-module such that there is a $G$-invariant vector subspace $V_0$ with codimension 1, i.e., we have an exact sequence of rational $G$-modules

$$0 \longrightarrow V_0 \longrightarrow V \longrightarrow V/V_0 = k \longrightarrow 0$$

or equivalently, we have

$$\rho = \begin{pmatrix} 1 & \tau \\ 0 & \rho' \end{pmatrix} , \ \rho' : G \longrightarrow GL(V_0) , \ \tau \in H^1(G,V_0) \ ;$$

then there is a positive integer $n$ such that the following exact sequence

$$0 \longrightarrow S^{p^n-1}(V)V_0 \longrightarrow S^{p^n}(V) \longrightarrow S^{p^n}(V/V_0) = k \longrightarrow 0$$

splits as rational $G$-modules, i.e., there is a non-zero $G$-invariant vector $v$ such that

$$S^{p^n}(V) = S^{p^n-1}(V)\cdot V_0 \oplus kv \ .$$

Remark 3. The above condition in Definition 2 is equivalent to any one of the following conditions; the corresponding conditions were also considered also in the case of char $k = 0$:

(a) Let $x\ ,\ldots,x_n$ be indeterminates over $k$, let $\rho : G \longrightarrow$

$GL(n+1)$ be any representation of $G$ such that

$$\rho = \begin{pmatrix} 1 & \tau_1,\dots,\tau_n \\ 0 & \rho' \end{pmatrix}$$

and let $G$ act on the polynomial ring $R = k[x_0,\dots,x_n]$ as follows:

$$x_0^g = x_0 + \sum_{i=1}^{n} \tau_i(g)x_i \quad \text{and} \quad x_i^g = \sum_{j=1}^{n} \rho'_{ij}(g)x_j \quad (1 \leqq i \leqq n).$$

Then there is a G-invariant homogeneous polynomial $f(x_0,\dots,x_n)$ which is monic in $x_0$, i.e., $f = x_0^m + \dots$ , where $m = \deg f$.

(b) Let $G$ act on the n-dimensional projective space $\mathbb{P}^n$ via the above linear representation $\rho$, for which $0 = (1,0,\dots,0)$ is a G-invariant point. Then there is a G-invariant hypersurface $S$ which does not contain the above fixed point $0$.

(c) Let $P$ be a standard parabolic subgroup of $GL(n+1)$,

$$P = \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ & & & \\ 0 & * & \cdots & * \end{pmatrix} \right\}$$

and let $G$ be a subgroup of $P$. Then there is a divisor $D$ on $GL(n+1)$ satisfying the following conditions:

(i) $D$ is invariant under the left multiplication of $G$ and the right multiplication of $P$ on $GL(n+1)$.

(ii) $D \cap P = \phi$.

<u>Remark</u> 4. (1) If we can take a G-invariant homogeneous polynomial $f$ with degree $1$ in the condition (b) then $G$ is linearly reductive.

(2) If char $k = 0$, linear reductiveness and geometrical reductiveness of $G$ coincide. Moreover, the existence of a G-invariant

closed subset $Y \not\supset 0$ in the condition (c), which may not be a divisor, implies the linear reductiveness of G (H. Sumihiro [91]).

As for geometrically reductive groups, we have

Theorem 2. (The former Mumford conjecture; W. Haboush [33]). A reductive algebraic group is geometrically reductive.

§ 2. Some properties of geometrically (or linearly) reductive groups

We shall show some important properties of linearly reductive and geometrically reductive groups from the point of view of invariant subrings.

Let G be a linear algebraic group and let A be a k-algebra. If the following conditions are satisfied then we say that G acts on A rationally:

(i) There exists a group homomorphism $\rho : G \longrightarrow \mathrm{Hom}_{alg}(A,A)$ such that $a^{g_1 g_2} = (a^{g_1})^{g_2}$ and $a^e = a$.

(ii) For any element $a \in A$, $\sum_{g \in G} a^g k$ is a finite-dimensional G-rational module.

Let X = Spec(A). Then G acts on A rationally if and only if G acts on X regularly.

As usual, we shall denote by $A^G$ the subring of G-invariant elements of A, i.e., $A^G = \{a \in A \mid a^g = a$ for all $g \in G\}$ and we call $A^G$ the G-invariant subring of A. Moreover, for any element $a \in A$, we shall denote by $\mathfrak{A}(a)$ the ideal generated by elements $\{a^g - a \mid g \in G\}$. Then $\mathfrak{A}(a)$ is a G-invariant ideal of A.

If $G$ is geometrically (or linearly) reductive then, for any element $a \in A$, there is a positive integer $m$ ($m = 1$ if $G$ is linearly reductive) such that

$$(*) \quad a^m+\alpha_1 a^{m-1}+\ldots+\alpha_m = c, \text{ where } \alpha_i \in \mathfrak{A}(a)^i \text{ and } c \in A^G.$$

In fact, consider the finite-dimensional rational G-modules

$$V = \sum_{g\in G} a^g k \quad \text{and} \quad W = V \cap \mathfrak{A}(a).$$

Then $V = ak + W$ and $a$ is G-invariant modulo $W$, i.e., we have the following representation of $G$

$$\rho = \begin{pmatrix} 1 & \tau \\ 0 & \rho' \end{pmatrix}.$$

Therefore there is a G-invariant element $c$ of $A$ by the geometrical reductiveness of $G$ such that

$$a^m+\alpha_1 a^{m-1}+\ldots+\alpha_m = c \quad \text{with} \quad \alpha_i \in \mathfrak{A}(a)^i.$$

Remark 5. If $G$ is linearly reductive then $c$ is uniquely determined by $a$ and we have an $A^G$-homomorphism

$$R : A \ni a \longmapsto c \in A^G \subset A$$

called the Reynolds operator, which is very important; for example, using this operator, we can show that $A^G$ is a direct summand of $A$ as $A^G$-modules.

