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On rational double points on quartic surfaces

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Introduction. The quartic surfaces in $\mathbb{C}P^3$: (x,y,z,w) with only rational double singular points are K3 surfaces of degree 4, when desingularized minimally (Brieskorn [2,3]). (The converse is not true, which is the sourse of difficulty of the problem below.) Since the intersection form on the middle homology group has index (+3,-19) for K3 surfaces, the sum of the ranks (the Milnor numbers) of the singular points cannot be greater than 19. If one restricts attension to a single rational double point on a quartic, one might naturally ask, how high could the rank be, i.e. whether there would really exist A_{19} or D_{19} on quartics. In this paper we shall answer this question, giving normal forms of quartic surfaces with A_k(or D_k) singularity at a fixed point of $\mathbb{C}P^3$ for any k. Our method is roughly described as follows: We first write the equation of the general quartic surface S with a double point at, say p=(0,0,0,1). The coefficients of the equation should be regarded as the parameters on which the surface depends. Now, for a fixed k, the condition for the singularity (S,p) to be A_ℓ(or D_ℓ) with $\ell \geq k$, defines a subvariety V_k in the parameter space; furthermore we see that V_k is defined in V_{k-1} as the zero locus of a polynomial (in the coordinates of the parameter space); thus we obtain a series of polynomials Γ_k which give us the varieties V_k. Fortunately, there is an efficient computation scheme for Γ_k's. Before computing them we should, of course, reduce some of the parameters by suitable projective transformations or by using some elementary local analytic transformations at p and so on. Even after all of this, the computor (REDUCE 2 on DEC 2020) gives us combersome polynomials Γ_{19}, Γ_{18}, etc. But, for lower k's, the polynomials Γ_k are not so complicated, and we can in fact go up little by little by introducing, at each step k, suitable new parameters for V_k and killing at each step one more parameter from among those chosen for V_{k-1}. In this way, as a side effect we have proved the rationality of irreducible components of the moduli space $\mathcal{M}_k(A)$ (or $\mathcal{M}_k(D)$) of projective equivalence classes of quartic surfaces having (at least one) A_ℓ (or D_ℓ) with $\ell \geq k$, observing the natural mappings $V_k \to \mathcal{M}_k(A)$ (or $\mathcal{M}_k(D)$):

Main Theorem. The irreducible components of $\mathcal{M}_k(A)$ or $\mathcal{M}_k(D)$ are rational for any $k \leq 19$. $\mathcal{M}_k(A)$ is irreducible except for $k=11,15,17$ and for $k=17$ resp. $15,11$ it has two resp. three irreducible compornents. $\mathcal{M}_k(D)$ is irreducible except for $k=12,16,19$. For $k=12,16$ $\mathcal{M}_k(D)$ has two irreducible components and $\mathcal{M}_{19}(D)$ is empty.

We know by the theory of period mapping (Kulikov [7], Saint-Donat [14], Shah [13]) that $\mathcal{M}_k(A)$ (or $\mathcal{M}_k(D)$) is purely $19-k$ dimensional provided it is non-empty. Thus, in particular, the quartic surface with A_{19} is unique and its explicit equation is given in Section 5. There is also given in Section 10, without any detailed discussion, the quartic surface with D_{18} and A_1 which is also unique. These add two examples to the K3 surfaces with the maximal Picard number, which Hirzebruch is particularly interested in. We should say here that the present method by itself is not adequate to enumerate all possible combinations of rational double points on quartics; at this point, the intrinsic theory of period mapping has a great advantage, although it seems to be a non-trivial problem to give explicit equations of the family of surfaces corresponding to a given combination, or to decide the question of rationality of the parameter space. Umezu and Urabe proved independently the existence of A_{19} and D_{18} and the non-existence of D_{19} on quartics, combining the theory of period and the theory of lattices in the second homology group of the K3 surface as in Nikulin [12]. They decided to publish their result as an appendix to this paper, for which we are grateful.

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Most of the results have already been announced in [11].

1. <u>General remarks.</u> Every quartic surface in $\mathbb{C}P^3$: (x,y,z,w) with a double point at $p = (0,0,0,1)$ is defined by an equation of the form:

$$S:\ f(x,y,z)w^2+g(x,y,z)w+h(x,y,z) = 0$$

where f,g,h are homogeneous polynomials of degree 2,3,4. If f, regarded as a quadratic form in x,y,z, is non-degenerate, then (S,p) is an A_1-singularity; so we exclude this case, assuming that the rank of f, denoted by $rk(f)$, is not greater than 2. We first want to show that, if $rk(f) = 2$, then (S,p) is either A_k for some $k \geqq 2$ or p is not isolated in the singular locus of S. But, before proceeding, we note a useful principle how to recognize the type of singularity: A power series $f(z) = \sum_\alpha a_\alpha z^\alpha$. $\alpha=(\alpha_1,\dots,\alpha_n)$, $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$ is said to define a *semi-quasi-homogeneous* isolated singularity at the origin $z=0$ with respect to the weight system $\omega = (\omega_1,\dots,\omega_n)$ (ω_i: positive rational numbers) if $a_\alpha \neq 0$ implies $|\alpha|_\omega := \sum_i \omega_i\alpha_i \geq 1$ and if $z=0$ is an isolated singular point of the hypersurface $f_0(z) := \sum_{|\alpha|_\omega=1} a_\alpha z^\alpha = 0$. Now, *if $(f_0^{-1}(0),0)$ is a rational double point, then $(f^{-1}(0),0)$ is also a rational double point of the same type*. This follows from Arnold [1] and it is used in Bruce-Wall [5]. Following [5], we call it the recognition principle (for rational double points). Now we turn to the case that $rk(f) = 2$. In this case we can assume $f = xy$ by making a suitable linear change of x,y,z and we operate in the affine coordinates (x,y,z) by setting $w=1$:

The equation of S is:

$$F(x,y,z) := xy+g(x,y,z)+h(x,y,z) = 0.$$

Then, by the implicit function theorem, the power series $\xi = \xi(z)$, $\eta = \eta(z)$ are uniquely determined by the condition:

$$(1.1)\quad \begin{cases} \partial F/\partial x(\xi,\eta,z) = \partial F/\partial y(\xi,\eta,z) = 0 \\ \xi(0) = \eta(0) = 0 \end{cases}$$

We set further:

$$\tilde{F}(x,y,z) := F(x+\xi(z),\ y+\eta(z),z).$$

Since we also have $\xi'(0) = \eta'(0) = 0$, $\tilde{F} = 0$ defines an equivalent singularity at the origin; $\tilde{F}$ has now an expansion of the following form:

$$\tilde{F} = xy+F(\xi(z),\eta(z),z)+\dots$$

where dots indicate terms of weights higher than 1 with respect to $\omega = (1/2,1/2,1/k)$ for any $k \geq 3$. Thus, if we set

$$\sum_\nu \Gamma_\nu z^\nu = F(\xi(z),\eta(z),z),$$

then the above principle proves:

__Lemma 1.1__. (S,p) *is a rational double point of type* A_k *if and only if* $\Gamma_3 = \dots = \Gamma_k = 0$, $\Gamma_{k+1} \neq 0$. *If all* Γ_k *vanish, then* p *is not isolated in the singular locus of* S.

Now we shall discuss the case $rk(f) = 1$. In this case we can assume $f = x^2$ by a suitable linear change of x,y,z. We first observe the binary cubic form:

$$p(y,z) = g(0,y,z)$$

If this form is separable, then (S,p) is of type D_4. In fact, in the affine coordinates (x,y,z) the defining polynomial $F = f+g+h$ has an expansion of the form:

$$x^2+p(y,z)+\dots$$

where dots indicate terms of weights higher than 1 with respect to $w = (1/2,1/3,1/3)$, and this is certainly D_4 since we can assume $p = y^3+z^3$ by a suitable linear change of y,z. Thus, to exclude this case, we assume that p has a multiple factor i.e. that p takes one of the following forms:

(1.2.1) $f = y^2z$

(1.2.2) $f = y^3$

(1.2.3) $f = 0$

In this section, however, we discuss only the case (1.2.1), the case which leads to rational double points of D type. Recall that the defining equation of (S,p) is now of the form

$$F(x,y,z) := x^2+y^2z+xq(x,y,z)+h(x,y,z) = 0$$

where $q(x,y,z)$ is a quadratic form. Also in this case we want to solve the equation $\partial F/\partial x(\xi,\eta,z) = \partial F/\partial y(\xi,\eta,z) = 0$. But, for this, it is convenient to blow up the ambient space with center $p : x = y = z = 0$ and consider the strict transform of S; namely, we introduce a new polynomial $\overline{F}$ by setting

$$\overline{F}(x,y,x) := F(xz,yz,z)/z^2$$

$$= x^2+y^2z+xz\overline{q}(x,y)+z^2\overline{h}(x,y)$$

where we have set $\overline{q}(x,y) = q(x,y,1)$, $\overline{h}(x,y) = h(x,y.1)$. We now have

$$\overline{F}_x = 2x+z\overline{q}_x+xz\overline{q}_x+z^2\overline{h}_x$$

$$z^{-1}\overline{F}_y = 2y+x\overline{q}_y+z\overline{h}_y$$

where we have used the abreviations $\overline{F}_x$, $\overline{q}_x$ etc. for $\partial\overline{F}/\partial x$, $\partial\overline{q}/\partial x$ etc. Thus, in particular, we can write

$$\overline{F}_x/2 = x-zG_1(x,y,z)$$

$$\overline{F}_y/2z -\overline{q}_y\overline{F}_x/4 = y-zG_2(x,y,z)$$

where G_1, G_2 are polynomials in x,y,z. This also implies that there also exist unique power series $\overline{\xi} = \overline{\xi}(z)$, $\overline{\eta} = \overline{\eta}(z)$ such that

$$(1.2)\qquad \begin{cases} \overline{\xi} = zG_1(\overline{\xi},\overline{\eta},z) \\ \overline{\eta} = zG_2(\overline{\xi},\overline{\eta},z) \end{cases}$$

Just as in the previous case we set now

$$\tilde{F}(x,y,z) = F(x+z\overline{\xi}(z),y+z\overline{\eta}(z),z)$$

which has the expansion:

$$x^2+y^2z+F(z\overline{\xi}(z),z\overline{\eta}(z),z)+\ldots$$

$$= x^2+y^2z+z^2\overline{F}(\overline{\xi},\overline{\eta},z)+\ldots$$

where dots indicate terms with weights higher than 1 with respect to any $w = (1/2,(k-2)/2(k-1),1/(k-1))$ $(k \geqq 5)$. We also set:

$$z^2\overline{F}(\overline{\xi},\overline{\eta},z) = \sum_\nu \overline{\Gamma}_\nu z^\nu$$

As before from the recognition principle we get the following:

<u>Lemma 1.2</u>. <u>With the notation and the assumption just as above</u>, (S,p) <u>is a rational double point of type</u> D_k <u>if and only if</u> $\overline{\Gamma}_4 = \ldots = \overline{\Gamma}_{k-2} = 0$, $\overline{\Gamma}_{k-1} \neq 0$. <u>If all</u> $\overline{\Gamma}_\nu$ <u>vanish, then</u> p <u>is not isolated in the singular locus of</u> S.

