数理解析研究所講究録
第 446 巻 1981 年 34-41

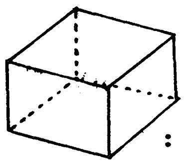
Applications of algebraic geometry to combinatorics

By

M. Hochster (Notes by Y. Matsuura)

In this lecture I would like to explain Stanley's idea in the application of algebraic geometry to the problem of characterizing the f-vectors of simplicial convex polytopes.

Let P be a compact convex polytope in the d-dimensional Euclidean space $\mathbb{R}^d$. P is called simplicial if every face of P except P itself is a simplex.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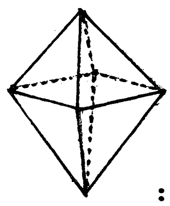

: simplicial .

Let $(f_0, f_1, \ldots, f_{d-1})$ be a sequence of non-negative integers. If there exists a simplicial convex d-polytope P with f_i faces of dimension i for $1 \leqslant i \leqslant d-1$, then we call $(f_0, \ldots, f_{d-1})$ the f-vector of P.

Problem. Characterize the possible f-vectors.

More precisely, we want to find necessary and sufficient conditions on a sequence of integers for the existence of a simplicial convex polytope with those number of faces.

Before describing an answer to this problem, we give some definitions. For an f-vector $(f_0, f_1, \ldots, f_{d-1})$ of a simplicial convex d-polytope P, we define

$$h_i = \sum_{j=0}^{i} \binom{d-j}{d-i} (-1)^{i-j} f_{j-1}$$

where we set $f_{-1} = 1$. The sequence of integers $(h_0, h_1, \ldots, h_d)$ is called the h-vector of P.

If k and i are positive integers, then k can be written uniquely in the form

$$k = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \ldots + \binom{n_j}{j}$$

where $n_i > n_{i-1} > \ldots > n_j \geq j \geq 1$. We define

$$k^{\langle i \rangle} = \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \ldots + \binom{n_j+1}{j+1}$$

with $0^{\langle i \rangle} = 0$.

A sequence $(k_0, k_1, \ldots, k_d)$ of non-negative integers is called an M-vector if $k_0 = 1$ and $0 \leq k_{i+1} \leq k_i^{\langle i \rangle}$ for $1 \leq i \leq d-1$.

The conjecture of McMullen, now a theorem, is stated as follows :

<u>Theorem</u>. A sequence $(h_0, h_1, \ldots, h_d)$ of integers is the h-vector of a simplicial convex d-polytope if and only if the following conditions are satisfied :

1) $h_i = h_{d-i}$ for $0 \leq i \leq d$ (the Dehn - Sommerville equations),

2) $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{[d/2]} - h_{[d/2]-1})$ is an M-vector.

The sufficiency of McMullen's conditions was proved by Billera and Lee [6]. Soon after, Stanley proved the necessity in 1980.

Remark. 1) The notions of f-vector and h-vector make sense for any triangulation of the sphere S^{d-1}.

2) The Dehn - Sommerville equation $h_0 = h_d$ for i = 0 is just the famous Euler - Poincaré formula.

3) It is known that the Dehn - Sommerville equations continue to hold for any triangulation of the sphere S^{d-1}.

Stanley's Proof of the necessity

We recall the following result essentially due to Macaulay:

(Macaulay). A sequence $(k_0, k_1, \ldots, k_d)$ of integers is an M-vector if and only if there exists a finitely generated commutative graded algebra R over a field K, generated by R_1, such that $\dim_K R_i = k_i$.

By this fact, it is sufficient to find a homogeneous algebra R such that $\dim_K R_i = h_i - h_{i-1}$ for $0 \leq i \leq [d/2]$.

We are now in a position to explain Stanley's argument. He realized this algebra R as the cohomology ring of a certain projective variety X obtained from a given simplicial convex polytope.