By the above-mentioned, rather simple fact (*), we can obtain the following fundamental result on G-invariant subrings.

Theorem 3 (M. Nagata [65]). Let $G$ be a geometrically (or linearly) reductive group and let $A, B, \ldots$ be k-algebras on which $G$ acts rationally.

(i) Let $\phi : A \longrightarrow B$ be a G-equivariant surjective homomorphism. Then $B^G$ is integral over $\phi(A^G)$, i.e., for any element $b \in B^G$, there is a positive integer $m$ such that $b^m = \phi(a)$ with $a \in A^G$. In particular, if $\mathfrak{a}$ is a G-invariant ideal of $A$ then $(A/\mathfrak{a})^G$ is integral over $A^G/A^G \cap \mathfrak{a}$. (If $G$ is linearly reductive then $B^G = \phi(A^G)$, hence $(A/\mathfrak{a})^G = A^G/A^G \cap \mathfrak{a}$.)

(ii) Let $\mathfrak{a}$ be an ideal of $A^G$. Then $\sqrt{\mathfrak{a}A \cap A^G} = \sqrt{\mathfrak{a}}$. Hence the canonical morphism $f : \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A^G)$ is surjective. (If $G$ is linearly reductive, $\mathfrak{a}A \cap A^G = \mathfrak{a}$.)

(iii) If $A$ is finitely generated over $k$ then $A^G$ is finitely generated over $k$.

(iv) Let $B$ be a flat $A^G$-algebra on which $G$ acts trivially. Then $(A \otimes_{A^G} B)^G = B$.

As for singularities of $A^G$, we have

Theorem 4 (M. Hochster and J. Roberts [36]). Let $G$ be a linearly reductive group and let $A$ be a regular k-algebra with a G-action. Then the G-invariant subring $A^G$ is Cohen-Macaulay.

## § 3. Quotient schemes after Mumford [116] and Seshadri [85, 86]

Let $X$ be a scheme and let $\sigma : G \times X \longrightarrow X$ be a G-action. For any element $x \in X$, we shall define

$O(x)$ = orbit of $x = \mathrm{Im}[G \times X \xrightarrow{\sigma} X] = \{x^g \mid g \in G\}$

$S_x$ = stabilizer group of $x$ = the fiber of the X-group scheme $(\sigma \times 1)^{-1}(\Delta)$ at $x$, where $\Delta$ is the diagonal of $X \times X$ and $\sigma \times 1 : G \times X \ni (g,x) \longmapsto (x^g, x) \in X \times X$.

In general, $O(x)$ is a locally closed subscheme of $X$ (not necessarily closed). Hence $O(x)$ contains an open subset of $\overline{O(x)}$ (the closure of $O(x)$). Let us make the following

Definition 3. With the above notation, the action $\sigma$ is said to be

(i) closed if $O(x)$ is closed in $X$ for all $x \in X$,

(ii) separated if $\mathrm{Im}[\sigma \times 1]$ is closed,

(iii) proper if $\sigma \times 1$ is a proper morphism,

(iv) free if $\sigma \times 1$ is a closed immersion.

Next we shall introduce several notions of quotient schemes.

Definition 4. For a given action $\sigma$ of $G$ on $X$, a pair $(Y,f)$ consisting of a prescheme $Y$ and a morphism $f : X \longrightarrow Y$ is called

(i) a categorical quotient if

(a) the following diagram commutes

$$\begin{array}{ccc} G \times X & \xrightarrow{\sigma} & X \\ \downarrow p_2 & f & \downarrow f \\ X & \longrightarrow & X \end{array} ,$$

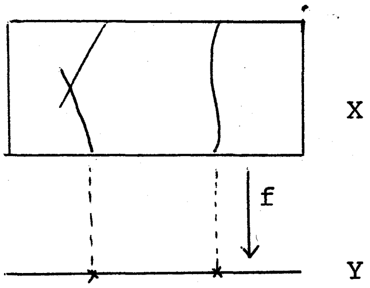
(b) given any pair $(Z,g)$ consisting of a prescheme $Z$ and a morphism $g : X \longrightarrow Z$ such that (a) holds for $(Z,g)$, then there is a unique morphism $h : Y \longrightarrow Z$ such that $g = h \cdot f$;

(ii) a good quotient if $(Y,f)$ is a categorical quotient and if

(a) $f$ is surjective and affine,

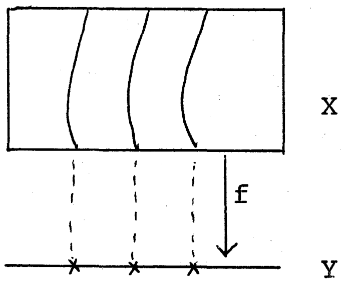
(b) $f_*(O_X)^G = O_Y$,

(c) $f(Z)$ is closed in $Y$ for every G-invariant subset $Z$ of $X$, and $f(Z_1) \cap f(Z_2) = \phi$ for G-invariant closed subsets $Z_1$, $Z_2$ with $Z_1 \cap Z_2 = \phi$, i.e., $f$ separates G-invariant closed subsets;




, $Y = X/\sim$, where $x_1 \sim x_2$ if and only if $\overline{O(x_1)} \cap \overline{O(x_2)} \neq \phi$

(iii) <u>a geometric quotient</u> if $(Y,f)$ is a good quotient and if $\mathrm{Im}[\sigma \times 1] = X \underset{Y}{\times} X$, i.e., for any points $x_1$, $x_2$ of $X$, $f(x_1) = f(x_2)$ if and only if $O(x_1) = O(x_2)$. Hence, for any point $x \in X$, $O(x) = f^{-1}(f(x))$ and is closed in $X$,




, $Y = X/\sim$, where $x_1 \sim x_2$ if and only if $O(x_1) = O(x_2)$.