In [10] it is proved that both in cases of (1.2.2) and (1.2.3) the singularity (S,p) is either of type E_k for some $6 \leqq k \leqq 8$ or a singular point with $p_g \geqq 1$ (The rational double points are characterized as the singularities with $p_g = 0$ [6]) and it is also trivial to determine the type of (S,p) in these cases provided it is rational double (see [10]). Thus, we can say that we have obtained a computational method for determining the type of a given rational double point on a quartic surface. The problem now is whether this can be done in practice by some good computor or not. But, before using a computor, we should of course minimize the computation by introducing an effective way of approximating power series $\xi(z)$, $\eta(z)$, $F(\xi,\eta,z)$, $z^2\bar{F}(\xi,\eta,z)$ or by eliminating as many parameters as possible which the defining equation depend on. Leaving the reduction of parameters to the next section, we devote the rest of this section to the appoximation theory of $\xi(z)$, $\eta(z)$ etc. Let us begin with the case $f = xy$ and let ξ, η be defined by (1.1). We first introduce a series of operations $\sum_n$ on power series in z. For and $f = f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ we set:

$$\sum_n(f) = \sum_{\nu=0}^{n} a_\nu z^\nu .$$

We set further:

$$\xi_n(z) = \sum_n(\xi(z))$$

$$\eta_n(z) = \sum_n(\eta(z))$$

$$\Phi_n(z) = \sum_n(F(\xi(z),\eta(z),z))$$

If we write the derivatives of F in the form:

$$F_x = y - G''(x,y,z)$$

$$F_y = x - G'(x,y,z),$$

then the polynomials G', G'' both consist of terms of degree higher or equal to 2. This implies $G'(\xi,\eta,z) \equiv G'(\xi_{n-1}, \eta_{n-1},z)$, $G''(\xi,\eta,z) \equiv G''(\xi_{n-1},\eta_{n-1},z) \bmod z^{n+1}$ where by convention $A \equiv B \bmod z^m$ means that $A-B$ is divisible by z^m in the power series ring. But this congruence implies further

$$(1.3) \qquad \begin{cases} \xi_n = \sum_n(G'(\xi_{n-1},\eta_{n-1},z)) \\ \eta_n = \sum_n(G''(\xi_{n-1},\eta_{n-1},z)) \end{cases}$$

Since $\xi_0 = \eta_0 = 0$, we can thus compute inductively ξ_n, η_n by (2.3). Now we obtain a very good approximation formula:

$$(1.4) \qquad \Phi_{2n+1} = \sum_{2n+1}(F(\xi_n,\eta_n,z))$$

This follows from (1.1) and the Taylor expansion of $F(\xi_n, \eta_n,z)$ at (ξ,η,z):

$$F(\xi_n,\eta_n,z) = F(\xi,\eta,z)$$

$$+F_x(\xi,\eta,z)(\xi_n-\xi)+F_y(\xi,\eta,z)(\eta_n-\eta)$$

$$+\tfrac{1}{2}F_{xx}(\xi,\eta,z)(\xi_n-\xi)^2+F_{xy}(\xi,\eta,z)(\xi_n-\xi)(\eta_n-\eta)$$

$$+\tfrac{1}{2}F_{yy}(\xi,\eta,z)(\eta_n-\eta)^2+\ldots$$

As remarked in the introduction we need only calculate Φ_{20} to determine the largest number k for which there is a quartic surface with A_k singular point. By the same reasoning we obtain also:

$$(1.5)\qquad \begin{cases} \overline{\xi}_n = \sum_n(zG_1(\overline{\xi}_{n-1},\overline{\eta}_{n-1},z)) \\ \overline{\eta}_n = \sum_n(zG_2(\overline{\xi}_{n-1},\overline{\eta}_{n-1},z)) \\ \overline{\Phi}_{2n+3} = \sum_{2n+3}(z^2\overline{F}(\overline{\xi}_n,\overline{\eta}_n,z)) \end{cases}$$

where we have put as before $\overline{\xi}_n = \sum_n(\overline{\xi})$, $\overline{\eta}_n = \sum_n(\overline{\eta})$, $\overline{\Phi}_n = \sum_n(z^2\overline{F}(\overline{\xi},\overline{\eta},z))$.

2. Reduction by projective transformations. So far we did not use any projective transformation in the space $\mathbb{C}P^3$: (x,y,z). But, using them, one can obviously anihilate or fix some of the coefficients of the polynomials g and h in the defining equation of the surface S. This is

important since an extra coefficient could increase the size of Γ_ν's, which are polynomials in the coefficients of g and h, more than ten times bigger. It is also helpful to compute by hand the first several terms of the series $\Gamma_3, \Gamma_4, \ldots$ or $\overline{\Gamma}_4, \overline{\Gamma}_5, \ldots$, which we however leave to the next section. Let us begin with the case of A type i.e. the case $f = xy$. If $g(0,0,z) \neq 0$, then (S,p) is of type A_2 since $F(x,y,z) = xy+z^3+\ldots$ where dots are term of weight higher than 1 with respect to $\omega = (1/2,1/2,1/3)$. We will, therefore, exclude this case and assume $g(0,0,z) = 0$, and we write g in the form $g_1(x,y)z^2+g_2(x,y)z+g_3(x,y)$. Now we can assume, by a scale change of x and y, that g_1 is written in one of the following forms: $g_1 = x+y$; $g_1 = x$; $g_1 = 0$; First we discuss the case $g_1 = x+y$. Since we want to use projective transformations, we come back to the homogeneous coordinates (x,y,z,w). Replacing z by a suitable $z+ax+by$, we can bring g_2 into the form cxy. We then replace w by a suitable $w+ax+by+cz$ so that g is of the form $(x+y)z^2+ax^3+by^3$. Next we discuss the case $g_1 = x$. But this case separates into two cases $g_2(0,y) \neq 0$ and $g_2(0,y) = 0$. In the first case we can assume by a scale change of coordinates that $g_2 = y^2+axy+bx^2$; we can moreover assume $b = 0$ by replacing z by $z+bx/2$ if necessary. But now we can achieve $g_3(0,y)=0$ by replacing z by a suitable $z+cy$. We finally bring g into the form $g = xz^2+y^2z+ax^3$ by replacing w by a suitable $w+bx+cy+dz$. In the latter case: $g_2(0,y) = 0$, we can put g into the form $g = xz^2+ax^3+by^3$ in a similar reasoning. In the case $g_1 = 0$; that is, the case g_1 vanishes identically we do not make any reduction here. To sum up, we obtain the following coarse normal forms for the singularity of A type:

(2.1.1) $xyw^2+\{(x+y)z^2+ax^3+by^3\}w+h(x,y,z) = 0$

(2.1.2) $xyw^2+\{xz^2+y^2z+ax^3\}w+h(x,y,z) = 0$

(2.1.3) $xyw^2+\{xz^2+ax^3+by^3\}w+h(x,y,z) = 0$

(2.1.4) $xyw^2+\{(a_1x^2+b_1y^2)z+(a_2x^3+b_2y^3)\}w+h(x,y,z) = 0$

More precisely we have:

Lemma 2.1. Any quartic surface with an A_k singularity at $p = (0,0,0,1)$ for $k \geq 3$ can be transformed into one of the forms (2.1.1)-(2.1.4) above by a suitable projective transformation (fixing p).

We only state the corresponding lemma for the D type without any detailed discussion:

Lemma 2.2. Any quartic surface with D_k singularity at $p = (0,0,0,1)$ for $k \geq 5$ can be transformed by a suitable projective transformation (fixing p) into one of the following forms:

(2.2.1) $x^2w^2+\{y^2z+2xz^2\}w+h(x,y,z) = 0$

(2.2.2) $x^2w^2+y^2zw+h(x,y,z) = 0$

3. Further reductions. In this section we will concern ourselves with the reduction of parameters which uses some simple analytic change of local coordinates near $p = (0,0,0)$ in the affine space (x,y,z). Let h be as in the previous section. Regarding h as a polynomial in z we set:

$$h(x,y,z) = \sum h_i(x,y)z^{4-i}$$

Then h_i are binary forms of degree i; in particular h_0 is constant. Let us now assume that (in the affine space) the surface S is defined by

$$F(x,y,z) = xy+\{(x+y)z^2+ax^3+by^3\}+h(x,y,z) = 0$$

By the coordinate change of the form $x = X-z^2$, $y = Y-z^2$, the surface S is transformed into the new surface S_1

$$xy+(h_0-1)z^4+\dots = 0$$

where dots indicate terms with weights higher than 1 with respect to $(1/2,1/2,1/4)$. Thus we see that, if $h_0 \neq 1$, then (S,p) is of type A_3; therefore we assume $h_0 = 1$. Then transformed surface S_1 is now of the form:

$$xy+h_1(1,1)z^5+\dots = 0$$

where dots indicate terms with weights higher than 1 with respect to (1/2,1/2,1/5). Thus we see that, if $h_1(1,1) \neq 0$, then (S,p) is of type A_4; so we further assume that $h_1(1,1) = 0$. Then $h_1(x,y) = s(x-y)$ for some constant s. Now we transform S by $x = X-z^2+sz^3$, $y = Y-z^2-sz^3$ (or equivalently S_1 by $x = X+sz^3$, $y = Y-sz^3$) to the new surface S_2:

$$xy+\{h_2(1,1)+s^2-a-b\}z^6+\ldots = 0$$

where dots denote terms with weights higher than 1 with respect to (1/2,1/2,1/6). Thus (S,p) is of type A_5 if $h_2(1,1) \neq a+b-s^2$; so we assume moreover that $h_2(1,1) = a+b-s^2$ which means now that $h_2(x,y)-ax^2-by^2+sxy$ is divisible by x-y; in other words, we can set:

$$h_2(x,y) = ax^2+by^2+(x-y)(ux+vy)-s^2xy$$

where u, v are constants; this implies that $F(x,y,z) - \{x+z^2+ay^2+x(sz+ux+vy)\}\{y+z^2+ax^2-y(sz+ux+vy)\}$ is of order ≤ 1 with respect to z. Thus, in the homogeneous coordinates, the equation of S now has the following form:

$$\{xw+z^2+ay^2+x\overline{\theta}(x,y,z)\}\{yw+z^2+ax^2-y\overline{\theta}(x,y,z)\}+$$

$$+\phi(x,y)z+\psi(x,y) = 0$$

where we have set $\overline{\theta}(x,y,z) = sz+ux+vy$. One sees immediately that, if S decomposes into two quadrics passing through p, then $\phi = \psi = 0$; and the converse is obviously true. In exactly this sense, it is likely that the coefficients of $\overline{\theta},\phi,\psi$ might be good parameters for the surface S. As we will see later, there is a sharp distinction between the case $s \neq 0$ and the case $s = 0$. If $s \neq 0$, then we can assume that $s = 1$ by using the $\mathbb{C}^*$-action $(x,y,z) \to (t^2x, t^2y, tz)$, $t \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$, and, if we assume this, then the parameters of the surface are almost rigid; in fact, there is only an involutive automorphism of the parameter space extending the transposition $x \leftrightarrow y$ in $\mathbb{C}P^3 : (x,y,z)$. But, if $s = 0$, this $\mathbb{C}^*$-action induces naturally a $\mathbb{C}^*$-action on the parameter space. At any rate we obtain two types of families of quartic surfaces corresponding to (2.1.1):

(I) $$F_1(x,y,z,w) := \{xw+z^2+xz+ay^2+x\theta(x,y)\}$$
$$\times\{yw+z^2-yz+bx^2-y\theta(x,y)\}+\phi(x,y)z+\psi(x,y) = 0$$

(II) $$F_2(x,y,z,w) := \{xw+z^2+by^2+x\theta(x,y)\}$$
$$\times\{yw+z^2+ax^2-y\theta(x,y)\}+\phi(x,y)z+\psi(x,y) = 0$$

Lemma 3.1. *If S is given by (2.2.1) and (S,p) is of type A_k $(k \geq 6)$, then S can be written in the form (I) or (II) above.*

We suppose now that the surface S is written in the form (2.1.2), and that h is also set to be $\sum h_i(x,y)z^i$.

By the transformation of the form $y = Y-z^2$, we see that, if $h_0 \neq 0$, then (S,p) is an A_3; and that, if we assume $h_0 = 0$, then, in the new coordinates, S is given by

$$xy+\{1-h_1(0,1)\}z^5+\ldots = 0$$

where dots indicate terms with weights higher than 1 with respect to $(1/2,1/2,1/5)$. This means that (S,p) is an A_4 if $h_0 = 0$, $h_1(0,1) \neq 1$; so now we assume $h_0 = 0$, $h_1(0,1) = 1$ and we set $h_1 = sx+y$. With the new x,y,z we make the transformation $x = X+z^3$, $y = Y-sz^3$, which turns the equation of S into:

$$xy+\{h_2(0,1)+s\}z^6+\ldots = 0$$

where dots stand for terms of weights higher than 1 with respect to $(1/2,1/2,1/6)$. Thus (S,p) is of type A_5 if $h_2(0,1)+s \neq 0$. Therefore we assume further $h_2(0,1)+s = 0$. But this means that we can set

$$h_2 = x\theta(x,y)-sy(sx+y)$$

where θ is a linear form of x,y. Now the difference $F(x,y,z)-\{x+(sx+y)z+x\theta\}\{y+z^2-syz+ax^2-y\theta\}$ is of first order in z; that is the surface is given by

$$\text{(III)} \qquad F_3(x,y,z,w) := \{xw+(sx+y)z+x\theta(x,y)\}$$
$$\times\{yw+z^2-syz+ax^2-y\theta\}+\phi(x,y)z+\psi(x,y) = 0$$

The same remark as before applies to this case: The surface decomposes into two quadrics if and only if $\phi = \psi = 0$. If $s \neq 0$, we can put $s = 1$; if $s = 0$, the family admits a $\mathbb{C}^*$-action extending $(x,y,z) \to (t^3x, t^2y, tz)$, $t \in \mathbb{C} \setminus \{0\}$. But we do not make any distinction between the two cases, so as not to introduce too many types of surfaces.

