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THE KOSZUL COMPLEX OF BUCHSBAUM MODULES

Naoyoshi Suzuki
Dept. of General Education
Shizuoka College of Pharmacy.

Let $(A,\mathfrak{M},k)$ be a Noetherian local ring and M be a finite A-module of dimension d. For simplicity we assume that A is complete. An element x in $\mathfrak{M}$ is called a parameter for M if $\dim(M/xM)<\dim(M)$. A system of elements in $\mathfrak{M}$ is refered to as a sub-system of parameters for M (s.s.o.p.), if it can be extended to a s.o.p. for M. $L_A(\ )$ denotes the length of an A-module, $h^i(\ )$ the length (or the dimension as a vector space) of the i-th local cohomology module $H^i_{\mathfrak{M}}(\ )$ and $h_i(\underline{x};M)$ the length of the Koszul homology module $H_i(\underline{x};M)$.

§1. The Koszul Homology and the local cohomology.

We start with the definition and some basic properties of Buchsbaum (abbreviated to Bbm) modules.

DEFINITION. (i) $x\in\mathfrak{M}$ is said to be weakly M-regular if

$$\mathfrak{M}(0:_M x) = 0.$$

(ii) $\underline{x} = \{x_1,\ldots,x_r\}$ is called a weak M-sequence if x_i is weakly $M/(x_1,\ldots,x_{i-1})M$ -regular for $i=1,\ldots,r$.

(iii) M is called a Buchsbaum module if any s.o.p. for M is a weak M-sequence.

THEOREM.(Stückrad, Vogel) The following are equivalent.

(i) M is a Bbm module.

(ii) For any s.o.p. $\underline{x}$ for M, the difference $L_A(M/(\underline{x})M)-e_0(\underline{x};M)$ is an invariant I(M) not depending on $\underline{x}$.

(iii) The natural limit map $H^i(\mathfrak{m};M) \longrightarrow H^i_{\mathfrak{m}}(M)$ is surjective for all $i < d=\dim(M)$.

From the historical point of view, the second charactrization seems to be the most meaningful, because the theory of Bbm-module has its origin in the following words of Buchsbaum: "It would of course be hoped that the difference between the length and multiplicity could be determined by the difference dim(R) - codim(R) and/or other invariants yet to be found." On the other hand, the third one implies that the local cohomology must play a role to be respected as well as the Koszul homology. Indeed in section 2 a new relation between the local cohomologies and the multiplicity will be stated.

Our first result comes from the following observation.

COROLLARY. If M is a Bbm module then $\mathfrak{m}H^i_{\mathfrak{m}}(M) = 0$ for all $i < d$.

The converse is not true. We call such a module satisfying the conclusion of the corollary a quasi-Buchsbaum module. We must make the difference of the modules clearer.

THEOREM(1.1). Let $x \in \mathfrak{m}^2$ be a parameter for M. If M is quasi-Buchsbaum, then so is M/xM.

If besides $L_A((0:x)_M) < \infty$, then the converse is also true.
Proof. We must prove that $\mathfrak{m}H^i_{\mathfrak{m}}(M/xM)=0$ for $i=0,\dots,d-2$. To begin with, note that $\mathfrak{m}^2 H^i_{\mathfrak{m}}(M/xM) = 0$ for all $i<d-1$. $M':=M/xM$. Set $\mathfrak{a}=(0 : H^0_{\mathfrak{m}}(M'))$ and suppose $\mathfrak{a} \subsetneq \mathfrak{m}$. There exists $z \in \mathfrak{m}$ such that z is not contained in $\mathfrak{a}$ and is a parameter for both M and M'. We show that $zH^0_{\mathfrak{m}}(M')=0$ contradicting the choice of z. Let $m' \in H^0_{\mathfrak{m}}(M')$. Then $m \in (xM :_M \mathfrak{m}^2)$. Since x is $M/H^0_{\mathfrak{m}}(M)$-regular, we have the following exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(M/(xM+H^0(M))) \longrightarrow H^1(M/H^0_{\mathfrak{m}}(M)) \xrightarrow{\ x\ } H^1(M/H^0_{\mathfrak{m}}(M))$$

and we have the isomorphisms

$$H^0_{\mathfrak{m}}(M/xM+H^0_{\mathfrak{m}}(M)) \cong H^1_{\mathfrak{m}}(M/H^0_{\mathfrak{m}}(M)) \cong H^1_{\mathfrak{m}}(M).$$