Now let $G$ be a geometrically reductive group and let $X = \mathrm{Spec}(A)$ be an algebraic scheme over $k$ with an action of $G$. By Theorem 2 we then see that:

(i) $Y = \mathrm{Spec}(A^G)$ is an algebraic scheme over $k$.

(ii) Let $f : X = \mathrm{Spec}(A) \longrightarrow Y$ be the canonical morphism. Then $f$ is surjective, affine and $f_*(O_X)^G = O_Y$.

(iii) If $Z = V(\mathfrak{a})$ is a G-invariant closed subset of $X$ with a G-invariant ideal $\mathfrak{a}$ of $A$ then $f(Z)$ is closed; in fact, $(A/\mathfrak{a})^G$ is integral over $A^G/A^G \cap \mathfrak{a}$; therefore, for any maximal ideal $\mathfrak{m}$ of $A^G/A^G \cap \mathfrak{a}$, $\mathfrak{m}A/\mathfrak{a}$ is a proper ideal of $A/\mathfrak{a}$ and so there is a maximal ideal $\mathfrak{m}'$ of $A/\mathfrak{a}$ lying over $\mathfrak{m}$. Let $Z_1 = V(\mathfrak{a}_1)$ and $Z_2 = V(\mathfrak{a}_2)$ be two G-invariant closed subschemes such that $Z_1 \cap Z_2 = \phi$, i.e., $\mathfrak{a}_1 + \mathfrak{a}_2 = A$. Then $f(Z_1) \cap f(Z_2) = \phi$; in fact, write $1 = a_1 + a_2$ with $a_1 \in \mathfrak{a}_1$ and $a_2 \in \mathfrak{a}_2$, i.e., $a_1$ vanishes on $Z_1$ and 1 on $Z_2$; then there is a positive integer m such that

$$a_1^m + \alpha_1 a_1^{m-1} + \ldots + \alpha_m = c \quad \text{with} \quad c \in A^G \quad \text{and} \quad \alpha_i \in \mathfrak{a}(a_1)^i;$$

then c vanishes on $Z_1$ and 1 on $Z_2$; hence c separates $f(Z_1)$ and $f(Z_2)$.

Therefore we see that (Y,f) is a good quotient of X by G. Moreover, if (Y,f) is a geometric quotient then O(x) is closed for any $x \in X$, whence $\sigma$ is closed. Conversely, if $\sigma$ is closed then $\mathrm{Im}[\sigma \times 1] = X \times_Y X$ because (Y,f) is a good quotient, hence (Y,f) is a geometric quotient of X by G.

Summing up the above results, we obtain

<u>Theorem</u> 5. Let G be a geometrically reductive group and let $X = \mathrm{Spec}(A)$ be an algebraic k-scheme on which G acts regularly. Then X has a good quotient (Y,f), where $Y = \mathrm{Spec}(A^G)$ and $f : X \longrightarrow Y$ is the canonical morphism. Moreover, the good quotient (Y,f) is a geometric quotient if and only if $\sigma$ is closed.

## § 4. Semistable and stable points

In this section, we shall introduce a very useful concept of semistable and stable points, due to Mumford, to construct quotient preschemes in more general cases.

For simplicity, we assume that:

$X \subset \mathbb{P}^n$ : a G-invariant locally closed subscheme

$$\begin{array}{ccc} \rho : G & \longrightarrow & PGL(n) \\ & \searrow & \uparrow \\ & & GL(n+1) \end{array}$$ : a representation of $G$ which lifts up to $GL(n+1)$ .

<u>Remark</u> 6. Let $X$ be a normal quasi-projective algebraic variety with a G-action $\sigma$ and $L$ be an ample line bundle on $X$. Then there is a positive integer $m$ such that:

$X \subset \mathbb{P}^n$ : a G-invariant locally closed subscheme embedded G-equivariantly,

$L^{\otimes m} = O_{\mathbb{P}^n}(1)|_X$,

$\rho : G \longrightarrow PGL(n)$ : a representation;

this results follows essentially from the fact that $Pic(G)$ is a finite group. Moreover, if $G$ has no nontrivial characters then $\rho$ factors through $GL(n+1)$,

$$\begin{array}{ccc} G & \xrightarrow{\rho} & PGL(n) \\ & \searrow & \uparrow \\ & & GL(n+1) \end{array}$$

and so $L^{\otimes m}$ is G-linearizable; this result follows essentially from the fact that every invertible function of $G$ is a product of a character and a nonzero constant.

$O_{\mathbb{P}^n}(1)$ is not PGL(n)-linearizable. However $O_{\mathbb{P}^n}(n+1)$ is PGL(n)-linearizable because the tangent bundle $T_{\mathbb{P}^n}$ of $\mathbb{P}^n$ is PGL(n)-linearizable vector bundle and $\overset{n}{\Lambda} T_{\mathbb{P}^n} \cong O_{\mathbb{P}^n}(n+1)$.

Combining the above results, we see that every action on normal quasi-projective algebraic varieties can be reduced to the action assumed at the beginning of this section after changing the embedding suitably.

(4.1) Semistable and stable points

Let $\pi : \mathbf{A}^{n+1}-\{0\} \longrightarrow \mathbf{P}^n$ be the canonical morphism and let $\hat{X}$ be the affine cone of $X$.

<u>Definition</u> 5. With the above notation, $x$ is said to be

(i) <u>semistable</u> if $\overline{O(\hat{x})} \not\ni 0$, where $\hat{x}$ is a point of $\hat{X}$ such that $\pi(\hat{x}) = x$ and $0$ is the origin of $\mathbf{A}^{n+1}$;

(ii) <u>stable</u> if $O(\hat{x})$ is closed in $\hat{X}$ and $S_{\hat{x}}$ is a finite group.