We suppose now that S is defined by (2.1.3). Just as in the discussion above, if $h_0 \neq 0$, then (S,p) is of type A_3, and if $h_0 = 0$, $h_1(0,1) \neq 0$, then (S,p) is of type A_4; so we assume $h_0 = h_1(0,1) = 0$ and we set $h_1 = sx$. By making the transformation $y = Y - z^2 - sz^3$, we see then that, in the new coordinates, S is given by

$$xy + \{h_2(0,1) - b\}z^6 + \ldots = 0$$

where dots are terms of weights higher than 1 with respect to $(1/2, 1/2, 1/6)$. If we assume, further, that $h_2(0,1) = b$, then we can set

$$h_2(x,y) = by^2 - s^2xy + x\theta(x,y)$$

where θ is a linear form. This implies as before that S is given by

$$\text{(IV)} \qquad F_4(x,y,z,w) := \{xw + sxz + byz + x\theta(x,y)\}$$

$$\times\{yw + z^2 - syz + ax^2 - y\theta(x,y)\} + \phi(x,y)z + \psi(x,y) = 0$$

For this type we also do not make a separate discussion of cases $s \neq 0$, $s = 0$.

Finally we suppose that S is defined by (2.1.4). Then (S,p) is an A_3 if $h_0 \neq 0$; so we assume $h_0 = 0$ and transform S by the coordinate change $x = X - h_1'' z^3$, $y = Y - h_1' z^3$ into the following form:

$$xy + h_1' h_1'' z^6 + \dots = 0$$

where we have put $h_1 = h_1' x + h_1'' y$ and dots indicate inessential terms with respect to $(1/2, 1/2, 1/6)$. Thus (S,p) is of type A_5 if $h_1' h_1'' \neq 0$; so we assume $h_1' h_1'' = 0$.

Then we may assume $h_1'' = 0$ by transposition of x,y if necessary; we set namely $h_1 = sx$. Now, if $s = 0$, then the line $x = y = 0$ is in the singular locus of S. Thus we can assume that $s \neq 0$ or even $s = 1$ by a scale change. Then, in the new coordinates, S is defined by:

$$xy + b_1 z^7 + \dots = 0$$

where dots are negligible with respect to $(1/2, 1/2, 1/7)$. This shows that (S,p) is an A_6 if $b_1 \neq 0$, and we assume $b_1 = 0$. The original equation of S is now:

$$\text{(V)} \qquad F_5(x,y,z,w) = xyw^2 + \{a_1 x^2 z + a_2 x^3 + b_2 y^3\} w + xz^3$$

$$+ h_2(x,y) z^2 + h_3(x,y) z + h_4(x,y) = 0$$

In this last case the discussion is particularly simple so that we can even compute by hand that only A_7, A_8, $\cdots$, A_9, A_{11} are possible as the isolated singularities of (S,p). But we will rather stop here.

Now we denote the rest of this section to the study of the D type. Assume that S is defined by (2.2.1). By making the transformation $x = X-z^2$ we see that, in the new coordinates, S is given by

$$x^2+y^2z+(h_0-1)z^4+\ldots = 0$$

where dots are terms of weights higher than 1 with respect to $(1/2,3/8,1/4)$. Thus (S,p) is of type D_5 if $h_0 \neq 1$; so we assume $h_0 = 1$. Setting $h_1 = h_1'x+h_1''y$, we further change the coordinate by $y = Y-h_1''z^2/2$. Then the new equation of S is:

$$x^2+y^2z-(h_1'+(h_1'')^2/4)z^5+\ldots = 0$$

with the rest negligible with respect to $(1/2,2/5,1/5)$. Thus (S,p) is of type D_6 if $h_1''^2+4h_1' \neq 0$. Therefore we set $h_1 = 2sy-s^2x$, excluding the case of D_6. If $s \neq 0$, we can assume that $s = 1$ by a suitable scale change; if $s = 0$, the family of surface still admits a non trivial $\mathbb{C}^*$-action, as one can easily see. We will now make a further transformation of the form $x = X-s^2z^3/2$ so that the resulting equation is of the form:

$$x^2+y^2z+(h_2(1,s)-s^4/4)z^6+\dots = 0$$

where dots are negligible with respect to $(1/2,5/12,1/6)$. Thus, if $h_2(1,s) \neq s^4/4$, then (S,p) is of type D_7. We thus arrive at the equation:

$$\text{(VI)} \qquad F_6(x,y,z,w) := x^2w^2+\{y^2z+2xz^2\}w+z^4+(2sy-s^2x)z^3$$

$$+h_2(x,y)z^2+h_3(x,y)z+h_4(x,y) = 0$$

with the condition:

$$\text{(3.2.1)} \quad h_2(1,s) = s^4/4$$

for the surface S with D_k singularity $(k \geqq 8)$ at $p = (0,0,0,1)$. Later we will discuss the cases $s \neq 0$, $s = 0$ separately for this type.

Now we deal with the case in which the surface S is defined by (2.2.2). With respect to weight system $(1/2,3/8,1/4)$ the essential part of the defining equation is $x^2+y^2z+h_0z^4$ in this case also; so we simply assume $h_0 = 0$ as before. The next step is to make the coordinate change $y = Y-h_1''z^2/2$ and this leads to the equation:

$$x^2+y^2z-(h_1'')^2z^5/4+\dots = 0$$

where dots are terms of weights higher than 1 with respect to $(1/2,2/5,1/5)$. Thus we should assume $h_1'' = 0$, excluding

D_6. We now make the transformation $x = X-h_1'z^3/2$ so that the equation of S is put into the form

$$x^2+y^2z-(h_1')^2z^6/4+\dots = 0$$

where dots are terms of weights higher than 1 with respect to $(1/2,5/12,1/6)$. Thus (S,p) is of type D_7 unless $h_1' = 0$. If we exclude D_7 also, then we have to set $h_0 = h_1 = 0$, which however implies that the line $x = y = 0$ in $\mathbb{C}P^3$ lies in the singular locus of S. Thus we see that we need not discuss this case any more.

4. Stratification of the parameter space given by singularities. As we have already seen partly, the singularities which the parametrized surface S can have at p introduce naturally a stratification in the parameter space. This might be a most useful object, like moduli space, for the visualization of the adjacency relation between the singularities. This also provides necessary notations for the formulation of our computational results more conceptually. To begin with, we regard the equations (I) - (VI) as families of quatic surfaces in $\mathbb{C}P^3$ from now on; so, for example, (I) is a fibre space whose fibres are quartic surfaces and whose base space is the affine space with coordinates a,b and the coefficients of the polynomials θ,ϕ,ψ. This base space will sometimes be called the parameter space of the family and denoted by $V(1)$. Similarly the base spaces of the families

(II),...,(VI) are denoted by V(2),...,V(6). The surface S defined for $v \in V(i)$ is denoted by S_v if we want to express that it is the fiber over v. Setting $F(x,y,z) = F_i(x,y,z,1)$ $1 \leq i \leq 5$, $\overline{F}(x,y,z) = F_6(x,y,z,1)$, we can define for each family the power series $F(\xi(z),\eta(z),z)$ $\overline{F}(\overline{\xi}(z),\overline{\eta}(z),z)$ and their coefficients Γ_ν, $\overline{\Gamma}_\nu$ in the way described in Section 1. We denote these Γ_ν or $\overline{\Gamma}_\nu$ by $\Gamma_\nu(i)$, expressing their dependence on the family. (Thus $\Gamma_\nu(6) = \overline{\Gamma}_\nu$.) $\Gamma_\nu(i)$ are polynomials whose variables are the coordinates of the affine space V(i); so we can introduce now a descending chain of subvarieties in V(i) by setting:

$$V_k(i) = \{v \in V(i);\ \Gamma_\nu(i)(v) = 0 \quad \text{for} \quad \nu \leqq k\} \quad 1 \leqq i \leqq 5$$

$$V_k(6) = \{v \in V(6);\ \Gamma_\nu(i)(v) = 0 \quad \text{for} \quad \nu \leqq k-2\}$$

According to Lemmas 1.1-1.2, $v \in V_k(i) \setminus V_{k+1}(i)$ if and only if (S_v, p) is of type A_k for $1 \leqq i \leqq 5$ and $v \in V_k(6) \setminus V_{k+1}(6)$ if and only if (S_v, p) is of type D_k. We set:

$$V_\infty(i) = \cap_k V_k(i) \qquad 1 \leqq i \leqq 6$$

$$V_k^*(i) = V_k(i) \setminus V_\infty(i) \qquad 1 \leqq i \leqq 6$$

For each i $V_\infty(i)$ is the set of points v of V(i) for which p is no longer isolated in the singular locus of S_v. Note also that $V_k(i) \setminus V_{k+1}(i) = V_k^*(i) \setminus V_{k+1}^*(i)$. Thus we can say that the main objective of this paper is to describe as clearly as possible the stratification of $V(i)$ given by the descending chain $V_k(i)$ of subvarieties. Among the subvarieties $V_k^*(i)$ some are irreducible, some are not. But, as we will see later, their irreducible components are always rational i.e. their function fields are all purely transcendental extensions of the ground field $\mathbb{C}$. Roughly speaking, for a fixed k, a choice of generators for the function fields of the components of $V_k(i)$ $1 \leqq i \leqq 5$ corresponds to a choice of the canonical form for quartic surfaces having A_k singularity at p = (0,0,0,1), although the choice should be in a way "good" and all types of such surfaces should be exhausted by the normal forms.

We end this section by remarking that $V_\infty(i) = V_{20}(i)$. To show this, consider the Milnor fiber of the hypersurface singularity (S,p), assuming that p is an isolated singular point of S. We obtain a natural morphism of the second homology group of this fiber into the K3 lattice by perturbing S to a regular quartic. This map preserves the intersection forms. Since (S,p) is rational double, the form on the Milnor fiber is negative definite. This implies that the mapping is injective and that the Milnor number can not be greater than 19 which is the number of negative eigenvalues of the form on the K3 lattice.

5. The Structure of the parameter spaces for the family (I). This is the most difficult case, involving the most complicated computation. Recall that the equation (I): $F_I = 0$ was written in such a way that the polynomials Γ_k, $k \leq 6$ vanish automatically; that is, the singularity (S,p) is higher than A_5 where S denotes the parametrized surface defined by (I). But, by computing first one or two of the remaining Γ_k's we notice that the homogeneous coordinates of $\mathbb{C}P^3$ are not the most convenient yet. In fact, we can make everything much simpler by replacing x, y by $x+y$, $x-y$ respectively. (We do not know if there is a deeper reason for that. For this is not the case for families (III) - (V).) By making a projective transformation of the form $w = W+\ell x+my+nz$, we obtain an equivalent representation of the family (I):

$$\text{(I)}' \quad F_I'(x,y,z,w):=\{(x+y)w-z^2+(x+y)z-(a-b)y^2 +(x+y)(ux+vy)\}\cdot\{(x-y)w-z^2-(x-y)z-(a+b)y^2 -(x-y)(ux+vy)\}+\phi(x,y)z+\psi(x,y) = 0.$$

We set, further:

$$\phi(x,y) = c_0x^3-y(c_1x^2+2c_2xy+c_3y^2)$$

$$\psi(x,y) = -(d_1x^4+d_2x^3y+d_3x^2y^2+d_4xy^3+d_5y^4).$$

Thus, for this family, the parameter space, denoted also by

$V(1)$ is the affine space with coordinates $(a,b,u,v;c_0,c_1,c_2,c_3,d_1,d_2,d_3,d_4,d_5)$, which are also thought of as the independent variables for the polynomials $\Gamma_k(1)$ $k=7,8,\cdots$. We now see the utility of the homogeneous coordinates of $\mathbb{C}P^3$ from the following proposition:

Proposition 5.1. *Up to a non-zero constant factor* $\Gamma_7(1)$ *coincides with* c_0, *and for* $8 \leq k \leq 12$ $\Gamma_k(1)$ *is of the form* $d_{k-7} - \varphi_k(a,b,u,v;c_0,\cdots,c_3;d_1,\cdots,d_{k-8})$ *up to a non-zero constant factor*.

From this we see that the structure of the varieties $V_7(1),V_8(1),\cdots,V_{12}(1)$ is particularly simple: $V_k(1)$ is parametrized by $(a,b,u,v;c_1,c_2,c_3;d_{k-6},\cdots,d_5)$ for $7 \leq k \leq 12$. Thus, to obtain singularities higher than A_{11} for (S,p), we have to assume that all the polynomials $\Gamma_7(1),\cdots,\Gamma_{12}(1)$ vanish; namely, we eliminate $d_1,d_2,\cdots,d_5$ by setting $d_{k-7} = \varphi_k$ $(7 \leq k \leq 12)$. Thus from now on we are operating in the affine space $V_{12}(1)$: (a,b,u,v,c_1,c_2,c_3).