$zm \in xM + H^0_{\mathfrak{m}}(M)$, i.e., $zm = xn + t$ for some $n \in M$ and $t \in H^0_{\mathfrak{m}}(M)$. $xzm = x^2 n$ and $n \in (zM:x^2)$. Since $\{z,x^2\}$ is a s.s.o.p. for M,

we have $(0:x^2)_{M/zM} \subset H^0_{\mathfrak{m}}(M/zM) \subset (0:\mathfrak{m}^2)_{M/zM} \subset (0:x)_{M/zM}$.
Consequently, $xn=zu$ for some $u \in M$ and $t=zm - xn = z(m-u) \in zM$.
It follows that $t \in H^0_{\mathfrak{m}}(M) \cap zM = (0)$, for $H^0_{\mathfrak{m}}(M) = (0:z^i)_M$ for any $i \geq 1$. We get $zm = xn \in xM$ and $zm' = 0$, as was required.

Now let i be ≥ 1. Since $H^0_{\mathfrak{m}}(M) = (0:x^j)_M$ for all $j \geq 1$, we have an exact sequence $0 \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow M/xM \to M/(xM+H^0_{\mathfrak{m}}(M)) \longrightarrow 0$ and isomorphisms $H^i_{\mathfrak{m}}(M/xM) \cong H^i_{\mathfrak{m}}(\overline{M}/x\overline{M})$ for $i \geq 1$ with $\overline{M}=M/H^0_{\mathfrak{m}}(M)$. We may assume that $\operatorname{depth}(M) > 0$ and hence x is M-regular.

Suppose $\mathfrak{a}:=\operatorname{ann}_A(H^i_{\mathfrak{m}}(M)) \subsetneq \mathfrak{m}$ and choose $\alpha \notin \mathfrak{a}$ so that α is a parameter for both M and M'=M/xM. Let $E^\bullet$ and $F^\bullet$ be the minimal injective resolutions of M and M', respectively. $e^\bullet$ and $f^\bullet$ be the differential maps. ${}^0E^\bullet := H^0_{\mathfrak{m}}(E^\bullet)$, ${}^0F^\bullet := H^0_{\mathfrak{m}}(F^\bullet)$,

${}^{o}e^{\bullet} := H^{0}_{\mathfrak{m}}(e^{\bullet})$ and ${}^{o}f^{\bullet} := H^{0}_{\mathfrak{m}}(f^{\bullet})$. Consider the exact sequence

$$0 \longrightarrow {}^{o}E^{\bullet} \xrightarrow{x} {}^{o}E^{\bullet} \longrightarrow {}^{o}F^{\bullet} \longrightarrow 0$$

of complexes induced from the exact sequence $0 \to M \xrightarrow{x} M \to M' \to 0$. Let $z \in {}^{o}E^{i}$ be arbitrary such that $z' = z \bmod x\,{}^{o}E^{i} \in \mathrm{Ker}({}^{o}f^{i})$. ${}^{o}e^{i}(z) = xw$ for some $w \in {}^{o}E^{i+1}$ with ${}^{o}e^{i+1}(w) = 0$. Since $\mathfrak{m}H^{i+1}_{\mathfrak{m}}(M) = 0$, there exists $u \in {}^{o}E^{i}$ such that

$$(\#) \quad \alpha w = {}^{o}e^{i}(u).$$

${}^{o}e^{i}(\alpha z) = \alpha\,{}^{o}e^{i}(z) = \alpha xw = {}^{o}e^{i}(xu)$ —— (##). Consequently $\alpha z - xu \in \ker {}^{o}e^{i}$. Since $\mathfrak{m}H^{i}_{\mathfrak{m}}(M) = 0$, we have $x^2u - \alpha zx \in \mathrm{Im}({}^{o}e^{i-1})$ and $x^2u \in \alpha\,{}^{o}E^{i} + \mathrm{Im}({}^{o}e^{i-1})$. Applying the same argument to the exact sequence $0 \to M \xrightarrow{\alpha} M \to M/\alpha M \to 0$, and the exact sequence of complexes $0 \to {}^{o}E^{\bullet} \xrightarrow{\alpha} {}^{o}E^{\bullet} \to {}^{o}F^{\bullet} \to 0$, we see that $x^2(u \bmod \alpha\,{}^{o}E^{i}) \in \mathrm{Im}({}^{o}f^{i-1})$ and (#) implies that $u \bmod \alpha\,{}^{o}E^{i} \in \mathrm{Ker}({}^{o}f^{i})$, i.e.,$(u \bmod \alpha\,{}^{o}E^{i}) \in \mathrm{Ker}({}^{o}f^{i}) \cap (\mathrm{Im}({}^{o}f^{i-1}) : x^2)$. Since $x \in \mathfrak{m}^2$, we have $H^{i}_{\mathfrak{m}}(M/\alpha M) \subset (0:\mathfrak{m}^2)_{H^{i}_{\mathfrak{m}}(M/\alpha M)} \subset (0:x)_{H^{i}_{\mathfrak{m}}(M/\alpha M)} \subset (0:x^2)_{H^{i}_{\mathfrak{m}}(M/\alpha M)} \subset H^{i}_{\mathfrak{m}}(M/\alpha M)$. Thus we have $x(u \bmod \alpha\,{}^{o}E^{i}) \in \mathrm{Im}({}^{o}f^{i-1})$, and there exists $v \in {}^{o}E^{i}$ such that $xu - \alpha v \in \mathrm{Im}({}^{o}e^{i-1})$ By (##) we have ${}^{o}e^{i}(\alpha v) = {}^{o}e^{i}(xu) = {}^{o}e^{i}(\alpha z)$, and

