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# A note on E. Michael's example and rectangular products

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The notion of a rectangular product was introduced by B. A. Pasynkov and is used for product theorem in dimension theory. Here we are concerned especially with a rectangular product with a metric factor, i.e. $X \times M$ where $M$ is a metric space. We give a brief introduction to recent results concerning a rectangular product with a metric factor.

In 1968, Y. Kodama proved the following:

THEOREM(Kodama). If $X \times M$ is normal and countably paracompact, then $\dim(X \times M) \leqslant \dim X + \dim M$.

Later in 1973, B. A. Pasynkov introduced the notion of a rectangular product and announced the following interesting theorem.

THEOREM(Pasynkov). If $X \times M$ is rectangular, then $\dim(X \times M) \leqslant \dim X + \dim M$.

A product space $X \times Y$ is said to be rectangular if every cozero subset of $X \times Y$ has a $\sigma$-locally finite (in $X \times Y$) covering consisting of cozero rectangles, i.e. products $U \times V$ of cozero subsets of $X$ and $Y$.

Pasynkov also observed the following:

THEOREM. If $X \times M$ is normal and countably paracompact, then the space $X \times M$ is rectangular.

Thus Kodama's theorem can also be obtained from Pasynkov's theorem. The following question naturally arises:

"Conversely, does the rectangularity of $X \times M$ imply normal and countable paracompactness of $X \times M$?"

The answer is no! For example:

EXAMPLE (Non normal rectangular product). This is suggested by T. Hoshina. For every space $X$, there exists an extremally disconnected space $E(X)$ called an absolute of $X$, and a perfect irreducible map $E: E(X) \longrightarrow X$. Let $X \times M$ be not normal, for example (E. Michael's line) $\times$ (Irrationals). Note that $E \times 1_M: E(X) \times M \longrightarrow X \times M$ is also a perfect map. Since normality is preserved under closed map, $E(X) \times M$ is not normal. H. Ohta proved that every product of a metric space and an extremally disconnected space is rectangular. Thus the space $E(X) \times M$ is the desired example.

A weaker condition for a product with a metric factor to be rectangular was given by H. Ohta. The condition is a generalization of normal and countable paracompactness. The following diagram will illustrate the situation.

| | | | | |
|---|---|---|---|---|
| A | normal and countably paracompact | $\longrightarrow$ | weak cb | A' |
| | $\downarrow$ | | $\downarrow$ | |
| B | normal | $\longrightarrow$ | weak normal | B' |

A normal space is countably paracompact iff for any decreasing sequence $\{F_n: n \in N\}$ of closed subsets with empty intersection, there exists a sequence $\{U_n: n \in N\}$ of open subsets with empty intersection such that $F_n \subset U_n$ for all $n \in N$.

A space is weak cb iff for any decreasing sequence $\{F_n: n \in N\}$ of regular closed subsets with empty inteesection, there exists a sequence $\{U_n : n \in N\}$ of cozero subsets with empty intersection such that $F_n \subset U_n$ for all $n \in N$.

A space is <u>weak normal</u> iff any regular closed subset $F$ and any zero set $Z$ with $F \cap Z = \phi$ can be completely separated.

His theorem is as follows:

<u>THEOREM</u>(Ohta). The following conditions for a space $X$ are equivalent.

(1) $X \times M$ is weak cb for any metric space $M$.

(2) $X \times M$ is weak normal for any metric space $M$.

(3) $X \times M$ is rectangular for any metric space $M$.

Sketch of the proof. The equivalence of (1) and (2) follows from the following well known theorem.

<u>THEOREM</u>. Let $C$ be a one point compactification of a countable discrete space, i.e. a converging sequence and the limit point. Then $X \times C$ is $B(B')$ if and only if $X$ is $A(A')$.

The proof of the implication (1)$\longrightarrow$(3) is similar to the implication (normal and countably paracompact)$\longrightarrow$(rectangular).

Thus the implication (3)$\longrightarrow$(1) is the most important. His method is a technic for constructing counterexamples.

So, now, let us mention counterexamples. Only a few non-rectangular products with a metric factor are known. They are as follows:

<u>EXAMPLES</u>(Non-rectangular products with a metric factor).

(1) (Wage's example and Przymusinski's example). $X \times M$ where $X$ is a first countable, separable and Lindelöf space and $M$ is a separable metric space. They showed that $\dim X = \dim M$

$= 0$ but $\dim X \times M > 0$.

(2) (Ohta) If $X$ is not weak cb, then there exists a metric space $M$ such that the product $X \times M$ is not rectangular. This result was used to prove $(3) \longrightarrow (1)$ of Ohta's theorem mentioned above.

(3) (Tamano) (Michael's line) $\times$ (Irrationals) is not rectangular. Let $R$ be the set of real numbers, $Q$ the set of rational numbers and $P = R \setminus Q$ the set of irrational numbers. Michael's line is the same as $R$ as a set. The open sets of $X$ is the form $U \cup A$, where $U$ is an open set of $R$ in the usual topology and $A$ is a subset of $P$.

Sketch of the proof. Let $D = \{d_n : n \in N\}$ be a countable dense subset of $P$ in the usual topology. Then $Z = \Delta_D = \{(d,d) : d \in D\}$ is a zero set of $X \times P$. Then $(X \times P) \setminus Z$ is a cozero set and it can be shown that it is not a union of $\sigma$-locally finite cozero rectangles.

If a space $X \times M$ is weak cb, then by Ohta's theorem, the product is rectangular. Therefore this space $X \times P$ is not weak cb. The following question naturally arises:

"Is this space $X \times P$ weak normal?"

But the answer is no! In fact $F = Cl_{X \times P}(D \times P) = (D \cup Q) \times P$ is a regular closed set, $Z' = \Delta_{P \setminus D} = \{(p,p) : p \in P \setminus D\}$ is a zero set and $F$ and $Z'$ cannot be completely separated. This fact is obtained by the standard use of Baire's category theorem.

But this fact is obtained more generally by the following remark.

REMARK(Ito). Let $M$ be a non discrete metric space. If $X \times M$ is normal, then $X$ is weak cb. So, if $X \times P = (X \times P) \times P$ is weak normal, then $X \times P$ is weak cb.

Now, we pose a question. I think the most interesting and important question about this topic is the following one.

QUESTION. If $X \times M$ is weak normal (and furthermore $M$ is not discrete), is $X \times M$ rectangular (weak cb)?

The following theorem also suggests this question.

THEOREM(Starbird). Let $M$ be a non-discrete metric space. If $X \times M$ is normal, then $X \times M$ is countably paracompact.

COROLLARY. If $X \times M$ is normal, then $X \times M$ is rectangular.

## REFERENCES


[1] Y. Kodama, On subset theorems and the dimension of products, Amer. J. of Math. 91 (1969) 486-497.

[2] H. Ohta, Some new characterization of metrizable spaces, preprint.

[3] _______, On the relation of rectangularity and weak normality of products of a metric space and a completely regular space.

[4] B. A. Pasynkov, On the dimension of rectangular products, Soviet Math. Dokl. 16 (1975) 344-347.

[5] T. C. Przymusinski, On the dimension of product spaces and an example of M. Wage, Proc. Amer. Math. Soc. 76 (1979) 315-321.

[6] M. Starbird, Normality in products with a compact or a metric factor, Ph.D. Thesis at Univ. Wisconsin, 1974.

[7] K. Tamano, A note on E. Michael's example and rectangular products, to appear in J. Math. Soc. Japan.

[8] M. L. Wage, The dimension of product spaces, Proc. Nat. Acad. Sci. U.S.A. 75 (1978) 4671-4672.