Remark. The system of equations $\Gamma_k(1) = 0$ $7 \leq k \leq 12$ can obviously by replaced by the equivalent system $c_0 = 0$, $d_{k-7} = \bar{\varphi}_k$ $8 \leq k \leq 12$, where $\bar{\varphi}_k$ do not contain any of $d_1,\cdots,d_5$. We then have:

$$\bar{\varphi}_8 = c_1$$

$$\bar{\varphi}_9 = 2c_2 - c_1 u$$

$$\overline{\varphi}_{10} = (4c_3-8c_2u-4c_1-4c_1v-4c_1u^2+c_1^2)/4$$

$$\overline{\varphi}_{11} = -bc_1-c_3u-2c_2-2c_2v-2c_2u^2+c_1c_2-2c_1uv-2c_1u^3+c_1^2u$$

$$\begin{aligned}\overline{\varphi}_{12} = &(-4ac_1-8bc_2-8bc_1u-4c_3-4c_3v-4c_3u^2-16c_2uv-16c_2u^3+4c_2^2\\ &+2c_1c_3+16c_1c_2u-4c_1v^2-4c_1u^2-24c_1u^2-20c_1u^4+c_1^2+4c_1^2v\\ &+16c_1^2u^2-c_1^3)/4.\end{aligned}$$

From these forms we see, in particular, that $c_1 = c_2 = c_3 = 0$ implies $d_1 = d_2 = \cdots = d_5 = 0$ i.e. $\phi = \psi = 0$. This means that S decomposes into two quadrics, both of which pass through p, which implies that p is not isolated in the singular locus of S. Thus the subvariety $\{c_1 = c_2 = c_3 = 0\}$ in $V_{12}(1)$ lies in $V_\infty(1)$. We state without a proof that this subvariety is in fact the 4-dimensional irreducible component of $V_\infty(1)$. Other than this, $V_\infty(1)$ has only one irreducible component which is 3 dimensional.

Now, partly for the sake of defining this 3 dimensional component, we blow up $V_{12}(1)$ with the center $\{c_1 = c_2 = c_3 = 0\}$. We restrict attention to the complement of the strict transform of the hyperplane $\{c_1 = 0\}$; namely we want to set:

$$(5.1)\qquad \begin{cases} c_1 = \ell_1 \\ c_2 = \ell_1\ell_2 \\ c_3 = \ell_1\ell_3. \end{cases}$$

But, before introducing this change of variables, we should check that the hyperplane $\{c_i = 0\}$ cuts out negligible sections from the varieties $V_k^*(1)$ $(k \geq 13)$. To make the meaning of "negligible" precise, we need the following:

<u>Definition</u> 5.1. Let V be an algebraic variety and W a locally closed (algebraic) subset of V. W is called a <u>thin</u> subset if it does not contain any Zariski open subset of V. Suppose that we are given a morphism h of W into some $V_k(i)$ and that an algebraic group G acts on the pull back of the family of quartic surfaces over $V_k(i)$ defined by $F_i = 0$. W is called <u>absolutely thin</u> with respect to h if V/G is of dimension less than $19-k$. We simply call W <u>absolutely thin</u> if W is a subset of $V_k(i)$ and h is the inclusion. We remark that the absolute thinness of a subset of $V_k(i)$ does not necessarily imply its thinness.

The importance of this notion consists in the following idea for proving the rationality of the components of $V_k^*(i)$ or, rather, of the moduli space $\mathcal{M}_k(A)$ (resp. $\mathcal{M}_k(D)$) of projective equivalence classes of quartic surfaces having (at least) one singular point of A (resp. D) type not lower than A_k (resp. D_k) for each given k: Obviously, we can throw away a finite number of thin subsets from $V_k^*(i)$ without affecting the function fields of its irreducible components. Next we assume that $h: W \to V_k^*(i)$ is as in the definition above and that W is absolutely thin with respect to h. The morphism h induces the natural map of W into the moduli space $\mathcal{M}_k(A)$ or $\mathcal{M}_k(D)$ according to whether $1 \leq i \leq 5$ or $i = 6$. But the absolute thinness of W implies that the image of W under this map is a union of a finite number of thin subsets of the moduli space.

We now turn to the sections by $\{c_1 = 0\}$:

<u>Proposition 5.2. For any</u> k, $V_k^*(1) \cap \{c_1 = 0\}$ <u>is</u> $18-k$ <u>dimensional, so that it is thin and absolutely thin in the sense of Definition</u> 5.1.

In order not to break the continuity of the discussion, we do not give a proof of this here, but leave it to the next section; in fact, it is rather longer than it ought to be.

Now we consider only the complements $V_k^{(2)}(1) := V_k^*(1) \setminus \{c_1 = 0\}$ for $k \geqq 12$, on which we can introduce new variables ℓ_1, ℓ_2, ℓ_3 instead of c_1, c_2, c_3 by (5.1). They are closed subvarieties of the open set $U_1 = \{\ell_1 \neq 0\}$ of the affine space $(a, b, u, v; \ell_1, \ell_2, \ell_3)$. Recall that, by taking the complement of $\{c_1 = 0\}$, the component $\{c_1 = c_2 = c_3 = 0\}$ of $V_\infty(1)$ is thrown away, but there still remains the 3 dimensional irreducible component of it. We set $V_\infty^{(2)}(1) := V_\infty(1) \setminus \{c_1 = 0\}$. It is now plausible that any system of variables in U_1, which suits to define $V_\infty^{(2)}(1)$, will also help to simplify the expressions of the polynomials $\Gamma_k(1)$ for $k \geqq 13$. Believing this, we change the variables a, b, u, v to p, q, s, t by the substitution:

$$(5.2)\quad \begin{cases} a = p-\ell_2^2-(\ell_3+1)^2/4 \\ b = q-\ell_2(\ell_3+1) \\ u = s-\ell_2 \\ v = t-\ell_1/2-\ell_2^2-(\ell_3+1)/2. \end{cases}$$

Then $V_\infty^{(2)}(1)$ is given by $p = q = s = t = 0$. We regard $\Gamma_k(1)$ as polynomials in p, q, s, t, ℓ_1, ℓ_2, ℓ_3. The variety $V_k^{(2)}(1)$ is now defined by $\Gamma_{13}(1) = \cdots = \Gamma_k(1) = 0$ for $k \geqq 13$ in the open set $U_1 = \{\ell_1 \neq 0\}$. By inspecting the expressions of $\Gamma_k(1)$, we notice that they contain many terms which are divisible by s. This suggests that the variable s has some special significance. In fact we have the following:

Proposition 5.3. For $13 \leqq k \leqq 16$ the section $V_k^{(2)}(1) \cap \{s = 0\}$ is $18-k$ dimensional, so that it is thin and absolutely thin in the sense of Definition 5.1. On the other hand $V_{17}^{(2)}(1) \cap \{s = 0\}$ is two dimensional.

As we will see later, the complement $V_{17}^{(2)} \setminus \{s = 0\}$ is also two dimensional, so that $V_{17}^{(2)}(1) \cap \{s = 0\}$ is an irreducible component of $V_{17}^{(2)}(1)$. It will turn out that this intersection is not absolutely thin either.

Computational Evidence for Proposition 5.3. $V_{13}^{(2)}(1)$ is defined by $q = \ell_2 t$ in $V_{12}^{(2)}(1) \cap \{s = 0\}$ $(V_{12}^{(2)}(1) = V_{12}^*(1) \setminus \{c_1 = 0\})$;

likewise $V_{14}^{(2)}(1) \cap \{s=0\}$ is defined by $p = t(\ell_2+2\ell_3-2t+2)/2$ in $V_{13}^{(2)}(1) \cap \{s=0\}$, $V_{15}^{(2)}(1) \cap \{s=0\}$ by $\ell_2 = 0$ in $V_{14}^{(2)}(1) \cap \{s=0\}$, and $V_{16}^{(2)}(1) \cap \{s=0\}$ by $\ell_3 = (8t-\ell_1-4)/4$ in $V_{15}^{(2)}(1) \cap \{s=0\}$; furthermore, $V_{17}^{(2)}(1) \cap \{s=0\} = V_{16}^{(2)}(1) \cap \{s=0\}$ and finally $V_{18}^{(2)}(1) \cap \{s=0\}$ is defined by $t=0$ in $V_{17}^{(2)}(1) \cap \{s=0\}$. This implies that $V_{18}^{(2)}(1) \cap \{s=0\}$ lies in $\{s=t=p=q=0\} = V_{\infty}^{(2)}(1) \subset V_{\infty}(1)$. Since $V_{18}^{(2)}(1) \subset V_{18}^{*}(1) = V_{18}(1) \setminus V_{\infty}(1)$, $V_{18}^{(2)}(1) \cap \{s=0\}$ must be empty.

Thus we need only consider the complements $V_k^{(3)}(1) := V_k^{(2)}(1) \setminus \{s=0\}$ for $k \geq 12$; they are closed subvarieties in the open set $V_{12}^{(3)}(1)$ of U_1 and they are already disjoint from $V_{\infty}^{(2)}(1)$. Now we are operating in $V_{12}^{(3)}(1)$, which is given by $s \neq 0$, $\ell_1 \neq 0$ in the affine space: $(p,q,s,t;\ell_1,\ell_2,\ell_3)$ so that we can introduce new variable p_1, q_1, t_1 instead of p,q,t by the following substitution:

$$(5.3) \qquad \begin{cases} p = sp_1 \\ q = sq_1 \\ t = st_1 . \end{cases}$$

<u>Proposition 5.4. With the variables $p_1, q_1, s, t_1, \ell_1, \ell_2, \ell_3$ above, the polynomial $\Gamma_{13}(1)$ has the form $2s^2\ell_1(p_1 - \varphi)$ where φ is a polynomial with no dependence on p_1. Thus $V_{13}^{(3)}(1)$ is a smooth rational variety whose function field is generated by the holomorphic parameters $q_1, s, t_1, \ell_1, \ell_2, \ell_3$.</u>

Remark. Explicitly, the φ above is the following:

$$2t_1-q_1t_1+2\ell_3t_1-4\ell_2+2\ell_2t_1^2+4\ell_2q_1-6\ell_2\ell_3-12\ell_2^2t_1$$
$$+16\ell_2^3+\ell_1t_1-4\ell_1\ell_2+3s-3st_1^2-3sq_1+4s\ell_3$$
$$+24s\ell_2t_1-40s\ell_2^2+3s\ell_1-10s^2t_1+30s^2\ell_2-7s^3.$$

By this proposition we identify $V_{13}^{(3)}(1)$ with the open subset $\{\ell_1 s\neq 0\}$ of the affine space: $(q_1,s,t_1,\ell_1,\ell_2,\ell_3)$, so that all $V_k^{(3)}(1)$ are closed subvarieties of this open set. The hypersurface $V_{14}^{(3)}(1)$ is now defined by $\Gamma_{14}(1)=0$; but the present form of $\Gamma_{14}(1)$ is too complicated to see the structure of $V_{14}^{(3)}(1)$: We should introduce new variables (q_2,k_2,k_3) instead of (q_1,ℓ_2,ℓ_3) by setting:

$$(5.4)\quad \begin{cases} \ell_2 = (3s+t_1+k_2)/4 \\ \ell_3 = (2k_3-k_2^2+t_1k_2-3sk_2+2st_1-2s^2)/2 \\ q_1 = (4q_2+4+8k_3-3k_2^2+2t_1k_2+t_1^2+4\ell_1-6sk_2+2st_1-3s^2)/4. \end{cases}$$

We obtain the simple expression:

$$\Gamma_{14}(1) = \ell_1 s^2\{\ell_1(k_2^2-1)+k_3^2-q_2^2\}$$

that is, $V_{14}^{(3)}(1)$ is defined in $V_{13}^{(3)}(1)$ by

$$(5.5)\quad \ell_1(k_2^2-1)+k_3^2-q_2^2 = 0$$

In the affine space $(q_2, s, t_1, \ell_1, k_2, \ell_3)$, this is no longer a smooth variety (the singular points are $\ell_1 = k_3 = q_2 = 0$, $k_2 = \pm 1$), but it remains rational and irreducible. In fact, outside of the 2 dimensional subsets $q_2 + k_3 = 0$, $k_2 = \pm 1$, we can define a new quantity r by $r = \ell_1 / 2(q_2 + k_3)$ or by $r = (q_2 - k_3)/2(k_2^2 - 1)$, and, if one of $q_2 - k_3 = 2r(k_2^2 - 1)$ and $\ell_1 = 2r(q_2 + k_3)$ holds, then the other holds also by (5.5). This shows that (5.5) is generically equivalent to the following substitution scheme:

$$
(5.6) \qquad \begin{cases} \ell_1 = 4q_3 r \\ q_2 = q_3 + (k_2^2 - 1) r \\ k_3 = q_3 - (k_2^2 - 1) r. \end{cases} \qquad (q_3 := (q_2 + k_3)/2)
$$

Thus we get two new variables q_3, r instead of ℓ_1, q_2, k_3, proving the rationality of $V_{14}^{(3)}(1)$. But recall that we should have checked the thinness of the sections of $V_k^{(3)}(1)$ $(k \geq 14)$ by the subvariety $\{q_2 + k_3 = k_2^2 - 1 = 0\}$. This is almost trivial; if we assume $q_2 + k_3 = k_2^2 - 1 = 0$, then $\Gamma_{15}(1)$ is $\ell_1 s^2 k_3$ up to a constant factor, so that $V_{15}^{(3)}(1) \cap \{q_2 + k_3 = k_2^2 - 1 = 0\}$ is thin in $V_{15}^{(3)}(1)$; if we further assume $k_3 = 0$, then $\Gamma_{16}(1)$ cannot be zero in $V_{15}^{(3)}(1)$ since it has only powers of ℓ_1's as its factors. Now we denote the complement $V_k^{(3)}(1) \setminus \{q_2 + k_3 = k_2^2 - 1 = 0\}$ by $V_k^{(4)}(1)$ for $k \geq 14$ and identify $V_{14}^{(4)}(1)$ with the open set $\{s q_3 r \neq 0\}$ in the affine space: (q_3, s, r, ℓ_1, k_2) by (5.6). The polynomials

$\Gamma_k(1)$ $(k \geqq 15)$ depend now on the coordinates of this affine space.

