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Shape fibrations for topological spaces

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The notion of a shape fibration $p:E \to B$ was first introduced in [4] for the case when p is a map of metric compacta. In order to generalize it so as to apply to maps of arbitrary topological spaces, the author has introduced the notion of a resolution of a space and that of a resolution of a map [3].

Definition 1. A resolution of a topological space E consists of an inverse system $\underline{E} = (E_\lambda, p_{\lambda\lambda'}, \Lambda)$ of topological spaces and of a map of systems $\underline{q} = (q_\lambda): E \to \underline{E}$ such that the following two conditions hold:

(R_1) If P is a polyhedron, $\mathcal{U}$ is an open covering of P and $h:E \to P$ is a map, then there exist a $\lambda \in \Lambda$ and a map $f:E_\lambda \to P$ such that the maps fp_λ and h are $\mathcal{U}$-near.

(R_2) If P is a polyhedron and $\mathcal{U}$ is an open covering of P, then there exists an open covering $\mathcal{V}$ of P with the following property: whenever $\lambda \in \Lambda$ and $f, f':E_\lambda \to P$ are maps such that the maps fp_λ and $f'p_\lambda$ are $\mathcal{V}$-near, then there exists a $\lambda' \geq \lambda$ such that the maps $fp_{\lambda\lambda'}$ and $f'p_{\lambda\lambda'}$ are $\mathcal{U}$-near.

If all E_λ are polyhedra, the resolution is called polyhedral.

Definition 2. A resolution of a map $p:E \to B$ consists of resolutions $\underline{q}:E \to \underline{E}$, $\underline{r}:B \to \underline{B}$ and of a map of systems $\underline{p}:\underline{E} \to \underline{B}$ such that

$$\underline{p}\,\underline{q} = \underline{r}\,p.$$

The resolution is polyhedral if $\underline{q}$ and $\underline{r}$ are polyhedral resolutions.

Definition 3. A level map of systems $\underline{p}=(p_\lambda):\underline{E} \to \underline{B}$ has the approximate homotopy lifting property (AHLP) provided for every $\lambda \in \Lambda$, normal covering $\mathcal{U}$ of E_λ and normal covering $\mathcal{V}$ of B_λ there exist a $\lambda' \geq \lambda$ and a normal covering $\mathcal{W}$ of $B_{\lambda'}$ such that the following condition holds. Whenever $h:X \to E_{\lambda'}$ and $H:X \times I \to B_{\lambda'}$ are maps such that H_0 and $p_{\lambda'}h$ are $\mathcal{W}$-near, then there exists a homotopy $\tilde{H}:X \times I \to E_\lambda$ such that $\tilde{H}_0$ and $q_{\lambda\lambda'}h$ are $\mathcal{U}$-near and $p_\lambda\tilde{H}$ and $r_{\lambda\lambda'}H$ are $\mathcal{V}$-near.

Definition 4. A map $p:E \to B$ is a shape fibration if there exists a polyhedral resolution $(\underline{q},\underline{r},\underline{p})$ of p such that $\underline{p}$ is a level map of systems with the AHLP.

The original definition of a shape fibration, given in [3] did not assume that $\underline{p}$ was a level map of systems. In that more general case the AHLP assumes a more complicated form. However, the notion of shape fibration remains the same. This was proved by Q. Haxhibeqiri in [1]. Moreover, his arguments together with the ones from [3] prove the following theorem.

Theorem 1. Every map $p:E \to B$ of topological spaces admits a polyhedral resolution $(\underline{q},\underline{r},\underline{p})$, where $\underline{p}$ is a level map of systems.

The next theorem is also a consequence of [1] and [3].

Theorem 2. Let $(\underline{q},\underline{r},\underline{p})$ and $(\underline{q}',\underline{r}',\underline{p}')$ be two polyhedral resolutions of the same map $p:E \to B$ and let $\underline{p}$ and $\underline{p}'$ be level maps of systems. If $\underline{p}$ has the AHLP, then so does $\underline{p}'$.

The next two theorms are proved in [2].

Theorem 3. Let $p:E \to B$ be a shape fibration, let $B_0 \subset B$ be a closed subset, let $E_0 = p^{-1}(B_0)$ and let $p_0 = p|E_0:E_0 \to B_0$. If B is normal, B_0 and E_0 are P-embedded in B and E respectively and p is a closed map, then p_0 is also a shape fibration.

Theorem 4. Let $p:E \to B$ be a shape fibration, let $e \in E$, $b = p(e)$, $F = p^{-1}(b)$. If B is normal, F is P-embedded in E and p is a closed map, then p induces an isomorphism of the homotopy pro-groups $\text{pro-}\pi_n(E,F,e) \to \text{pro-}\pi_n(B,b)$. Moreover, the following sequence of pro-groups is exact

$$\cdots \to \text{pro-}\pi_n(F,e) \to \text{pro-}\pi_n(E,e) \to \text{pro-}\pi_n(B,b) \to \text{pro-}\pi_{n-1}(F,e) \to \cdots.$$

In the proof of these theorems one uses the following result from [5].

Theorem 5. Let X be a space and $X_0 \subset X$ a subspace. If X_0 is P-embedded in X, then there exists an inverse system of polyhedral pairs $(\underline{X},\underline{X}_0)$ and a map of systems $\underline{p}:(X,X_0) \to (\underline{X},\underline{X}_0)$ such that $\underline{p}:X \to \underline{X}$ and $\underline{p}_0 = \underline{p}|X_0:X_0 \to \underline{X}_0$ are resolutions.

References

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