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Local rings with multiplicity two

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Let $(A,\underline{m},k)$ be a Noetherian local ring and let $e(A)$ be the multiplicity of A. It is well known that A is regular if and only if A is unmixed and $e(A) = 1$. But, in general, a local ring with multiplicity 2 is not a hypersurface even if it is unmixed.

Example. Let k be a field, $d \geq 2$ an integer and $X_1,..,X_d,Y_1,..,Y_d$ indeterminates over k. We put

$$A = k[[X_1,..,X_d,Y_1,..,Y_d]]/(X_1,..,X_d) \cap (Y_1,..,Y_d).$$

Then, A is unmixed and $e(A) = 2$, but A is not a hypersurface. Note that A does not satisfy (S_2).

In a recent work [1], S. Goto studied Buchsbaum rings with multiplicity 2. Inspired by [1], K. Watanabe raised the following questions.

(1) Is a local ring with multiplicity 2 satisfying (S_2) a hypersurface?

(2) Is a local ring with multiplicity 3 satisfying (S_2) Cohen-Macaulay?

In this note we give an affirmative answer to the question (1) under some additional coditions and we give a counter example to the question (2).

Throughout this note a ring means a commutative Noetherian ring with a unit.

1. Preliminaries.

First we recall basic properties of the multiplicity of local rings. Let $(A,\underline{m},k)$ be a local ring. We put

$$\mathrm{Assh}(A) = \left\{ p \in \mathrm{Ass}(A) \;\middle|\; \dim A/p = \dim A \right\} .$$

The following result can be found in [2].

Proposition 1.

(1) $$e(A) = \sum_{p \in \mathrm{Assh}(A)} \underline{1}(A_p)e(A/p)$$

(2) Let $p \in \mathrm{Spec}(A)$. If $\mathrm{ht}\, p + \dim A/p = \dim A$ and A/p is analytically unramified, then $e(A_p) \leq e(A)$.

The notion of ideal transform plays an important rôle in the sequel. We recall the definition: let R be a ring, I an ideal of R and M a finitely generated R-module; we define

$$D_I(M) = \varinjlim_n \mathrm{Hom}_R(I^n, M)$$

and call it the ideal transform of M with respect to I.

Proposition 2. Let R, I and M be as above. Then,

(1)
$$H_I^i(D_I(M)) = \begin{cases} (0) & \text{for } i \leq 1 \\ H_I^i(M) & \text{for } i \geq 2 \end{cases}$$

(2) we have the following exact sequence

$$0 \longrightarrow H_I^0(M) \longrightarrow M \longrightarrow D_I(M) \longrightarrow H_I^1(M) \longrightarrow 0$$

and

(3) $D_I(M)_p = M_p$ for $p \notin V(I)$.

We need a result of M. Hochster, the "direct summand conjecture" (cf. [3]).

Proposition 3. Let R be a regular ring containing a field and let S be a module-finite extension algebra of R. Then, R is a direct summand of S as an R-module.

2. Local rings with multiplicity 2.

First we give an affirmative answer to the question (1) under the condition that the local ring is complete and contains a field.

Theorem 4. Let $(A,\underline{m},k)$ be a complete local ring containing a field. Assume that A satisfies (S_2) and $e(A) = 2$. Then A is a hypersurface with multiplicity 2.

(Proof). It is sufficient to prove that A is Cohen-macaulay. If $\dim A \leq 2$ there is nothing to prove. We will prove the assertion by induction on dim A. It is easy to see that $e(A_p) \leq e(A)$ for all $p \in \mathrm{Spec}(A)$. By the induction hypothesis we may assume that A_p is Cohen-Macaulay for $p \in \mathrm{Spec}(A) - \{\underline{m}\}$. In particular, we may assume that $\underline{l}(H^i_{\underline{m}}(A)) < \infty$ for $0 \leq i < \dim A$. Assume that $\dim A = 3$. We may assume that k is an infinite field, so that there exists an S.O.P. a_1, a_2, a_3 such that $\underline{m}^n = (a_1, a_2, a_3)\underline{m}^{n-1}$ for some positive integer n. Set $S = k[[a_1, a_2, a_3]]$. Then S is a regular local ring and A is a module finite extension of S. Using Proposition 3, we get an exact sequence