$$\alpha(v - z) \in \ker({}^{o}e^{i}) \cap \alpha({}^{o}E^{i}) \subset \mathrm{Im}({}^{o}e^{i-1}),$$

for $(0:\alpha^j)_{H^{i}_{\mathfrak{m}}(M)} = H^{i}_{\mathfrak{m}}(M)$ for all $j \geq 1$.
We therefore have $\alpha z - xu = (\alpha z - \alpha v) + (\alpha v - xu) \in \mathrm{Im}({}^{o}e^{i-1})$. Namely $\alpha(z \bmod x\,{}^{o}E^{i}) \in \mathrm{Im}({}^{o}f^{i-1})$ and α kills $H^{i}_{\mathfrak{m}}(M/xM)$, contradicting the choice of α.

The proof of the latter half is just a slight modification of one of Vogel's Non-Zero-Divisor characterization of Buchsbaum modules [2].

As an easy consequence of the theorem, the following is proved.

COROLLARY(1.2)([3]). The following are equivalent.

(i) M is a quasi-Bbm module.

(ii) Any s.o.p. for M contained in $\mathfrak{m}^2$ forms a weak M-seq..

(iii) There exists a weak M-sequence of length d = dim(M) in $\mathfrak{m}^2$.

The next lemma connects the local cohomology to the Koszul homology and played an essential role in the proof of theorem (1.1).

LEMMA(1.3). Let M be a generalized Bbm module (i.e., for i<d $L_A(H^i_{\mathfrak{m}}(M))<\infty$) and x be a parameter for M. Then

$$(0:x)_M \subseteq H^0_{\mathfrak{m}}(M).$$

Considering the long exact sequence of local cohomology we easily see that M/xM is also a generalized Bbm module for any parameter x for M. Since the Koszul complex is obtained by the successive construction of mapping cylinder, taking the lemma into account, we see that the Koszul homology $H_+(\underline{x};M)$ with respect to any s.s.o.p. $\underline{x}$ for a generalized Bbm module M has finite length and it is not hard to see the following.

PROPOSITION(1.4). Let M be a generalized Bbm module and $\underline{x}$ = $\{x_1,\dots,x_d\}$ be a s.o.p. for M. Then

(i) $h^p(M/(x_1,\dots,x_r)M) \leq \sum_{i=0}^{r}\binom{r}{i}h^{i+p}(M)$ for r = 1,...,d-1.

(ii) $h_p(x_1,\dots,x_r;M) \leq \sum_{i=0}^{r-p}\binom{r}{p+i}h^i(M)$ for any p ≧ 1.

(iii) $L_A(M/(\underline{x})M) - e_0(\underline{x};M) = L_A((0:x_d)_{M/(x_1,\dots,x_{d-1})M})$.

If M is Bbm, then equality holds in (i). Moreover each

Koszul homology module is a vector space (and its length is expressed in terms of local cohomology). The fact conversly characterizes the Buchsbaum modules.

THEOREM(1.5)(Suzuki, Schenzel) The following are equivalent.

(i) M is a Bbm module.

(ii) $\mathfrak{m}H_1(\underline{x};M) = 0$ for any s.o.p. (resp. s.s.o.p.) $\underline{x}$ for M.

(iii) $\mathfrak{m}H_+(\underline{x};M)=0$ for any s.o.p. (resp. s.s.o.p.) $\underline{x}$ for M.