We can restate the above definitions as follows:

<u>Theorem</u> 6 (and <u>Definition</u> 6). A point $x \in X$ is

(i) semistable if there is a non-constant G-invariant homogeneous polynomial $f$ such that $f(x) \neq 0$ and $X_f := \{x \in X \mid f(x) \neq 0\}$ is affine; thus the set of semistable points is a G-invariant open subset of $X$ which we shall denote by $X^{ss}$;

(ii) stable if there is a non-constant G-invariant homogeneous polynomial $f$ such that $f(x) \neq 0$, $X_f$ is affine and the action of $G$ on $X_f$ is closed; then the set of stable points is a G-invariant open subset of $X$ which we shall denote by $X^s$.

<u>Remark</u> 6. (1) $X-X^{ss} = \bigcap_f V(f)$, where $f$ runs over all non-constant G-invariant homogeneous polynomials. Hence we cannot separate the points in $X-X^{ss}$ by G-invariant homogeneous polynomials.
(2) There are examples of bad actions of $G$ for which $X^{ss} = \phi$ or $X^{ss} \neq \phi$ and $X^s = \phi$. As for examples with $X^{ss} = \phi$, see

Kimura and Sato [166]. In Example 1, $X^{ss} = \{x \in \mathbb{P}^2 \mid x_1^2 + x_0 x_2 \neq 0\}$ and $X^s = \phi$.

(3) Let $\nu_m : \mathbf{P}^n \longrightarrow \mathbf{P}^N$ be an m-th Veronese embedding and let G act on $\mathbb{P}^N$ via the symmetric tensor representation of degree m. It may occur that $\nu_m(x)$ is not stable even if x is stable in $\mathbb{P}^n$.

We have the following,very useful criterion for a point $x \in X$ to be semistable or stable by using 1-PS's (= one-parameter subgroups) of G. Let $\lambda : G_m \longrightarrow G$ be a 1-PS of G and let x be a point of X. Since $\mathbb{P}^n$ is complete, the limit point

$$x_0 = \lim_{t \to 0} \lambda(t) \cdot x$$

exists and $x_0$ is a $G_m$-invariant point. Therefore $G_m$ acts on the line $\ell(x_0) = \hat{x}_0 \cdot k$, where $\hat{x}_0 \in \mathbb{A}^{n+1}$ and $\pi(\hat{x}_0) = x_0$. Since the action of $G_m$ on $\mathbb{A}^1$ is the multiplication, there is an integer r such that

$$\lambda(t) \cdot \hat{x}_0 = t^r \hat{x}_0 \text{ for all } t \in G_m.$$

Let us define the following integer with respect to $\lambda$ and x.

Definition 7. With the above notation, $\mu(\lambda, x) := -r$.

Example 2. Let

$$\lambda(t) = \begin{pmatrix} t^{r_0} & & \\ & \ddots & \\ & & t^{r_n} \end{pmatrix} \text{ for all } t \in G_m \text{ and } x = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}.$$

Then we have $\mu(\lambda, x) = -\min_i \{r_i \mid x_i \neq 0\}$.

Theorem 8. Let x be a point of X. Then x is

(i) semistable if and only if $\mu(\lambda, x) \geq 0$ for all 1-PS $\lambda$ ;

(ii) stable if and only if $\mu(\lambda, x) > 0$ for all 1-PS $\lambda$.

The above theorem follows essentially from the fact (due to Hilbert) that: Let $\hat{x}$ be a point of $\hat{X}$ such that $\pi(\hat{x}) = x$; then we have

(i)' $0 \in \overline{O(\hat{x})}$ if and only if, for some 1-PS $\lambda$, $\lambda(t)\hat{x}$ specializes to $0$ as $t \to 0$;

(ii)' the morphism $\phi : G \ni g \longmapsto g\hat{x} \in \hat{X}$ is proper if and only if, for some nontrivial 1-PS $\lambda$, the morphism $\phi_\lambda : G_m \ni t \longmapsto \lambda(t)\hat{x} \in \hat{X}$ is proper, i.e., $\lambda(t)\hat{x}$ has no specialization in $\mathbb{A}^{n+1}$ as $t \to 0$.

<u>Example</u> 3 (of semistable and stable points) (Mumford [118]).

Let $G = SL(n+1)$ act on $\mathbb{P}^n$ in the standard fashion and let $X = \{$hypersurfaces in $\mathbb{P}^n$ with degree $m\}$. Then $X \cong \mathbb{P}^N$ with $N = {}_{n+1}H_m - 1$ and $X$ has a canonical $G$-action via the m-th symmetric tensor representation. Consider the semistable and stable points for smaller $n$ and $m$.

(1) $n = 1$. Then $G = SL(2)$ and $X = \{D = \Sigma m_i P_i$ with $m = \Sigma m_i\}$ (the set of effective 0-cycles of degree $m$ on $\mathbb{P}^1$). We have

$$X^{ss} = \{D \mid m_i \leqq m/2 \text{ for all } i\} \text{ and } X^s = \{D \mid m_i < m/2 \text{ for all } i\}.$$

In fact, take homogeneous coordinates $x_0$, $x_1$ of $\mathbb{P}^1$ such that $P_0 : x_0 = 0$ and let $D$ be defined by $f = \Sigma a_i x_0^{m-i} x_1^i$. Let $\lambda$ be a 1-PS such that

$$\lambda : G_m \ni t \longmapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in G$$

$$-\lambda : G_m \ni t \longmapsto \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \in G \quad .$$