The following variable change $t_1 \to t_2$ is needed to simplify $\Gamma_{15}(1)$:

(5.7) $\quad t_1 = t_2+3k_2+5s+4k_2r.$

From this we obtain:

(5.8) $\quad \Gamma_{15}(1) = -8s^2q_3^2r\{q_3t_2+(t_2+4s)r(k_2^2-1)\}.$

We now consider the sections of $V_k^{(4)}(1)$ by the hyperplane $\{t_2=0\}$:

<u>Proposition 5.5.</u> <u>For any</u> $k \geqq 15$, $V_k^{(4)}(1) \cap \{t_2=0\}$ <u>is</u> 18-k <u>dimensional, so that it is thin and absolutely thin in the sense of Definition</u> 5.1.

The computational evidence for this is the following: $V_{15}^{(4)}(1) \cap \{t_2=0\}$ is defined by $k_2^2=1$ in $\{t_2=0\}$; further $V_{16}^{(4)}(1) \cap \{t_2=0\}$ is defined by $sk_2 = q_3(r+1)/4$ in $V_{15}^{(4)}(1) \cap \{t_2=0\}$, $V_{17}^{(4)}(1) \cap \{t_2=0\}$ by $q_3=-4r^2$ in $V_{16}^{(4)}(1) \cap \{t_2=0\}$, and $V_{18}^{(4)}(1) \cap \{t_2=0\}$ by $r=-2$ in $V_{17}^{(4)}(1) \cap \{t_2=0\}$. Thus $V_{18}^{(4)}(1) \cap \{t_2=0\}$ consists of one point, at which $\Gamma_{19}(1)$ does not vanish; that implies $V_{19}^{(4)}(1) \cap \{t_2=0\}$ is empty.

Now we can operate in the complement of $\{t_2=0\}$; we consider only the complements $V_k^{(5)}(1) = V_k^{(4)} \setminus \{t_2=0\}$ $(k \geqq 14)$.

Since $t_2 \neq 0$ in $V_{14}^{(5)}(1)$, the equation (5.8) = 0 is equivalent to the substitution:

$$\text{(5.9)} \qquad \begin{cases} q_3 = -w(t_2+4s)(k_2^2-1) \\ r = t_2 w \qquad \text{(w: new variable).} \end{cases}$$

Thus we can identify $V_{15}^{(5)}(1)$ with the open subset $\{st_2w(t_2+4s)(k_2^2-1) \neq 0\}$ of the affine space: (s,t_2,w,k_2) and we regard $V_k^{(5)}(1)$ as closed subvarieties of this open subset. We introduce new variable t_3, s_1 by

$$\text{(5.9)'} \qquad \begin{cases} t_2 = 4st_3 \\ s = 4(k_2^2-1)w/s_1. \end{cases}$$

$V_{15}^{(5)}(1)$ is now identified with the open subset $\{s_1t_3(t_3+1)w(k_2^2-1) \neq 0\}$ of the affine space: (s_1,t_3,w,k_2) and the polynomial $\Gamma_{16}(1)$ has the following form:

$$\text{(5.10)} \qquad \Gamma_{16}(1) = 256\, s^5 q_3^2 r(k_2^2-1)w[4s_1+t_3\{(t_3+1+2k_2w)^2-4w^2\}]$$

where we have kept the old variables s,q_3,r which appear as factors in the expression since they do not vanish in the domain $V_{15}^{(5)}(1)$. We therefore eliminate s_1 by

$$\text{(5.11)} \qquad s_1 = -t_3\{(t_3+1+2k_2w)^2-4w^2\}/4$$

and identify $V_{16}^{(5)}(1)$ with the open subset $\{t_3(t_3+1)w(k_2^2-1)\times$ $\times\{(t_3+1+2k_2w)^2-4w^2\}\neq 0\}$ of the affine space (t_3,w,k_2).

Now we regard $k_2+(t_3+1)/2w$ as a new variable, replacing k_2 i.e. we eliminate k_2 by setting:

$$(5.12) \qquad k_2 = \bar{k}-(t_3+1)/2w.$$

Then $V_{16}^{(5)}(1)$ is the open subset $\{t_3(t_3+1)w(k_2^2-1)(\bar{k}^2-1)\neq 0\}$ of the affine space: $(t_3,w,\bar{k})$ and $\Gamma_{17}(1)$ has the form:

$$(5.13) \qquad \Gamma_{17}(1) = 2048t_3(t_3+1)s^6q_3^2rw^3(k_2^2-1)\{1-(4t_3+1)\bar{k}^2\}.$$

As before we have kept the factors s,q_3,r,w,k_2^2-1 which do not vanish in the domain $V_{16}^{(5)}(1)$. We see from (5.13) that $V_{17}^{(5)}(1)$ is already a closed subvariety in the open subset $V_{16}^{(5)}(1)\setminus\{(4t_3+1)\bar{k}\neq 0\} = \{t_3(t_3+1)(4t_3+1)w\bar{k}(k_2^2-1)(\bar{k}^2-1)\neq 0\}$ of the affine space: $(t_3,w,\bar{k})$. Again by (5.13) $V_{17}^{(5)}(1)$ can be identified with the open subset $\{t_3(t_3+1)w(k_2^2-1)(\bar{k}^2-1)\bar{k}$ $\neq 0\}$ of the affine space: $(w,\bar{k})$ by eliminating t_3 by

$$(5.14) \qquad t_3 = (1-\bar{k}^2)/4\bar{k}^2.$$

Without writing explicitly the non-zero factors of $\Gamma_{18}(1)$ in terms of $w,\bar{k}$ we have:

$$(5.15) \qquad \Gamma_{18}(1) = 2^{34}t_3^9(t_3+1)^3s^{13}\bar{k}^4w^9\{3\bar{k}^2+1-4\bar{k}^3(1-\bar{k}^2)w\}.$$

Since $\bar{k}(1-\bar{k}^2) \neq 0$ in $V_{17}^{(5)}(1)$, we can eliminate w in $V_{18}^{(5)}(1)$ by setting:

$$(5.16) \qquad w = (3\bar{k}^2+1)/4\bar{k}^3(1-\bar{k}^2)$$

and $V_{18}^{(5)}(1)$ is then identified with the open subset $\{\bar{k}(\bar{k}^2-1)(3\bar{k}^2+1)(\bar{k}^2+k+2)(\bar{k}^2-\bar{k}+2) \neq 0\}$ of the 1 dimensional affine space $\bar{k}$. We have simply written down the condition for t_3, t_3+1, w, k_2^2-1, $\bar{k}^2-1$ not to vanish as rational functions of $\bar{k}$.

We have:

$$(5.17) \qquad \Gamma_{19}(1) = -2^{22}t_3(4t_3+1)^9(2t_3+1)(t_3+1)^{13}s^{14}w$$

where all the factors except for $2t_3+1$ do not vanish in $V_{18}^{(5)}(1)$. Thus $V_{19}^{(5)}(1)$ is defined by $2t_3+1 = \frac{\bar{k}^2+1}{2\bar{k}^2} = 0$; that is,

$$(5.18) \qquad V_{19}^{(5)}(1): \quad \bar{k} = \pm\sqrt{-1}.$$

We have thus completely described the varieties $V_k^*(1)$, although so far the description might be called set-theoretic. We can sum up the results in the following form:

<u>Theorem 5.6</u>. <u>The variety</u> $V_k^*(1)$ <u>introduced in Section</u> 4 <u>is an irreducible rational variety of dimension</u> $19-k$ <u>for</u> $k \leq 16$ <u>and</u> $k = 18$. $V_{17}^*(1)$ <u>is purely two dimensional variety with</u>

exactly two irreducible components, both of which are rational. $V^*_{19}(1)$ consists of two points, which give projectively equivalent surfaces.

We are now interested in the image of $\hat{V}^*_k(1)$ in the moduli space $\mathcal{M}_k(A)$ under the natural morphism. We have already noticed in Section 3 that the family (I) admits an involution automorphism. On the modified family (I') of this section, this automorphism acts by changing the sign for each of y,z,b,u,c_0,c_2,d_2,d_4 and leaving all $x,w,a,v,c_1,c_3,d_1,d_3,d_5$ invariant. We denote by $\{\pm 1\}$ the group generated by this involution. We can now easily prove the following:

Proposition 5.7. If, for two points v,v' of the parameter space $V(1)$ of the family (I'), the surfaces S_v, $S_{v'}$ are transformed into one another by a projective transformation fixing $p=(0,0,0,1)$, then v and v' are transposed by $\{\pm 1\}$ and the transformation is the one induced by $\{\pm 1\}$.

As a converse of this, we have:

Proposition 5.8. For $k \geq 10$ and for $v,v' \in \hat{V}^*_k(1)$, every projective transformation, which maps S_v to $S_{v'}$, necessarily fixes $p=(0,0,0,1)$. If v,v' are sufficiently general members of $\hat{V}^*_k(1)$, then the same statement holds even for $k \leq 9$.

If $k \geqq 10$, $v \in V_k^*(1)$, then p is the unique isolated singular point of S_v for which the number of negative eigenvalues of the intersection form on the Milnor fiber is maximal. The first statement follows immediately from this. The second statement follows from the fact that, for a general point v of $V_k^*(1)$ with $k \leqq 18$, p is the unique isolated singular point of S_v. Since $V_k^*(1)$ is irreducible for $k \leqq 18$ and since $V_{19}^*(1) \subseteq V_k^*(1)$, it suffices to show that for $v \in V_{19}^*(1)$ p is the unique singular point of S_v; so suppose $v \in V_{19}^*(1)$. Then (S_v, p) is A_{19} so p is the unique isolated singular point of S_v. S_v is irreducible since any surface in $\mathbb{C}P^3$ of degree $\leqq 3$ cannot have A_k with $k \leqq 7$. But an irreducible quartic surface with 1-dimensional singular locus cannot have any isolated singular points, as follows from the classification of such surfaces. (This last fact follows from the study of surfaces S_v, $v \in V_\infty(i)$ $(1 \leqq i \leqq 6)$ and a few other types of surfaces.)

According to Propositions 5.7 - 5.8, the natural mapping of $V_k^*/\{\pm 1\}$ into $\mathcal{M}_k(A)$ is injective for $k \geqq 10$ and generically injective for $k \leqq 9$. This implies that, for any irreducible component of $V_k^*(1)/\{\pm 1\}$, the closure of its image in $\mathcal{M}_k(A)$ is an irreducible component of $\mathcal{M}_k(A)$ and the natural map induces a birational mapping between these components. (Note that $V_k^*(1)$ and $\mathcal{M}_k(A)$ are both 19-k dimensional.)

Proposition 5.9. The irreducible components of $V_k^*(1)/\{\pm 1\}$ are rational. $V_k^*(1)/\{\pm 1\}$ is irreducible except when $k = 17$.

$V^*_{17}(1)/\{\pm 1\}$ consists of two irreducible components. All irreducible components of $\mathcal{M}_k(A)$ coming from $V^*_k(1)$ are rational.

For any k and for any irreducible component of $V^*_k(1)$ we have given above a system of generators of its function field. One can check directly that any element of the system either changes its sign or remains invariant under $\{\pm 1\}$. This proves the rationality of the quotient of the component by $\{\pm 1\}$, provided it is mapped into itself by $\{\pm 1\}$, which is in fact the case if $k \leq 18$. One can also check that the two points of $V^*_{19}(1)$ are interchanged by $\{\pm 1\}$.