Schenzel's proof uses the dualizing complex, while that of the author's is elementary. Note also that $H_+(\underline{x};M)$ is the socle of $K_+(\underline{x};M)/B_+(\underline{x};M)$.

COROLLARY(1.6). Let $\underline{x}$ be a s.s.o.p. for a Bbm module M. Then

$$h_p(x_1,\dots,x_r;M) = \sum_{i=0}^{r-p}\binom{r}{p+i}h^i(M),$$

hence for any s.o.p. $\underline{x} = x_1,\dots,x_d$ for a Bbm module M, we have

$$L_A(M/(\underline{x})M)-e_0(\underline{x};M) = h^0(M/(x_1,\dots,x_{d-1})M)=\sum_{i=0}^{d-1}\binom{d-1}{i}h^i(M),$$

which is the invariant stated in Theorem (1.1).

In spite of the above facts, we must be careful when we consider the Koszul homology of weak sequences.

REMARK. (i) If $\underline{x}$ is contained in $\mathfrak{m}^2$, then $\mathfrak{m}H_1(\underline{x};M) = 0$ implies that $\underline{x}$ is a weak M-sequence.

(ii) If $\underline{x}$ is an unconditioned weak sequence (i.e., after any permutation it is still a weak sequence), it is not necessarily true that $H_1(\underline{x};M)$ is a vector space.

We close this section with the following which is quoted partly from the recent results by M. Steurich.

THEOREM.([5]). Let $x_1,\dots,x_n$ be a sequence of elements generating minimally the ideal $(x_1,\dots,x_n)$. Then the following are equivalent.

(i) $x_1,\dots,x_n$ is an unconditioned weak sequence.

(ii) $Z_1(x_i, i\in I)/(\mathfrak{M}(\underline{x})K_1(x_i, i \in I, i\neq i_o)\cap Z_1(x_i, i\in I)+B_1(x_i, i\in I))$ is a vector space for any $I\subset\{1,2,\dots,n\}$ and $i_o\in I$.

Note that if $\underline{x}$ is besides an unconditioned relatively $\mathfrak{M}$-regular sequence with respect to $\mathfrak{M}(\underline{x})$ in the sence of Fiorentini [10], the module in (ii) coincides with the usual Koszul homology.

§2. Bounds for the multiplicity of Buchsbaum modules.

Our next purpose is to prove the following

THEOREM. Let M be a Buchsbaum module of dimension d (≥ 2) and $\underline{x}$ be a system of parameters for M. Then we have

$$e_o(\underline{x};M) \geq \sum_{i=1}^{d-1}\binom{d-2}{i-1}h^i(M).$$

If A is a Buchsbaum ring of dimension d (≥ 2), then

$$e_o(\underline{x};A) \geq 1 + \sum_{i=1}^{d-1}\binom{d-2}{i-1}h^i(A).$$

Notation. $D^p(\ ) := \mathrm{Hom}_A(H^p_{\mathfrak{M}}(\ ),E_A(k))$. $D^d(M)$ is the so-called canonical module of M.

COROLLARY(2.0) If $e_o(\underline{x};M) < d-2$ for some s.o.p. $\underline{x}$ for M, then $h^i(M) = 0$ for $i = 2,\dots,d-1$ and $D^d(M)$ is a C.-M.module with $e_o(\underline{x};M) = L_A(D^d(M)/(\underline{x})D^d(M))$.

If $e_o(\underline{x};A) < d-1$, then $h^i_{\mathfrak{M}}(M)=0$ for $i=2,\dots,d-2$ and $h^{d-1} \leq 1$.

LEMMA(2.1). Let M be a generalized Bbm module of dimension $d \geq 2$. Then there exists an exact sequence

$$0 \longrightarrow H^0_{\mathfrak{M}}(M) \longrightarrow M \longrightarrow D^dD^d(M) \longrightarrow H^1_{\mathfrak{M}}(M) \longrightarrow 0$$

and isomorphisms $D^pD^d(M) \cong D^0(D^{d-p+1}(D))$ for $p=2,\dots,d-1$, and $D^0D^d(M)=D^1D^d(M)=0$.

LEMMA(2.2). Let M be any finitely generated A-module and $\underline{x}$ be a s.o.p. for M. Then $e_o(\underline{x};D^d(M)) = e_o(\underline{x};M)$.

