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On products of countable tightness

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Introduction and definitions: The results in this paper are due to a joint paper with Gary Gruenhage (Auburn Univ., U.S.A.), " Products of k-spaces and spaces of countable tightness ", to appear in Trans. Amer. Math. Soc.

If a space Y has countable tightness, not much can be said about the tightness of Y^2. We consider what must be true if Y^2 has countable tightness and Y is a closed image of some " nice " space X under a map f. We prove some fairly general theorems concerning the behavior of the map f, and then we apply these results to more special cases.

All our spaces are assumed to be regular and T_1. We recall some basic definitions.

A space X has the <u>weak topology</u> with respect to a cover $\mathcal{C}$ of (not necessarily closed) subsets, if a subset A of X is closed in X whenever $A \cap C$ is closed in C for each $C \in \mathcal{C}$.

A space X is a <u>k-space</u> (resp. <u>sequential space</u>), if X has the weak topology with respect to its compact (resp. compact metric) subsets. Thus every sequential space is k.

The <u>tightness</u> [1], t(X), of a space X is the least cardinal α such that whenever $A \subset X$ and $x \in \overline{A}$, there is a subset $B \subset A$ with $|B| \leq \alpha$ and $x \in \overline{B}$. If $t(X) \leq \omega$, then

X is said to have countable tightness. It is easy to show that $t(X) \leq \omega$ if and only if X has the weak topology with respect to its countable subsets. Thus every sequential space has countable tightness.

A space X is strongly collectionwise Hausdorff if whenever $\{ x_\alpha ; \alpha \in A \}$ is a closed discrete subset of X, there exists a discrete collection $\{ U_\alpha ; \alpha \in A \}$ of open subsets such that $x_\alpha \in U_\alpha$ for each $\alpha \in A$. Note that every collectionwise normal space is strongly collectionwise Hausdorff.

1. General results. Let c denote the cardinality of the continuum. A space X is called c-compact, if every subset of A with cardinality c has an accumulation point in X.

Theorem 1.1. Suppose that $f: X \to Y$ is closed, with X strongly collectionwise Hausdorff. If $t(Y^2) \leq \omega$, then the boundary, $\partial f^{-1}(y)$, of $f^{-1}(y)$ is c-compact.

We remark that, if Y^2 is a k-space with $t(Y) \leq \omega$, then $t(Y^2) \leq \omega$. Thus, assuming the continuum hypothesis (CH), we have

Corollary 1.2. (CH). Suppose that $f: X \to Y$ is closed, with X paracompact. Then each $\partial f^{-1}(y)$ is Lindelöf if either Y^2 is a k-space with $t(Y) \leq \omega$, or $t(Y^2) \leq \omega$.

We don't know whether the CH assumption in Corollary 1.2 can be omitted or not. However, in case where Y is sequential, we can omit (CH) as will be seen in Corollary 1.4.

Theorem 1.3. Suppose that $f\colon X \to Y$ is closed with X strongly collectionwise Hausdorff and Y is sequential. If $t(Y^2) \leq \omega$, then each $\partial f^{-1}(y)$ is ω_1-compact.

Corollary 1.4. Suppose that $f\colon X \to Y$ is closed with X paracompact, Y sequential. If $t(Y^2) \leq \omega$, then each $\partial f^{-1}(y)$ is Lindelöf.

The following example shows that the assumption " Y^2 is a k-space " is not sufficient to obtain " $\partial f^{-1}(y)$ is Lindelöf " in Corollary 1.2.

Example 1.5. There exists $f\colon X \to Y$ closed with X locally compact and paracompact, such that Y^2 is a k-space, but $\partial f^{-1}(y)$ is not Lindelöf for some $y \in Y$.

Indeed, for each $\alpha < \omega_1$, let $S(\alpha)$ be a copy of ordinal space $[0, \omega_1]$. Let X be the free union of $\{ S(\alpha);\ \alpha < \omega_1\}$. Let Y be the space obtained from X by identifying the point ω_1 in each copy to a single point ∞. Let $f\colon X \to Y$ be the quotient map. Then X is paracompact and locally compact, f is closed, and $f^{-1}(\infty)$ is not Lindelöf. We can prove that Y^2 is a k-space.

2. Applications. A collection $\mathcal{N}$ of (not necessarily open) subsets of a space X is a k-network for X if, whenever $C \subset U$ with C compact and U open, then $C \subset \bigcup \mathcal{F} \subset U$ for some finite subcollection $\mathcal{F}$ of $\mathcal{N}$. An $\aleph_0$-space is a space with a countable k-network, and an $\aleph$-space is a space with a σ-locally finite k-network. The concept of $\aleph_0$-spaces;

$\aleph$-spaces is introduced by E. Michael (5); P. O'Meara (8).

We say that X is a locally $\aleph_0$-space if each point of X has a neighborhood which is an $\aleph_0$-space.

Theorem 2.1. (CH). Let f: $X \rightarrow Y$ be a closed map. Let X be a paracompact, locally $\aleph_0$-space. Then the following are equivalent:

(a) $t(Y^2) \leq \omega$.

(b) each $\partial f^{-1}(y)$ is Lindelöf.

(c) Y is locally $\aleph_0$.

(d) Y is locally separable.

Corollary 2.2. Let f: $X \rightarrow Y$ be a closed map with X locally separable, metric. Then the following are equivalent:

(a) $t(Y^2) \leq \omega$.

(b) each $\partial f^{-1}(y)$ is Lindelöf.

(c) Y is locally separable.

(d) Y is locally Lindelöf.

(e) Y is an $\aleph$-space.

A decreasing sequence (A_n) in a space X is a k-sequence (7), if $K = \bigcap_{n=1}^{\infty} A_n$ is compact and every neighborhood of K contains some A_n. A space Y is a bi-k-space (7) if, whenever a filter base $\mathfrak{F}$ accumulating at $y \in Y$, then there exists a k-sequence (A_n) in Y such that $y \in \overline{F \cap A_n}$ for all $n \in N$ and all $F \in \mathfrak{F}$. It is shown that (7) Y is a bi-k-space if and only if Y is a bi-quotient image of a paracompact M-space X. Then, by (13), spaces of pointwise countable type are bi-k.

Recall that a space X is a k_ω-space (6), if it has the weak topology with respect to a countable covering of compact subsets of X. For a space Y, we shall say that Y is a locally k_ω-space, if each point of Y has a neighborhood whose closure is a k_ω-space.

Theorem 2.3. (CH). Let f: $X \to Y$ be a closed map with X paracompact bi-k. If $t(Y) \leq \omega$, then the following are equivalent. When Y is sequential, the CH assumption can be omitted.

(a) Y^2 is a k-space.

(b) Y is locally k_ω, or each $\partial f^{-1}(y)$ is compact.

(c) Y is locally k_ω, or bi-k.

Corollary 2.4. Let f: $X \to Y$ be a closed map with X or Y sequential. Let X be a paracompact space of pointwise countable type. Then Y^2 is sequential if and only if Y is locally k_ω, or bi-k.

Theorem 2.5. Let f: $X \to Y$ be a closed map with X a paracompact $\aleph$-space. Then Y^2 is a k-space if and only if Y is metrizable, or Y is an $\aleph$-space which is locally k_ω.

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