Remark. We have thus seen that the quartic surface with A_{19} is unique. In fact, by replacing y by $\sqrt{-1}y$ or $-\sqrt{-1}y$, we can bring the two surfaces given by $V^*_{19}(1)$ to the following form:

$$\begin{aligned}16(x^2+y^2)w^2+8\{4xz^2+5(x+y)y^2\}w \\ +16z^4-32yz^3+8(2x^2-2xy+5y^2)z^2 \\ +8(2x^3-5x^2y-6xy^2-7y^3) \\ +20x^4+44x^3y+65x^2y^2+40xy^3+41y^4=0.\end{aligned}$$

Perhaps the reader will feel that the discussion of this section might be too detailed and too explicit. But this is because we wanted to discuss the case of the family (I) as a model which would show also how to argue in the other cases (II) - (VI) for which we will not give detailed explanations.

6. Supplementary discussion for the family (I)'. Just after we had anihilated $\Gamma_k(1)$ $k \leqq 12$ by Proposition 5.1 we had to check the case $c_1 = 0$ before introducing the substitution (5.1); namely we had to prove Proposition 5.2, which was postponed until this section. At present, everything sits in the affine space: (a,b,u,v,c_2,c_3). We begin with the following:

Proposition 6.1. The section $V_k^*(1) \cap \{c_1=c_2=0\}$ is rational and $17-k$ dimensional for $k \leqq 16$. $V_{17}^*(1) \cap \{c_1=c_2=0\}$ $= V_{16}^*(1) \cap \{c_1=c_2=0\}$. $V_{18}^*(1) \cap \{c_1=c_2=0\}$ is empty. In particular $V_k^*(1) \cap \{c_1=c_2=0\}$ are all thin and absolutely thin subsets of $V_k^*(1)$ in the sense of Definition 5.1.

Proof. In addition to $c_1=0$ we assume $c_2=0$, so that $\Gamma_k(1)$ are polynomials in a,b,u,v,c_3. We can of course assume $c_3 \neq 0$ since $c_1=c_2=c_3=0$ leads to a non-isolated singularity. By an explicit computation one can check the following steps:

(6.1.1) $\Gamma_{13}(1)=0 \iff b=-2u(u^2+v)$

(6.1.2) $\Gamma_{14}(1)=0 \iff a=c_3/4-u^4-2u^2v-u^2-v^2$ under $\Gamma_k(1)=0$ $(k \leqq 13)$

(6.1.3) $\Gamma_{15}(1)=0 \iff u=0$ under $\Gamma_k(1)=0$ $(k \leqq 14)$

(6.1.4) $\Gamma_{16}(1)=0 \iff v=0$ under $\Gamma_k(1)=0$ $(k \leqq 15)$

(6.1.5) $\Gamma_k(1)=0$ $(k \leqq 16) \Rightarrow \Gamma_{17}(1)=0$

(6.1.6) $\quad \Gamma_k(1)=0 \quad (k \leq 18) \Longrightarrow \Gamma_k(1)=0$ for all k.

Q.E.D.

According to this proposition, we may restrict attention to the open set $c_2 \neq 0$, so that we can make the following variable change $(c_2, c_3) \to (\ell_2, \ell_3)$:

$$(6.1) \quad \begin{cases} c_2 = \ell_2 \\ c_3 = \ell_2 \ell_3. \end{cases}$$

Now $\Gamma_k(1)$ are polynomials in $a, b, u, v, \ell_2, \ell_3$. We can of course throw away ℓ_2 from these polynomials every time it appears as a multiplicative factor. First, we have:

$$(6.2) \quad \Gamma_{13}(1)=0 \Longleftrightarrow a = \ell_2\ell_3/2 - b\ell_3/2 - 2bu + 2\ell_2 u - \ell_3 u^3 - \ell_3 uv - 5u^4 - 6u^2 v - u^2 - v^2$$

by which a is eliminated. Next, for simplifying $\Gamma_{14}(1)$, we make the change of variables $(v, \ell_3) \to (v_1, k_3)$:

$$(6.3) \quad \begin{cases} \ell_3 = 4(k_3 - u) \\ v = v_1 + 2k_3^2 - 2k_3 u - u^2 - 1/2. \end{cases}$$

$$(6.4) \quad \Gamma_{14}(1) = \ell_2[\ell_2 n + 4v_1\{\ell_2 + 4k_3 u^2 - 2v_1 u - nu - b\}]$$

where n is the abreviation for $4k_3^2 - 1$. (6.4) suggests to us to introduce $n/4v_1$ as a new variable r: but if we do, we must next eliminate v_1 by $v_1 = n/4r$. This makes it necessary to check the cases $v_1=0$ and $n=0$ separately so that we can introduce r and assume $r \neq 0$. We begin with the case $v_1=0$. There we have the following steps:

(6.5.1.1) $\Gamma_{14}(1)=0 \iff n=0$ i.e. $4k_3^2-1 = 0$

(6.5.1.2) $\Gamma_{15}(1)=0 \iff b = \ell_2+4k_3u^2$ under $\Gamma_{14}(1) = 0$

(6.5.1.3) $\Gamma_k(1)=0$ $(k \leq 15) \Rightarrow \Gamma_{16}(1)=0$

(6.5.1.4) $\Gamma_k(1)=0$ $(k \leq 17) \Rightarrow \Gamma_k(1)=0$ for all k.

We are thus finished with the case $v_1=0$. Next, we assume that $n = 4k_3^2-1 = 0$ and $v_1 \neq 0$. Then we obtain:

(6.5.2.1) $\Gamma_{14}(1)=0 \iff b = \ell_2+4k_3u^2-2v_1u$

(6.5.2.2) $\Gamma_k(1)=0$ $(k \leq 15) \iff \ell_2=0 \implies \Gamma_k(1)=0$ for all k.

We are done in this case also. (Check that we have obtained only thin and absolutely thin sections of $V_k^*(1)$ by assuming $v_1=0$ or $n=0$.)

Now, without the loss of generality, we make the following change of variables $v_1 \to r$, assuming $r \neq 0$:

(6.5) $v_1 = n/4r = (4k_3^2-1)/4r.$

In view of (6.4) we then eliminate b by the following.

(6.6) $b = (r+1)\ell_2 + 4k_3u^2 - (2r+1)nu/2r.$

The expression for the next term is:

$$(6.7) \qquad \Gamma_{15}(1) = 2\ell_2^2(\ell_2 r^3 - 2k_3 nr - k_3 n - nru)/r.$$

In view of this, we eliminate ℓ_2 by

$$(6.8) \qquad \ell_2 = n(2k_3 r + k_3 - ru)/r^3.$$

We will however keep ℓ_2 every time it appears as a factor in the expression for $\Gamma_k(1)$ $(k \geq 16)$.

To simplify $\Gamma_{16}(1)$ we introduce the change of variables $u \to \bar{u}$ by

$$(6.9) \qquad u = (8rk_3 + 6k_3 - \bar{u})/4r.$$

Then we have:

$$(6.10) \qquad \Gamma_{16}(1) = \ell_2^2 n(\bar{u}^2 - 1 + 2rn)/4r^2.$$

We want to anihilate this. We first note that $\bar{u}+1$ is not allowed to vanish: in fact, $\bar{u}^2-1+2rn = \bar{u}+1 = 0$ leads to $r=0$ or $n=0$ which contradicts the assumption above. This allows us to introduce new variable $w = (\bar{u}+1)/2n$ and to assume $w \neq 0$. We now anihilate $\Gamma_{16}(1)$ by setting:

$$(6.11) \qquad \begin{cases} \bar{u} = 2nw-1 \\ r = 2w(1-nw) \end{cases} \qquad (n := 4k_3^2 - 1).$$

Finally, we make the variable change $(w, k_3) \to (\bar{w}, \bar{k})$, to simplify $\Gamma_{17}(1)$, $\Gamma_{18}(1)$, by the following:

(6.12) $$\begin{cases} w = (\bar{w}+2)/3 \\ k_3 = \bar{k}+(2\bar{w}+1)/6. \end{cases}$$

We have:

(6.13) $$\Gamma_{17}(1) = -\ell_2^2\bar{k}n(\bar{w}^2+\bar{w}+1)/3r^3$$

(6.14) $$\Gamma_{18}(1) = -\ell_2^2\bar{k}n(9\bar{k}+2\bar{w}+1)/9r^4.$$

We see now that $\bar{k} = 0$ leads to a non-isolated singularity near p. Thus we have to set $\bar{w}^2+\bar{w}+1 = 0$ to annihilate $\Gamma_{17}(1)$ and $9\bar{k}+2\bar{w}+1 = 0$ to annihilate $\Gamma_{18}(1)$. We do not have any more parameters at this stage, where $\Gamma_{19}(1) \neq 0$. Thus we have proved Proposition 5.2 completely.

7. Structure of the parameter spaces for type (II).

As in the case of the family (I) in Section 5, we should modify the family (II) given in Section 3 to the following form:

$$\begin{aligned}(\mathrm{II})' \quad F_2'(x,y,z,w) = {} & (x^2-y^2)w^2-2\{xz^2+(ax+by)y^2\}w \\ & -z^4-\{2uxy+2(v-a)y^2\}z^2 \\ & -(c_1x^3+c_2x^2y+c_3xy^2+c_4y^3)z \\ & -(d_1+u^2)x^4-(d_2+2uv)x^3y \\ & -(d_3+2bu-u^2+v^2)x^2y^2 \\ & -(d_4+2au+2bv-2uv)xy^3 \\ & -(d_5-v^2+2av-a^2+b^2)y^4 = 0.\end{aligned}$$

Everything works much simpler in this case; in fact we can gradually kill $\Gamma_k(2)$ $k \geq 7$ in the following steps.

(7.1) $\Gamma_7(2) = 0 \iff c_1=0$

(7.2) $\Gamma_8(2) = 0 \iff d_1=0$ (under $\Gamma_7(2)=0$)

(7.3) $\Gamma_9(2) = 0 \iff (c_2=0$ or $u=0)$ (under $\Gamma_k(2)=0$ $(k\leq 8)$).

We now discuss separately the case (7.3.1): $c_1=d_1=c_2=0$, $u\neq 0$ and the case (7.3.2): $c_1=d_1=u=0$. For (7.3.1) we obtain the following substeps:

(7.3.1.1) $\Gamma_{10}(2) = 0 \Longleftrightarrow d_2=0$

(7.3.1.2) $\Gamma_{11}(2) = 0 \Longleftrightarrow c_3=0$ (under $d_2=0$)

(7.3.1.3) $\Gamma_{12}(2) = 0 \Longleftrightarrow d_3=0$

(From now on, we shall omit the similar assumption "under$\cdots$".)

(7.3.1.4) $\Gamma_{13}(2) = 0 \Longleftrightarrow c_4=0$

(7.3.1.5) $\Gamma_{14}(2) = 0 \Longleftrightarrow d_4=0$

(7.3.1.6) $\Gamma_{15}(2) = 0$ automatically

(7.3.1.7) $\Gamma_{16}(2) = 0 \Longleftrightarrow d_5 = 0.$

Obviously $c_1=\cdots=c_4=d_1=\cdots=d_5=0$ implies that the surface $F_2' = 0$ decomposes into two quadrics passing through $p = (0,0,0,1)$; so at the final stage of an isolated singularity in this case, we obtain a family of A_{15} with five parameters a, b, u, v, d_5. Now recall that the above family admits the following one-parameter action:

$$(x,y,z,w;\ a,b,u,v;\ c_1,\cdots,c_4;\ d_1,\cdots,d_5)$$
$$\to (t^2x,t^2y,tz,w;\ t^{-2}a,\cdots,t^{-2}v;t^{-3}c_1,\cdots,t^{-3}c_4;t^{-4}d_1,\cdots,t^{-4}d_5).$$

Reducing the number of parameters of the family by this action, we obtain a family of A_{15} with just 4 parameters. We can in fact rigidify this family by setting $d_5=1$. The family then admits only the finite automorphism group denoted by $G(2)$ which is generated by the restriction to $\{t;\ t^4=1\}$ of the above $\mathbb{C}^*$ action and the involution induced by $y \leftrightarrow -y$. As in Section 5 we can also prove that any two members of the family are projectively equivalent if and only if they are transformed into one another by $G(2)$. We have thus obtained a natural map of $\{(a,b,u,v);\ u\neq 0\}/G(2)$ into $\mathcal{M}_{15}(A)$ which is injective. Namely, we obtain:

Proposition 7.1. *In addition to the irreducible component coming from* $V^*_{15}(1)$, *there is an irreducible component of* $\mathcal{M}_{15}(A)$ *which arises from* $V^*_{15}(2)$.

The other families of isolated singularities obtained above are absolutely thin, so that we need not worry about them.