Then we have

$$f^{\lambda}(t) = \Sigma\ a_i t^{m-2i} x_0^{m-i} x_1^i$$
$$f^{-\lambda}(t) = \Sigma\ a_i t^{2i-m} x_0^{m-i} x_1^i$$

and hence $\mu(\lambda,1) = -\min\{\ m-2i \mid a_i \neq 0\}$ and $\mu(-\lambda,1) = -\min\{2i-m \mid a_i \neq 0\}$. Let $j = \min\{i \mid a_i \neq 0\}$ and $k = \max\{i \mid a_i \neq 0\}$. Then $\mu(\lambda,1) = -(m-2k) \geqq 0$ (resp. $> 0$) and $\mu(-\lambda,1) = -(2j-m) \geqq 0$ (resp. $> 0$), i.e., $k \geqq m/2$ (resp. $> m/2$) and $j \leqq m/2$ (resp. $< m/2$) if $D$ is semistable (resp. stable). Therefore,

$$f = \Sigma\ a_i x_0^{m-i} x_1^i = a_j x_0^{m-j} x_1^j + \dots + a_k x_0^{m-k} x_1^k = x_0^{m-k} x_1^j (a_j x_0^{k-j} + \dots + a_k x_1^{k-j})$$

has multiplicity $\leqq m/2$ (resp. $< m/2$) at $P_0$ if $D$ is semistable (resp. stable). Conversely, if $D = \Sigma\ m_i P_i$ with $m_i \leqq m/2$ (resp. $< m/2$) for every $i$ then we easily see, by the same argument as above, that $\mu(\lambda,1) \geqq 0$ (resp. $\mu(\lambda,1) > 0$) for any 1-PS $\lambda$, hence $D$ is semistable (resp. stable).

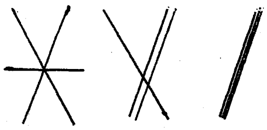
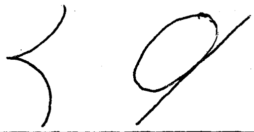
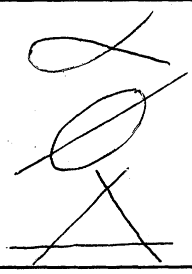
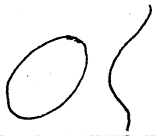
(2) $n = 2$. Then $G = SL(3)$ and $X = \{$plane curves of degree $m\}$. Then we have the following table:

$m = 1$ : $X^{ss} = \phi$.

$m = 2$ : $X = \{$quadric curves$\}$, and

| Type of singularities | Stability |
|---|---|
| nonsingular | semistable (not stable) |
| singular | unstable |

$m = 3$ : $X = \{$cubic curves$\}$, and

| Configulation | Type of singularities | Stability |
|---|---|---|
|  | Triple point | unstable |
|  | cusp or two components tangent at a point | unstable |
|  | ordinary double points; (this includes the reducible cases: a conic and a transversal line or a triangle of lines | semistable (not stable) |
|  | smooth cubic | stable |

(4.2) Semistable points and quotient schemes

Let G be a geometrically reductive group and let X be a quasi-projective algebraic variety with a G-action such that:

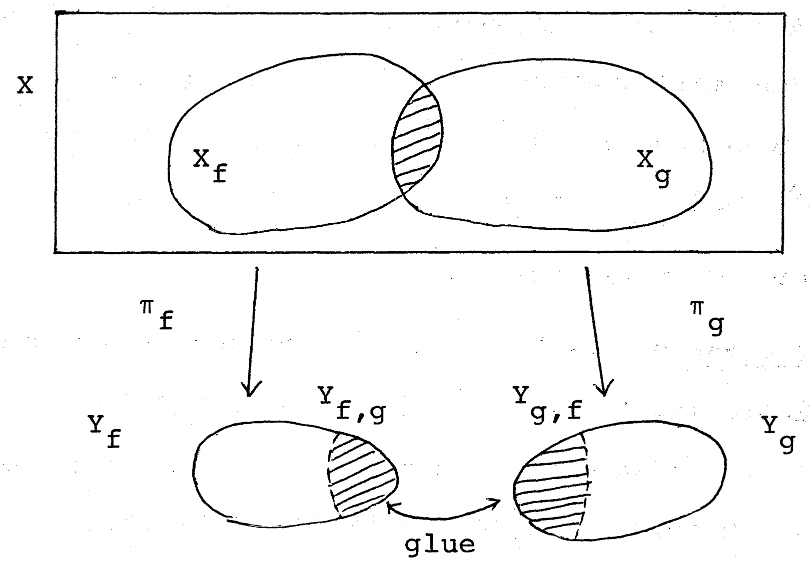
$$X \hookrightarrow \mathbb{P}^n \ : \quad \text{a locally closed subscheme,}$$

$$\rho : G \longrightarrow PGL(n) \ : \quad \text{a representation.}$$

$$\searrow \quad GL(n+1) \nearrow$$

We shall show that there exist a good quotient $(Y,\pi)$ of $X^{ss}$ by G and a geometric quotient $(Y^s,\pi')$ of $X^s$ by G. We may assume that X is closed by replacing X by its closure. Let $\mathfrak{a}$ be the defining ideal of X in $\mathbb{P}^n$. Then $X^{ss} = \bigcup X_f$, where f's are non-constant G-invariant homogeneous polynomials and $X_f = \mathrm{Spec}(A_f)$

with $A_f = k[x_0,\ldots,x_n]/\mathfrak{a}_{(f)}$. By Theorem 5, $X_f$ has a good quotient $(Y_f,\pi_f)$, where $Y_f = \mathrm{Spec}(A_f^G)$ and $\pi_f : X_f \longrightarrow Y_f$ is the canonical morphism. We can patch these good quotients $(Y_f,\pi_f)$ as follows: Let f, g be non-constant G-invariant homogeneous polynomials with deg f = r and deg g = s.