PROPOSITION(2.3)([6]). Let M be a Bbm module of positive depth and a be any M-regular element. Let U(aM) denote the unmixed component of the primary decomposition of aM in M:

$$U(aM) = \bigcap N(\mathfrak{p}) \quad \text{where } \mathfrak{p} \in \operatorname{ass}(M/aM) \text{ with } \dim(A/\mathfrak{p}) = d-1.$$

Then $U(aM) = (aM:b)_M = (aM:\mathfrak{M})_M$ for any parameter b for M/aM, and it is a Bbm module of dimension d. Furthermore we have an exact sequence

$$0 \longrightarrow M \xrightarrow{j} a^{-1}U(aM) \longrightarrow H^1_{\mathfrak{M}}(M) \longrightarrow 0 \qquad \text{with } j(m)= a^{-1}(am).$$

COROLLARY(2.4). For a Bbm module M of positive depth, we have an isomorphism of A-modules; $a^{-1}U(aM) \cong D^dD^d(M)$.

Consequently $a^{-1}U(aM)$ is a Bbm module over A and does not depend on the element a.

LEMMA(2.5) Let M be a Bbm module of $\dim(M) \geq 2$. Then the number $v_A(D^dD^d(M))$ of minimal generators of $D^dD^d(M)$ is not less than $h^1(M)$.

If A is a Bbm ring of dimension $d \geq 2$, then

$v_A(D^dD^d(A)) = 1 + h^1(A)$.

PROOF. The first inequality follows easily from (2.3). As to the second assertion, since we can choose $a \notin \mathfrak{m}^2$ and $U(aA) \subset \mathfrak{m}$, we can prove that $a^{-1}A \subsetneqq (a^{-1}U(aA))$.

We are now ready to prove our main theorem.

Since $L_A(H^0_{\mathfrak{m}}(M)) < \infty$, we may assume that M is of positive depth. Let $\underline{x} = \{x_1, \ldots, x_d\}$ be any s.o.p. and $M' := M/(x_3, \ldots, x_d)M$. Since $\dim(M') = 2$, by (2.1) $D^2(M')$ is a C.-M.module. $e_0(\underline{x};M) = e_0(x_1,x_2;M') = e_0(x_1,x_2;D^2(M')) = L_A(D^2(M')/(x_1,x_2)D^2(M))$ and the last term is not less than the dimension of the socle, which coincides with $v_A(D^2D^2(M')$ and by (2.5) which is not less than $h^1(M')$ (resp. $= 1 + h^1(A)$ in the ring case.) On the other hand since M is a Bbm module we have $h^1(M') = \sum_{i=1}^{d-1}\binom{d-2}{i-1}h^i(M)$ by (1.4).

EXAMPLES. (i) Let $(R,\mathfrak{m},k)$ be a regular local ring of dim d. Then $M := \mathfrak{m}$ is a d-dimensional Bbm module with depth = 1 and $h^1 = 1$ with $h^i = 0$ for $i \neq 1, d$. For the minimal generators $\underline{x}$ of $\mathfrak{m}$, we have $e_0(\underline{x};M) = e_0(\underline{x};R) = 1$. This is the case where the equality holds in the theorem.

(ii) Let $(R,\mathfrak{m},k)$ be as above. $A := R \ltimes \mathfrak{m} \supset \mathfrak{N} = \mathfrak{m} \ltimes \mathfrak{m}$, $\mathfrak{m} = (x_1, \ldots, x_d)$, and $y_i := (x_i, 0)$ for $i = 1, \ldots, d$. Then $h^1(A) = 1$ and $h^i(A) = 0$ for $i \neq 1, d$. On the other hand $L_A(A/(\underline{y})A) = d+1$, and we have $e_0(\underline{y};A) = d \geq 1 + h^1(A)$. Equality holds if d=2.

(iii) In the ring case S.Goto proved

$$e_0(\underline{x};A) \geq 1 + \sum_{i=1}^{d-1}\binom{d-1}{i-1}h^i(A).$$

Let d=3 above, and L be the 2nd syzygy of k:

$0 \longrightarrow L \longrightarrow R^3 \longrightarrow R \longrightarrow k \longrightarrow 0$, and $A := R\ltimes L$. Then A is a Buchsbaum ring of dimension three with $h^2(A)=1$ and $h^i=0$ for $i\neq 2,3$. $e_0(\underline{y};A) = 3 = 1+2h^2(A)$. This is the example where the equality holds in Goto's inequality. He also asserts that if equality holds, A must be of maximal embedding dimension. Indeed the ring above has maximal embedding dimension.

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