Let P be a simplicial convex d-polytope. We may 1) assume P is embedded in $\mathbb{R}^d$, 2) assume the origin is in the interior and 3) change the vertices a little so that they have rational coordinates. --- (This does not affect the combinatorial structure of P.) --------- For each face of P, we consider the rational convex polyhedral cone, obtained as the union of all rays from the origin through points of that face. These cones form a complete (the union is $\mathbb{R}^d$) simplicial fan. Then, by the theory of toric varieties, we obtain a projective variety X of dimension d:

$$X =_{\text{dfn.}} \bigcup_C \operatorname{Spec}(R_C)$$

where C runs over all cones of the above fan

and $R_C = \mathbb{C}[x_1^{t_1} \cdot x_2^{t_2} \cdot \ldots \cdot x_d^{t_d} \mid c=(c_1, c_2, \ldots, c_d) \in C,$

$\sum_i c_i t_i \geq 0]$.

X is locally a quotient of $\mathbb{C}^d$ by a finite group action. Hence, in general, X is singular. Fortunately, it is known that there are two nice facts on the structure of the cohomology groups of X:

The first is :

(Danilov [1]). 1) $H^{2i+1}(X, \mathbb{C}) = 0$ for $0 < 2i+1 < 2d$,

2) $\dim_{\mathbb{C}} H^{2i}(X, \mathbb{C}) = h_i$ for $0 \leqslant 2i \leqslant 2d$, where $(h_0, h_1, \ldots, h_d)$ is the h-vector of P.

From Poincare duality for the complex cohomology of X it follows that the Dehn - Sommerville equations hold for the h-vector of P.

By setting $A_i = H^{2i}(X, \mathbb{C})$ we have a commutative graded algebra $A = \bigoplus_i A_i$ over $\mathbb{C}$, generated by A_1, such that $\dim_{\mathbb{C}} A_i = h_i$.

The second is :

(Steenbrink [5]). The hard Lefschetz theorem holds for X.

This theorem means that there is an element $w \in H^2(X, \mathbb{C}) = A_1$ (the cohomology class of an ample divisor on X) such that $w^{d-2i}: H^{2i}(X, \mathbb{C}) \longrightarrow H^{2d-2i}(X, \mathbb{C})$ given by the cup product with w^{d-2i} is an isomorphism for $0 \leqslant i \leqslant [d/2]$. In particular,

$w : A_i \longrightarrow A_{i+1}$ is injective for $0 \leqslant i \leqslant [d/2]$. Now let $R = A/wA$. Then $\dim_{\mathbb{C}} R_i = h_i - h_{i-1}$ for $i \leqslant [d/2]$. This completes Stanley's proof.

The upper bound conjecture

There is a theorem, conjectured by Motzkin, giving an upper bound for the number of i-dimensional faces f_i, fixing f_0, of a finite simplicial complex.

Theorem (The upper bound conjecture). Let Δ be a triangulation of the sphere S^{d-1} with n vertices and $(h_0, h_1, \ldots, h_d)$ be the h-vector of Δ. Then we have

$$h_i \leqslant \binom{n-d+i-1}{i} \qquad \text{for} \quad 0 \leqslant i \leqslant d.$$

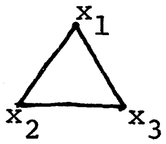
(McMullen proved a somewhat more general statement in 1970.) Stanley gave a different proof for this result by using the "Reisner - Stanley ring" theory in 1975. The sketch of his argument is as follows:

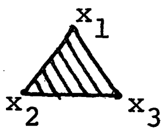
Let Δ be an abstract simplicial complex with vertices $\{x_1, x_2, \ldots, x_n\}$ and K be a field. We associate with Δ a graded K-algebra $K[\Delta]$, called the Reisner - Stanley ring, in the following manner: Let $K[x_1, x_2, \ldots, x_n]$ be the polynomial ring over K on the vertices of Δ. Let I_Δ be the ideal generated by all monomials $x_{i_1} x_{i_2} \ldots x_{i_s}$ with $i_1 < i_2 < \ldots < i_s$ and $\{x_{i_1}, x_{i_2}, \ldots, x_{i_s}\}$ not a face of Δ. We define

$$K[\Delta] = K[x_1, x_2, \ldots, x_n] / I_\Delta$$

with deg $x_i = 1$ for all i.