Here $g^r/f_s \in A_f^G$ and $f^s/g^r \in A_g^G$. If we set $Y_{f,g} := \mathrm{Spec}((A_f^G)_{g^r/f^s})$ and $Y_{g,f} := \mathrm{Spec}((A_g^G)_{f^s/g^r})$ then $Y_{f,g} = Y_{g,f}$ because $(A_f^G)_{g^r/f^s}$ $= (A_g^G)_{f^s/g^r} = ((A_f)_{g^r/f^s})^G = ((A_g)_{f^s/g^r})^G$ by virtue of Theorem 2. Patch up $Y_f$ and $Y_g$ along this open subset. Let Y be the prescheme obtained by the glueing of this kind and let $\pi : X^{ss} \longrightarrow Y$ be the morphism such that $\pi|_{X_f} = \pi_f$ for every f. Then Y is an algebraic scheme and $(Y,\pi)$ is the desired good quotient of $X^{ss}$ by G.

Furthermore, if we set $R := k[x_0,\ldots,x_n]/\mathfrak{a}$ (hence X = Proj R) and $R^G$ := the G-invariant subring of R (hence $R^G$ is a graded ring) then we can see that $Y = \mathrm{Proj}\, R^G$ and the canonical rational mapping $\xi : X \longrightarrow Y$ induced by the inclusion $R^G \hookrightarrow R$ is regular on $X^{ss}$ and $\pi = \xi|_{X^{ss}}$. Moreover, if $X$ is normal then $Y$ is normal. On the other hand, $X^s = \bigcup X_f$, where f's are non-constant G-invariant homogeneous polynomials and the G-action on $X_f$ is closed. By the same argument as above, we obtain an algebraic scheme $Y^s$ and a morphism $\pi' : X^s \longrightarrow Y^s$ such that $(Y^s, \pi')$ is the geometric quotient of $X^s$ by G. Since $X^s$ is a G-invariant open subscheme of $X^{ss}$, $Y^s$ is an open subscheme of $Y$ and $X^s = \pi^{-1}(Y^s)$ and $\pi' = \pi|_{X^s}$.

Thus we have shown

<u>Theorem</u> 9. With the above notation, there exists a good quotient $(Y, \pi)$ of $X^{ss}$ by G such that:

(i) $Y$ is a quasi-projective algebraic variety, which is normal if $X$ is so, and $O_X(m)$ descends to an ample line bundle on $Y$ for a suitable positive integer $m$;

(ii) there is an open subscheme $Y^s$ of $Y$ such that $X^s = \pi^{-1}(Y^s)$ and $(Y^s, \pi|_{X^s})$ is the geometric quotient of $X^s$,

$$\begin{array}{ccccccc} \mathbb{P}^n & \supset & X & \supset & X^{ss} & \supset & X^s \\ & & & \pi & \Big\downarrow \text{good quot.} & & \Big\downarrow \text{geom. quot.} \\ & & & & Y = X^{ss}/G & & Y^s = X^s/G \end{array} .$$

<u>Example</u> 4 (Binary quartics; cf. Example 3). Let $G = SL(2,k)$ and $X = \{D = \Sigma\, m_i P_i \mid \Sigma\, m_i = 4\}$ (the set of effective 0-cycles of degree 4 of $\mathbb{P}^1$). As we have seen earlier, we have

$X^s = \{D = P_1+P_2+P_3+P_4$ with $P_i \neq P_j$ if $i \neq j\}$

$X^{ss} = \{ D = 2P_1+P_2+P_3$ or $D = 2P_1+2P_2$ with $P_i \neq P_j$ if $i \neq j\}$.

Let $f$ be a homogeneous polynomial of degree 4 and write it in the form:

$$f = a_o x_o^4 + 4a_1 x_o^3 x_1 + 6a_2 x_o^2 x_1^2 + 4a_3 x_o x_1^3 + a_4 x_1^4 \ .$$

We have two invariants:

$$I = a_o a_4 - 4a_1 a_3 + 3a_2^2$$

$$J = a_o a_2 a_4 + 2a_1 a_2 a_3 - a_o a_3^2 - a_1^2 a_4 - a_2^3$$

and we put $\Delta = I^3 - 27J^2$. If $f$ has a simple root then $f$ is equivalent to

$$f_o = x_o^3 x_1 + a x_o x_1^3 + b x_1^4 \ .$$

for which $I = -\frac{a}{4}$ and $J = -\frac{b}{16}$. Moreover, we have

the discriminant of $f_o$ = const.$\cdot(4a^3+27b^2)$ = const.$\cdot(I^3-27J^2)$.

Then we see that:

(i) $\Delta = 0$ if and only if $f = 0$ has a multiple root, whence $X^s = \mathbf{P}^4_\Delta$ .

(ii) $I = J = 0$ if and only if $f = 0$ has a triple root or a quadruple root, whence $X^{ss} = \mathbf{P}^4_I \cup \mathbf{P}^4_J$.

(iii) $X^s/SL(2) = \mathbf{A}^1 = \mathrm{Spec}(k[J^2/\Delta])$. In fact, let $P_1+P_2+P_3+P_4 \in X^s$. Then there exists an element $g \in SL(2)$ such that $gP_1 = (1,0)$, $gP_2 = (0,1)$, $gP_3 = (1,1)$ and $gP_4 = (1,\lambda)$ ($\lambda \neq 0, 1$), i.e., $g(P_1,P_2,P_3,P_4) = (0,\infty,1,\lambda)$. The cross-ratio $\lambda$ is one of the following, which varies depending on choices of $g$ and orders $P_1,P_2,P_3,P_4$):

$$\lambda,\ 1-\lambda,\ \frac{1}{\lambda},\ \frac{(\lambda-1)}{\lambda},\ \frac{\lambda}{\lambda-1},\ \frac{1}{1-\lambda}\ .$$

Let $\mu = \left\{\frac{(2\lambda-1)(\lambda-2)(\lambda+1)}{\lambda(\lambda-1)}\right\}^2$. Then $\mu = 3^6 J^2/\Delta$. Hence $X^s/SL(2)$ $= \mathrm{Spec}(k[\mu])$.