Now we discuss the case (7.3.2): $c_1=d_1=u=0$, for which we have a much simpler way of annihilating $\Gamma_k(2)$'s ($k\geq 10$)

(7.3.2.1) $\Gamma_{10}(2) = 0 \iff c_2=0$

(7.3.2.2) $\Gamma_{11}(2) = 0$ automatically

(7.3.2.3) $\Gamma_{12}(2) = 0 \iff d_2=0$.

But now $c_1=d_1=c_2=d_2=u=0$ implies that the surface has a non-isolated singularity along its intersection with the plane y=0. We obtain, at the final stage of an isolated singularity in this case, family of A_{11} with nine parameters a, b, v, c_3, c_4, d_2, d_3, d_4, d_5, among which we can reduce one, for example d_2, by setting $d_2=1$ to kill the $\mathbb{C}^*$ action. Thus, by the same reasoning as above, we obtain:

Proposition 7.2. In addition to the irreducible component coming from $V^*_{11}(1)$, there is an irreducible component of $\mathcal{M}_{11}(A)$ which arises from $V^*_{11}(2)$. All $V^*_k(2)$ $k \neq 11$, 15 are absolutely thin.

8. Structure of the parameter spaces for type (III). In contrast to the cases (I) and (II) we need not change the homogeneous coordinates of $\mathbb{C}P^3$: (x,y,z,w) in this case.

So the parametrized surface is given by:

$$F_3(x,y,z,w) = \{xw-(bx-y)z-x(ux+vy)\}$$

$$\times\{yw+z^2+byz+ax^2+y(ux+vy)\} + \phi(x,y)z + \psi(x,y) = 0$$

(where the letter b is used where formerly — in equation (III) — we used s, for the sake of later convenience) where we set:

$$\phi(x,y) = c_1x^3 + c_2x^2y + c_3xy^2 + c_4y^3$$

$$\psi(x,y) = d_1x^4 + d_2x^3y + d_3x^2y^2 + d_4xy^3 + d_5y^4$$

so that the parameter space $V(3)$ is the affine space: $(a,b,u,v;\ c_1,c_2,c_3,c_4;\ d_1,d_2,d_3,d_4,d_5)$. We will first remark that the family admits the following $\mathbb{C}^*$ action:

$$(x,y,z,w;\ a,b,u,v;\ c_1,c_2,c_3,c_4;\ d_1,d_2,d_3,d_4,d_5)$$
$$\to(t^3x,t^2y,tz,w;\ t^{-4}a,t^{-1}b,t^{-3}u,t^{-2}v;$$
$$t^{-5}c_1,t^{-4}c_2,t^{-3}c_3,t^{-2}c_4;$$
$$t^{-7}d_1,t^{-6}d_2,t^{-5}d_3,t^{-4}d_4,t^{-3}d_5)$$

$$t\in\mathbb{C}^*.$$

As before, we will annihilate step by step, the polynomials $\Gamma_7(3)$, $\Gamma_8(3), \cdots$ to describe the subvarieties $V_k(3)$ or $V_k^*(3)$ in $V(3)$. We begin with the following change of variables $(a,u,c_1,c_2,c_3) \to (p,s,\ell_1,\ell_2,\ell_3)$:

$$(8.1) \qquad \begin{cases} c_3 = \ell_3 \\ c_2 = \ell_2 - 2b\ell_3 \\ c_1 = \ell_1 - 4v\ell_3 \\ u = s + \ell_3 - 2bv \\ a = p - 4bs - 4v^2 \end{cases}$$

The following steps annihilate $\Gamma_k(3)$ up to $k=12$:

$$(8.2) \qquad \Gamma_7(3) = 0 \iff c_4 = 0$$

$$(8.3) \qquad \Gamma_8(3) = 0 \iff d_5 = -\ell_3$$

$$(8.4) \qquad \Gamma_9(3) = 0 \iff d_4 = -\ell_2 + 4b\ell_3$$

$$(8.5) \qquad \Gamma_{10}(3) = 0 \iff d_3 = -\ell_1 - 4b^2\ell_3 + 2b\ell_2 + 6v\ell_3$$

$$(8.6) \qquad \Gamma_{11}(3) = 0 \iff d_2 = -12bv\ell_3 + 2b\ell_1 + 2v\ell_2 + \ell_3^2 + 2\ell_3 s$$

$$(8.7) \qquad \Gamma_{12}(3) = 0 \iff d_1 = p\ell_3 - 4bs\ell_3 - 8v^2\ell_3 + 2v\ell_1 + 2\ell_2 s.$$

In the equivalence $\Gamma_k(3) = 0 \iff \cdots$ we have assumed, exactly as in the previous section, $\Gamma_{k-1}(3) = \cdots = \Gamma_7(3) = 0$. We will often do the same thing in the discussion below. The following change of variables $\ell_2 \to k_2$ is for simplification of $\Gamma_{13}(3)$:

(8.8) $$\begin{cases} \ell_2 = k_2 + p/2 \\ \Gamma_{13}(3) = 2\ell_1 s - k_2^2 + p^2/4. \end{cases}$$

Before annihilating $\Gamma_{13}(3)$, we will examine the case $s=0$; so, in the following steps (8.9.*). We assume $s=0$.

(8.9) $$\Gamma_{13}(3) = 0 \Longleftrightarrow (p+2k_2)(p-2k_2) = 0.$$

In the steps (8.9.1.*) we furthermore, assume that $2\ell_2 = p+2k_2 = 0$ while, in the steps (8.9.2.*), $p-2k_2 = 0$ is assumed.

(8.9.1.1) $\Gamma_{14}(3) = 0 \Longleftrightarrow k_2\ell_1 = 0$

(8.9.1.2) $\Gamma_{15}(3) = 0 \Longleftrightarrow \ell_1 = 0$

(8.9.1.3) $\Gamma_{16}(3) = 0 \Longleftrightarrow k_2\ell_3 = 0.$

But both $s=\ell_1=\ell_2=\ell_3=0$, $s=\ell_1=\ell_2=p=0$ lead to a non-isolated singularity at $(0,0,0,1)$.

(8.9.2.1) $\Gamma_{14}(3) = 0 \Longleftrightarrow k_2(\ell_1+4bk_2) = 0$

(8.9.2.2) $\Gamma_{15}(3) = 0 \Longleftrightarrow \ell_1 = 0$

$$(8.9.2.3)\quad \begin{cases} \Gamma_k(3) = 0 \quad \text{for all } k \text{ if } \ell_1 = k_2 = 0 \\ \Gamma_{16}(3) = 0 \iff k_2\ell_3 = 0 \quad \text{if } \ell_1 = \ell_1+4bk_2 = 0. \end{cases}$$

Thus we are done in this case, too. Anyway, by setting $s=0$ we have obtained not greater than 19-k dimensional sections of $V_k^*(3)$, whose dimensions will only decrease further after dividing by the $\mathbb{C}^*$ action.

We now assume $s \neq 0$ and introduce a new variable $r = (2k_2-p)/4s$ to annihilate $\Gamma_{13}(3)$: namely we eliminate p, ℓ_1 by the following:

$$(8.10)\quad \begin{cases} p = 2(k_2-2sr) \\ \ell_1 = 2r(k_2-sr). \end{cases}$$

Since $s\neq 0$, we can also make the following change of variables $(k_2,v) \to (\bar{k}_2,v_1)$:

$$(8.11)\quad \begin{cases} k_2 = s(\bar{k}_2+r) \\ v = v_1+(r^2+\bar{k}_2r+2br+2b\bar{k}_2)/4. \end{cases}$$

We have:

$$(8.12)\quad \Gamma_{14}(3) = -4s^2(\ell_3+4\bar{k}_2v_1).$$

To annihilate this, we eliminate ℓ_3 by:

$$(8.13)\quad \ell_3 = -4\bar{k}_2v_1.$$

Now we remark that $\bar{k}_2 = 0$ implies $\ell_1=\ell_2=\ell_3=0$, which, together with (8.1)-(8.7), implies $c_1=\cdots=c_4=d_1=\cdots=d_5=0$; hence, the quartic surface $F_3=0$ has a non-isolated singularity at $(0,0,0,1)$. Thus we should assume $\bar{k}_2\neq 0$. To simplify $\Gamma_{15}(3)$ we make the change of variables $(\bar{k}_2,v_1) \to (\bar{\bar{k}}_2,v_2)$:

$$(8.14)\qquad \begin{cases} \bar{k}_2 = \bar{\bar{k}}_2-2(b+r) \\ v_1 = v_2+(2b\bar{\bar{k}}_2+r\bar{\bar{k}}_2-4b^2-6br-2r^2)/4. \end{cases}$$

Then we have:

$$(8.15)\qquad \Gamma_{15}(3) = 8s^2\bar{\bar{k}}_2(s+\bar{\bar{k}}_2v_2)$$

so we eliminate s by setting:

$$(8.16)\qquad s = -\bar{\bar{k}}_2v_2.$$

Since we assumed that $s\neq 0$, we have $\bar{\bar{k}}_2\neq 0$, $v_2\neq 0$. The following change of variables $(v_2,b,\bar{\bar{k}}_2,r) \to (v_3,b_1,\tilde{k}_2,r_1)$ will simplify $\Gamma_{16}(3)$:

$$(8.17)\qquad \begin{cases} v_2 = v_3\bar{\bar{k}}_2 \\ b = b_1-r_1+4v_3 \\ r = 2r_1-b_1-4v_3 \\ \bar{\bar{k}}_2 = \tilde{k}_2+2r_1 \end{cases}$$

$$(8.18)\qquad \Gamma_{16}(3) = 4s^2\bar{\bar{k}}_2^2\tilde{k}_2(\tilde{k}_2b_1+16v_3^2).$$

We see that $\Gamma_{16}(3)=b_1=0 \Longrightarrow v_3=0 \Longrightarrow v_2=0$. Since $v_2\neq 0$, we can assume $b_1\neq 0$, so that we can introduce $w=-v_3/b_1$. We can thus eliminate v_3 and $\tilde{k}_2$ by setting:

$$(8.19)\qquad \begin{cases} v_3 = -b_1 w \\ \tilde{k}_2 = -16b_1 w^2. \end{cases}$$

This annihilates $\Gamma_{16}(3)$. Since $v_3\neq 0$, we have $b_1\neq 0$, $w\neq 0$. We obtain:

$$(8.20)\qquad \Gamma_{17}(3) = 128s^2 b_1 \bar{\bar{k}}_2^2 w^2 (r_1+32b_1w^3-40b_1w^2+8b_1w).$$

To annihilate this, we eliminate r_1 by setting

$$(8.21)\qquad r_1 = -8b_1 w(4w^2-5w+1).$$

We obtain:

$$(8.22)\qquad \Gamma_{18}(1) = 8192s^2 b_1^4 \bar{k}_2 \bar{\bar{k}}_2^2 w^4 (2w-1)^2(8w^2-2w+1)$$

$$(8.23)\qquad \Gamma_{19}(1) = 65536s^2 b_1^5 \bar{k}_2 \bar{\bar{k}}_2^2 w^5 (2w-1)^2(-16w^2+6w-1).$$

We see immediately that $2w=1$ leads to a non-isolated singularity at $(0,0,0,1)$ and that the equations $8w^2-2w+1 = 0$ and $16w^2-6w+1 = 0$ do not have any common root.

To sum up, we have proved:

Proposition 8.1. The parameter spaces $\check{V}_k^*(3)$ are themselves absolutely thin in the sense of Definition 5.1.