Examples.

$\Delta =$  $K[\Delta] = K[x_1,x_2,x_3]/(x_1\, x_2\, x_3)$

$\Delta =$  $K[\Delta] = K[x_1,x_2,x_3]$.

Spec $K[\Delta]$ is the union of various coordinate t-planes with various t's. ~~planes~~.

To describe Reisner's theorem, we recall that if $\sigma \in \Delta$, then the link of σ is the subcomplex

$$\operatorname{link}\sigma =_{\text{dfn.}} \{\tau \in \Delta \mid \tau \cup \sigma \in \Delta,\ \tau \cap \sigma = \emptyset\}$$

In particular, $\operatorname{link}\emptyset = \Delta$.

(Reisner). The following are equivalent:

1) $K[\Delta]$ is a Cohen-Macaulay ring,

2) $\tilde{H}^i(\operatorname{link}(\sigma), K) = 0$ for $0 \leqslant i \leqslant \dim(\operatorname{link}(\sigma))-1$ and for all $\sigma \in \Delta$,

where $\tilde{H}^i(*, K)$ is the reduced cohomology group with coefficients in K.

As an immediate consequence of this, we have the following result:

Corollary. If the geometric realization $|\Delta|$ of Δ is a sphere, then $K[\Delta]$ is Cohen-Macaulay.

Let $H_{K[\Delta]}(m)$ $(m \in N)$ denote the Hilbert function of $K[\Delta]$. It is known that

$$H_{K[\Delta]}(m) = \begin{cases} 1 & m = 0 \\ \sum_{i=0}^{d-1} \binom{m-1}{i} f_i & m > 0 \end{cases},$$

where $(f_0, f_1, \ldots, f_{d-1})$ is the f-vector of Δ . Hence, it is easy to see that

$$(1-t)^d \sum_{m=0}^{\infty} H_{K[\Delta]}(m) t^m = h_0 + h_1 t + h_2 t^2 + \ldots + h_d t^d .$$

By another result essentially due to Macaulay, it follows that if $K[\Delta]$ is Cohen-Macaulay then $(h_0, h_1, \ldots, h_d)$ is an M-vector with $h_1 = n-d$. Finally Stanley showed that if $K[\Delta]$ is Cohen-Macaulay then the upper bound conjecture holds for Δ by using the notion of "an order ideal of monomials". In particular, if Δ is a triangulation of the sphere S^{d-1}, then the upper bound conjecture holds for Δ .

Open problems. 1) Find a simple proof of the necessity of McMullen's conditions.

2) Is $(h_0, h_1-h_0, \ldots, h_{[d/2]} - h_{[d/2]-1})$ always an M-vector for any triangulation of the sphere S^{d-1} ? (Note that we used the Hard Lefschetz theorem for the projective toric variety X. A triangulation of S^{d-1} not coming from a simplicial polytope also gives rise to a toric variety X in a similar manner. But X need not be projective in this case ; hence Stanley's proof does not go through in general.)

References

[1] V.I.Danilov, The geometry of toric varieties, Russian Math. Surveys, 33(1978), 97-154.

[2] P. McMullen, The numbers of faces of simplicial polytopes, Israel J. Math. 9(1971), 559-570.

[3] R. Stanley, The upper bound conjecture and Cohen-Macaulay rings, Studies in Applied Math. 54(1975), 135-142.

[4] R. Stanley, The number of faces of a simplicial convex polytope, Advances in Math. 35(1980), 236-238.

[5] J.H.M.Steenbrink, Mixed Hodge structure on the vanishing cohomology, in "Real and complex singularities, Oslo 1976" (P. Holm, Ed.), 525-563, Sijthoff and Noordhoff, Alphen den Rijn, The Netherlands, 1977.

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