(iv) $X^{ss}/SL(2) = \mathbb{P}^1$.

(4.3) Stability of Chow form (after D. Mumford [118])

We shall look for a condition for a Chow form to be semiatable or stable. Assume that:

$\rho : G \longrightarrow PGL(n)$ : a representation

$X \subset \mathbb{P}^n$ : an effective cycle with degree = d and dim = r

$C_d^r$ : Chow variety

$F \in C_d^r$ : the Chow form associated with X.

Then there is the canonical action of G on $C_d^r$.

Definition 9. With the above notation, a cycle X is said to be Chow semistable or Chow stable if F is semistable or stable.

As for hypersurfaces with small n and d, we have observed Chow semistability and Chow stability. We shall look for the conditions in the case of more general subvarieties. For this purpose, we need the following definitions:

(1) Reduced degree: Let $X \subset \mathbb{P}^n$ be an effective cycle with degree d and dim. r. Then we put

$$\text{red.deg } X = \frac{\deg X}{n+1-r}$$

and call it the reduced degree of X.

(2) Let L be an (n-m-1)-dimensional linear subspace of $\mathbb{P}^n$ and let $p_L : \mathbb{P}^n \longrightarrow \mathbb{P}^m$ be the projection with center L. Let X be an irreducible subvariety of $\mathbb{P}^n$ such that $X \not\subset L$. Then the restriction of $p_L$ onto X has the fundamental locus $\subseteq X \cap L$. Let I be the defining ideal of L in $\mathbb{P}^n$ and let J be the

homomorphic image of $I$ under the canonical homomorphism $O_{\mathbb{P}^n} \longrightarrow O_X$. If we blow up $X$ along $J$ then the base points of $p_L$ are eliminated and there is a unique morphism $q : \hat{X} \longrightarrow \mathbb{P}^m$ such that

$$\begin{array}{ccc} \hat{X} & \xrightarrow{q} & \mathbb{P}^m \\ \downarrow & \nearrow_{p_L} & \\ X & & \end{array} .$$

We shall define $p_L(X)$ to be the cycle $q(\hat{X})$, i.e., $q(\hat{X})$ with multiplicity equal to the degree of $q$ if $\dim \hat{X} = \dim q(\hat{X})$ and 0 otherwise.

<u>Definition</u> 10. With the above notation, $X \subset \mathbb{P}^n$ is said to be <u>linearly stable</u> (resp. <u>linearly semistable</u>) if, for any linear subspace $L^{n-m-1} \subset \mathbb{P}^n$ such that the image cycle $p_L(X)$ of $X$ under the projection $p_L : \mathbb{P}^n \longrightarrow \mathbb{P}^m$ has dimension $r$, we have

$$\text{red.deg } p_L(X) > \text{red.deg } X \quad (\text{resp. } \geq ).$$

This concept works efficiently for smooth curves. Indeed, we have the following criterions:

<u>Theorem</u> 10. If a smooth curve $C \subset \mathbb{P}^n$ is linearly stable (resp. linearly semistable) then $C$ is Chow stable (resp. Chow semistable).

<u>Theorem</u> 11. If $C \subset \mathbb{P}^n$ is a smooth curve of genus $g$ embedded by $\Gamma(C,L)$, where $L$ is a line bundle of degree $d$, then we have:

(i) $C$ is linearly stable if $d > 2g > 0$;

(ii) $C$ is linearly semistable if $d \geq 2g \geq 0$.

Combining these results, we obtain the following

<u>Theorem</u> 12. If $C$ is a smooth curve of genus $g \geq 1$ embedded

into a projective space by a complete linear system of degree $d > 2g$ then $C$ is Chow stable.

## § 5. Moduli problems

As an application of the geometric invariant theory, we shall consider the construction of various moduli spaces.

### (5.1) Moduli functors and Moduli schemes (or spaces)

Let (Sch/k) be the category of k-schemes. For any $S \in$ (Sch/k), consider _a family of algebraic objects_ $A(S)$ parametrized by $S$ such that, for any morphism $\phi : S' \longrightarrow S$, there is a morphism $\phi^* : A(S) \longrightarrow A(S')$ satisfying the following properties,

$(id)^* = id$

$(\phi\cdot\psi)^* = \psi^*\cdot\phi^*$ for morphisms $\phi : S' \longrightarrow S$ and $\psi : S'' \longrightarrow S'$,

and _an equivalence relation_ " $\sim$ " on $A(S)$ such that if $X \sim X'$ ($X, X' \in A(S)$) and $\phi : S' \longrightarrow S$ then $\phi^*(X) \sim \phi^*(X')$. If $\phi : U \longrightarrow S$ is an open immersion then we denote $\phi^*(X)$ by $X(U)$. Then we call the following contravariant functor $F$ a moduli functor,

$$F : (\mathrm{Sch}/k) \ni S \longmapsto A(S)/\sim \ \in (\mathrm{Sets}) .$$

_Example_ 5 (Smooth algebraic curves). For any k-scheme $S$, a curve over $S$ with genus $g$ is a smooth proper morphism $\pi : X \longrightarrow S$ such that $C_s := \pi^{-1}(s)$ is an irreducible smooth curve of genus $g$ for every $s \in S$. Let $A(S) = \{$curves over $S$ with genus $g\}$. For two curves $\pi : X \longrightarrow S$ and $\pi' : X' \longrightarrow S'$, we shall define a relation $X \sim X'$ if there are isomorphisms $f : S' \longrightarrow S$ and $g : X' \longrightarrow X$ such that $\pi\cdot g = f\cdot\pi'$. Then $F(S) :=$

$A(S)/\sim$ is a moduli functor and $F(k) = \{$nonsingular projective curves of genus $g$ over $k\}/(\text{Isomorphisms})$.