$$0 \longrightarrow S^{n-1} \xrightarrow[M]{} S^{n} \longrightarrow A/S \longrightarrow 0 ,$$

where $M = (a_{ij})$ is an $(n-1) \times n$ matrix with $a_{ij} \in \underline{n}$ and $\underline{n}$ is the maximal ideal of S . We want to show that A is Cohen-Macaulay. Assume the contrary. Since $(A/S)_p$ is free for $p \in \mathrm{Spec}(S) - \{\underline{n}\}$ the ideal generated by the maximal minors of M is an $\underline{n}$-primary ideal of height at most 2. This is a contradiction. Let $\dim A \geq 4$. Choose a non zero divisor x such that $e(A/xA) = 2$. We have an exact sequence

$$0 \longrightarrow A/xA \longrightarrow D_{\underline{m}}(A/xA) \longrightarrow H^1_{\underline{m}}(A/xA) \longrightarrow 0.$$

The ideal transform $D_{\underline{m}}(A/xA)$ is a finite product of complete local rings with multiplicity two and satisfies (S_2) by Proposition 2. Hence $D_{\underline{m}}(A/xA)$ is Cohen-Macaulay by the induction hypothesis. It is easy to see that

$H^i_{\underline{m}}(A/xA) = (0)$ for $2 \leq i < \dim A/xA$. From the exact sequence

$$0 \longrightarrow A \xrightarrow{x} A \longrightarrow A/xA \longrightarrow 0$$

we get the exact sequence

$$0 \longrightarrow H^1_{\underline{m}}(A/xA) \longrightarrow H^2_{\underline{m}}(A) \xrightarrow{x} H^2_{\underline{m}}(A) \longrightarrow 0.$$

Since $\underline{l}(H^2_{\underline{m}}(A)) < \infty$, we have $H^2_{\underline{m}}(A) = (0)$ by Nakayama's lemma. Thus, A is Cohen-Macaulay as required.

For local rings not containing a field we have the following result.

Theorem 5. Let $(A,\underline{m},k)$ be a complete local ring which is not a domain. Assume that $e(A) = 2$ and A satisfies (S_2). Then, A is a hypersurface.

The following result is the main theorem of [1].

Corollary 6. Let $(A,\underline{m},k)$ be a Buchsbaum ring with $\dim A \geq 2$ and $e(A) = 2$. Then, $H^i_{\underline{m}}(A) = (0)$ for $2 \leq i < \dim A$ and $\underline{l}(H^1_{\underline{m}}(A)) \leq 1$.

If A contains a field we can give a simple proof of this result by Theorem 4. Another consequence of Theorem 4 is:

Corollary 7. Let R be a regular local ring containing a field and let I be an ideal of R such that $e(R/I) = 2$ and $pd_{R/I} I/I^2 < \infty$. Then I is generated by an R-sequence.

Example. Let k be a field and let $X_1, X_2, X_3, Y_1, Y_2, Y_3$ be indeterminates over k. We put

$$A = k[[X_1,X_2,X_3,Y_1,Y_2,Y_3]]/(X_1Y_1 + X_2Y_2 + X_3Y_3 , (Y_1,Y_2,Y_3)^2).$$

Then A satisfies (S_2) and $e(A) = 2$. But A is not Cohen-Macaulay

During the symposium C. Huneke and S. Goto communicated to me the following generalization of Theorem 4.

Theorem. Let A be a complete local ring containing a field. Assume that

(1) A satisfies (S_n), $n \leq \dim A$

(2) $e(A) \leq n$.

Then A is Cohen-Macaulay.

Reference

[1] S. Goto, Buchsbaum rings with multiplicity 2, preprint.

[2] M. Nagata, Local rings, Interscience, New York, 1962.

[3] M. Hochster, Contracted ideals from integral extensions of regular rings, Nagoya Math. J. 51 (1973), 25-43.