Definition 11. If a moduli functor $F$ is representable, i.e., there exist a k-scheme $M$ and a functor-isomorphism $\Phi : F \cong h_M$ then $M$ is said to be a fine moduli scheme (space) of $F$.

If $F$ has a fine moduli scheme $M$ then there is a universal family of algebraic objects $X_M$ parametrized by $M$, i.e., for any $S \in (Sch/k)$, there is a functorial one-to-one correspondence

$$F(S) \ni X \underset{1:1}{\longleftrightarrow} \phi : S \longmapsto M \quad \text{such that} \quad X \sim \phi^*(X_M).$$

In general, moduli functors cannot be represented by such nice fine moduli schemes. Therefore, we shall make the following

Definition 12. If there are a scheme $M$ and a functor morphism $\Phi : F \longrightarrow h_M$ satisfying the following conditions, we say that $M$ is a coarse moduli scheme (space) of $F$:

(1) $\Phi(k) : F(k) \cong h_M(k) = M(k)$, a bijection;

(2) for any morphism of functors $\phi : F \longrightarrow h_N$, where $N$ is a k-scheme, there is a unique morphism $\xi : h_M \longrightarrow h_N$ such that $\phi = \xi \cdot \Phi$.

If there exists a coarse moduli scheme of $F$ then it is unique up to isomorphisms. For simplicity, we shall say that $M$ is a moduli scheme (space) of $F$ if $M$ is a coarse moduli scheme of $F$.

(5.2) A construction of moduli schemes

If a given moduli functor $F$ satisfies the following conditions then we can construct a moduli scheme of $F$ as a quotient scheme:

(i) There exists a reductive algebraic group $G$ and a quasi-projective algebraic variety $Y$ such that

$$Y \subset \mathbf{P}^n : \text{ a locally closed subvariety with a } G\text{-action induced from a } G\text{-action on } \mathbf{P}^n,$$

$$Y = Y^s.$$

(ii) For any $S \in (Sch/k)$ and $X \in A(S)$, there are open covering $\{U_i\}$ $(i \in I)$ of $S$ and a family of morphisms $\{f_i : U_i \longrightarrow Y\}$ $(i \in I)$, which we call, for simplicity, a family of local data of $X$, such that:

(a) Let $X' \in A(S')$ and let $\{U_j'\}$, $\{f_j'\}$ $(j \in J)$ be a family of local data of $X'$. Then $X' \sim X$ implies

$$O(f_i(s)) = O(f_j'(s)) \quad \text{for all} \quad s \in U_i \cap U_j'.$$

(b) Let $\phi : S' \longrightarrow S$ be a morphism and let $\{V_j\}$, $\{g_j\}$ be a family of local data of $\phi^*(X)$. Then

$$O(g_j(s')) = O(f_i(\phi(s'))) \quad \text{for all} \quad s' \in V_j \cap \phi^{-1}(U_i).$$

(c) When $S = \mathrm{Spec}(k)$, every algebraic object $X \in A(S)$ determines a (k-rational) point of $Y$. Denote it by $f(X)$. Then, for $X, X' \in A(k)$, $X \sim X'$ if and only if $O(f(X)) = O(f(X'))$.

(d) There exists an algebraic object $X_Y$ parametrized by $Y$ such that a family of local data of $X_Y$ induces the identity morphism of $Y$ and, for any $S$ and $X \in A(S)$,

$$X|_{U_i} \sim f_i^*(X_Y) \quad \text{for all} \quad i \in I,$$

where $\{U_i\}$, $\{f_i\}$ is a family of local data of $X$.

Then the quotient scheme $Z = Y/G$ is a moduli scheme of $F$. In fact, we can argue as follows:

(1) For any $X \in A(S)$, there is a morphism $\Phi(X) : S \longrightarrow Z$, and if $X \sim X'$ then $\Phi(X) = \Phi(X')$ by (ii)(a). Moreover, $\Phi(X)$ is functorial by (ii)(b). Hence there is a functor morphism $\Phi : F \longrightarrow h_Z$.

(2) By (i) and (ii)(c), we see that $\Phi(k) : F(k) \longrightarrow h_Z(k)$ is bijective. Let $\phi : F \longrightarrow h_W$ be a functor morphism, where $W$ is a k-scheme. Then the existence of an algebraic object $X_Y$ parametrized by $Y$ implies that there is an equivariant morphism $\psi : Y \longrightarrow W$, where $G$ acts trivially on $W$. Hence we have a unique morphism $\xi : Z \longrightarrow W$ and $\phi = \xi \cdot \Phi$ by (ii)(d).

Example 6 (D. Mumford [116]). Let $F$ be the moduli functor of smooth algebraic curves of genus $g \geqq 2$. Then $F$ has a moduli scheme, which is quasi-projective and of dimension $3g-3$, because $F$ satisfies the above conditions.

Furthermore, the following important results on moduli problems have been obtained;

(i) principally polarized abelian varieties with a level structure (D. Mumford [116]);

(ii) nonsingular projective surfaces of general type modulo birational equivalence (D. Gieseker [104]);

(iii) stable algebraic vector bundles with a fixed Hilbert polynomials (D. Mumford [115], C.S. Seshadri [141], D. Gieseker [103] M. Maruyama [113]).