9. Results for types (IV) and (V). First, we discuss the family (IV); namely:

$$\text{(IV)} \quad F_4(x,y,z,w) = \{xw + by^2 - x(rz + ux + vy)\} \times \{yw + z^2 + ax^2 + y(rz + ux + vy)\} + \phi(x,y)z + \psi(x,y) = 0$$

where we set:

$$\phi(x,y) = c_1x^3 + c_2x^2y + c_3xy^2 + c_4y^3$$

$$\psi(x,y) = d_1x^4 + d_2x^3y + d_3x^2y^2 + d_4xy^3 + d_5y^4$$

so that the parameter space $V(4)$ is the affine space with coordinates a, b, r, u, v, c_1, c_2, c_3, c_4, d_1, d_2, d_3, d_4, d_5. The subvarieties $V_k^*(4)$, as defined by the polynomials $\Gamma_k(4)$ in Section 4, have particularly simple structures:

(9.1) $\quad \Gamma_7(4) = 0 \iff c_4 = 0$

(9.2) $\quad \Gamma_8(4) = 0 \iff d_5 = 0$

(9.3) $\quad \Gamma_9(4) = 0 \iff c_3 = 0 \quad \text{or} \quad b = 0$

Here $b = 0$ implies that the plane $x = 0$ is an irreducible component of the surface S: $F_4 = 0$; so we should assume $b \neq 0$ in what follows:

(9.4) $\quad \Gamma_{10}(4) = 0 \iff d_4 = 0$

(9.5) $\Gamma_{11}(4) = 0 \iff c_2 = 0$

(9.6) $\Gamma_{12}(4) = 0 \iff d_3 = 0$

(9.7) $\Gamma_{13}(4) = 0 \iff c_1 = 0$

(9.8) $\Gamma_{14}(4) = 0 \iff d_2 = 0$

(9.9) $\Gamma_{15}(4) = 0$ if $\Gamma_k(4) = 0 \quad k \leq 14$

(9.10) $\Gamma_{16}(4) = 0 \iff d_1 = 0$

where, as before, to prove the equivalence $\Gamma_k(4) \iff *$, we need to assume $\Gamma_{k-1}(4) = \cdots = \Gamma_7(4) = 0$. Since $c_1 = \cdots = c_4 = d_1 = \cdots = d_5 = 0$ implies that S decomposes into two quadrics, this case is finished. Recall that the family (IV) admits the following action of $\mathbb{C}^* \times \mathbb{C}^*$: (s,t):

$$(x,y,z,w;\ a,b,u,v;\ r;\ c_1,c_2,c_3,c_4;\ d_1,d_2,d_3,d_4,d_5)$$

$$\to (sx,t^2y,tz,w;\ s^{-2}t^2a,\ st^{-4}b,\ s^{-1}u,t^{-2}v;\ t^{-1}r;$$

$$s^{-2}t^2c_1,\ s^{-1}c_2,\ t^{-2}c_3,\ st^{-4}c_4;$$

$$s^{-3}t^2d_1,\ s^{-2}d_2, s^{-1}t^{-2}d_3, t^{-4}d_4, st^{-6}d_5)\ .$$

Dividing by this $\mathbb{C}^* \times \mathbb{C}^*$ action, we get:

<u>Proposition 9.1</u>. <u>The subvarieties</u> $V_k^*(4)$ $k \leq 14$ <u>are themselves absolutely thin in the sense of Definition</u> 5.1. <u>There is an irreducible component of</u> $\mathcal{M}_{15}(A)$ <u>coming from</u> $V_{15}^*(4)$. <u>For</u> $k \geq 16$, $V_k^*(4)$ <u>is empty</u>.

Next, we discuss the family (V):

$$\text{(V)}\quad F_5(x,y,z,w) = xyw^2 + \{a_1x^2z + a_2x^3 + a_3y^3\}w$$
$$+ xz^3 + (b_1x^2 + b_2xy + b_3y^2)z^2$$
$$+ (c_1x^3 + c_2x^2y + c_3xy^2 + c_4y^3)z$$
$$+ d_1x^4 + d_2x^3y + d_3x^2y^2 + d_4xy^3 + d_5y^4 = 0.$$

First we remark that we can eliminate parameters b_1 and b_2 by setting $b_1 = b_2 = 0$: in fact, to do this, it suffices to replace z by $z + \alpha x + \beta y$ for suitable α, β. We remark also that the family admits the following action of $\mathbb{C}^* \times \mathbb{C}^*$: (x,t):

$$(x,y,z,w;\ a_1,a_2,a_3;\ b_3;\ c_1,c_2,c_3,c_4;\ d_1,d_2,d_3,d_4,d_5)$$
$$\to (sx,t^3y,tz,w;\ s^{-1}t^2a_1,s^{-2}t^3a_2,st^{-6}a_3;$$
$$st^{-5}b_3;\ s^{-2}t^2c_1,s^{-1}t^{-1}c_2,t^{-4}c_3,st^{-7}c_4;$$
$$s^{-3}t^3d_1,s^{-2}d_2,s^{-1}t^{-3}d_3,t^{-6}d_4,st^{-9}d_5)$$

We now have the following equivalences and the same remark as for (9.1) - (9.10) applies below:

(9.11) $\quad \Gamma_8(5) = 0 \iff b_3 = 0$

(9.12) $\quad \Gamma_9(5) = 0 \iff a_3 = 0$

(9.13) $\quad \Gamma_{10}(5) = 0 \iff c_4 = 0$

(9.14) $\Gamma_{11}(5)$ vanishes automatically if $\Gamma_k(5) = 0 \quad k \leqq 10$

(9.15) $\Gamma_{12}(5) = 0 \Longleftrightarrow d_5 = 0.$

But now $b_3 = a_3 = c_4 = d_5 = 0$ implies that the plane $x = 0$ is an irreducible component of the surface S: $F_5 = 0$. Thus, at the final stage of on isolated singularity at $(0,0,0,1)$, we obtain the ten-parameter family $V^*_{11}(5)$ of A_{11} with the coordinates $a_1, a_2; c_1, c_2, c_3; d_1, d_2, d_3, d_4, d_5$ $(d_5 \neq 0)$. We can of course set $d_5 = 1$, killing one parameter because of the above $\mathbb{C}^* \times \mathbb{C}^*$ action. We thus obtain a $\mathbb{C}^*$ action on this family by setting $s = t^9$:

$$(a_1, a_2; c_1, c_2, c_3; d_1, d_2, d_3, d_4)$$
$$\to (t^{-7}a_1, t^{-15}a_2; t^{-16}c_1, t^{-10}c_2, t^{-4}c_3;$$
$$t^{-24}d_1, t^{-18}d_2, t^{-12}d_3, t^{-6}d_4).$$

We can also show that two members of this last family are projectively equivalent if and only if they are transformed into one another by this $\mathbb{C}^*$ action. Thus we obatin:

<u>Proposition 9.2. The subvarieties</u> $V^*_k(5)$ $k \leqq 10$ <u>are thin in the sense of Definition</u> 5.1. $V^*_{11}(5)$ <u>gives another component of</u> $\mathcal{M}_{11}(A)$. <u>This component is rational</u>.

<u>Remark</u>. By (9.14), $V^*_{10}(5)$ and $V^*_{11}(5)$ coincide with each other; but we should distinguish between the

absolute thinness of $V^*_{10}(5)$ and that of $V^*_{11}(5)$; the former means that its image in $\mathcal{M}_{10}(A)$ is thin while the latter means that its image in $\mathcal{M}_{11}(A)$, which is of codimension 1 in $\mathcal{M}_{10}(A)$, is thin. Hence, the above coincidence does not contradict the statement of Proposition 9.2.

We have thus discussed all the types of A_k singular points on quartic surfaces. Recall tha $\mathcal{M}_k(A)$ denotes the moduli space of projective equivalence classes of quartic surfaces having at least one A_ℓ singular point with $\ell \geqq k$, and it is purely 19-k dimensional. We have the natural inclusion $\mathcal{M}_1(A) \supset \mathcal{M}_2(A) \supset \cdots \supset \mathcal{M}_{19}(A)$:

Corollary 9.3. $\mathcal{M}_k(A)$ is irreducible except for $k = 11, 15, 17$. For $k=17$ resp. 15, 11 it has two resp. three irreducible components. All the components of $\mathcal{M}_k(A)$ are rational.

10. The Structure of the parameter spaces for type (VI). In this section we will discuss the last family (VI) of Section 3; namely the case of rational double points of D type. But we study only the generic case $s\neq 0$. (We might leave the case $s=0$ to the reader as a pleasant and amusing exercise.) Without the loss of generality we can assume that $s=1$, so that the family is given by

$$\text{(VI)}' \qquad \begin{aligned} F_6(x,y,z,w) = {} & x^2w^2+\{(x+y)^2z+2xz^2\}w+z^4 \\ & +(x+2y)z^3+(b_1x^2+b_2xy+b_3y^2)z^2 \\ & +(c_1x^3+c_2x^2y+c_3xy^2+c_4y^3)z \\ & +(d_1x^4+d_2x^3y+d_3x^2y^2+d_4xy^3+d_5y^4) = 0 \end{aligned}$$

where we have replaced y by $x+y$ in the equation in Section 3. Then the condition (3.2.1) can be expressed by:

$$(10.1) \qquad b_1 = 1/4.$$

We introduce a change of parameters:

$$(10.2) \qquad \begin{cases} b_2 = 1-2s \\ b_3 = (6-a)/4 \\ c_1 = \ell_1 - s^2 \\ c_2 = \ell_2 - as/2 \\ c_3 = \ell_3 - 2s \\ c_4 = \ell_4 - s/2 \end{cases}$$

$$\begin{cases} d_1 = m_1 - as^2/4 \\ d_2 = m_2 - s^2 \\ d_3 = m_3 - s^2/4 \\ d_4 = m_4 \\ d_5 = m_5 \end{cases}$$

so that $(b_2,b_3;\ c_1,\ldots,c_4;\ d_1,\ldots,d_5)$ is replaced by $(s,a;\ \ell_1,\ldots,\ell_4;\ m_1,\ldots,m_5)$. Then we have to set:

(10.3) $\qquad \ell_1 = 0$

to annihilate $\Gamma_7(6)$; in order to annihilate $\Gamma_8(6)$, we also set:

(10.4) $\qquad m_1 = s\ell_2.$

We have:

(10.5) $\qquad \Gamma_9(6) = sm_2 - s^2\ell_3 - \ell_2^2/4.$

Before introducing a new variable $r=\ell_2/4s$, we should check what happens in the case $s=0$. If we assume $s=0$, ℓ_2 should be zero for $\Gamma_9(6)=0$ vanish; $\Gamma_{10}(6)$ vanishes automatically if $\ell_2=s=0$ and $\Gamma_{11}(6)=0$ if and only if $m_2=0$; now $s=\ell_2=m_2=0$ leads to a non-isolated singularity at $(0,0,0,1)$. As we will see later, $V^*_{12}(6)\cap\{s\neq 0\}$ is 7-dimensional, so we obtain:

Proposition 10.1. The sections $V^*_k(6) \cap \{s=0\}$, $k \leq 11$ are thin and absolutely thin subsets of $V^*_k(6)$, $k \leq 11$ in the sense of Definition 5.1. $V^*_{12}(6) \cap \{s=0\}$ is an irreducible component of $V^*_{12}(6)$. $V^*_k(6) \cap \{s=0\}$ is empty for $k \geq 13$.

Next, study the complements $V^*_k(6) \cap \{s \neq 0\}$. Assume:

$$s \neq 0 \tag{10.6}$$

and annihilate $\Gamma_9(6)$, introducing the variable r mentioned above and killing ℓ_2 and m_2 by

$$\begin{cases} \ell_2 = 4sr \\ m_2 = s(\ell_3 + 4r^2) \, . \end{cases} \tag{10.7}$$

We have:

$$\Gamma_{10}(6) = 16s^2(m_3 - s\ell_4 - 2\ell_3 r + 8r^3 - ar^2) \tag{10.8}$$

so we annihilate this by eliminating m_3 as follows:

$$m_3 = s\ell_4 + 2\ell_3 r - 8r^3 + ar^2 . \tag{10.9}$$

To simplify the form of $\Gamma_{11}(6)$ we need the change of variables $(\ell_3, m_4) \to (k_3, n_4)$:

$$\begin{cases} \ell_3 = k_3 + 12r^2 - ar \\ m_4 = n_4 + 2\ell_4 r + 4r^2 \end{cases} . \tag{10.10}$$

We obtain:

(10.11) $\Gamma_{11}(6) = s^2(4sn_4 - k_3^2)/4.$

Since we have assumed $s \neq 0$, we can annihilate $\Gamma_{11}(6)$ by the following substitution.

(10.12) $$\begin{cases} k_3 = 4sw \\ n_4 = 4sw^2 \end{cases} .$$

We get:

(10.13) $$\Gamma_{12}(6) = s^4(m_5 - r^2 + 24rw^2 - 8rw - aw^2 - 2\ell_4 w)$$

and we eliminate m_5 by setting

(10.14) $$m_5 = r^2 - 24rw^2 + 8rw + aw^2 + 2\ell_4 w \quad .$$

The following substitution simplifies the form of $\Gamma_{13}(6)$:

(10.15) $$\begin{cases} r = r_1 + 4w^2 - 2w \\ \ell_4 = k_4 + 24rw - 4r - aw \end{cases}$$

and we have

(10.16) $$\Gamma_{13}(6) = -s^4(k_4^2 + 8r_1 sw)/4 \quad .$$

Before introducing the substitution scheme to annihilate this, we should check the case $w=0$; if we assume $w=0$, then obviously $\Gamma_{13}(6)=0$ is equivalent to $k_4=0$ and if $k_4=0$ then $\Gamma_{14}(6)$ vanishes automatically and $\Gamma_{15}(6)=0$ if and only if $r_1=0$; but now $w=k_4=r_1=0$ leads to a non-isolated singularity at $(0,0,0,1) \in \mathbb{C}P^3$. As we will see later, the complement $V^*_{16}(6) \setminus \{w=0\}$ is 3 dimensional, we thus obtain:

<u>Proposition 10.2. The sections $V^*_k(6) \cap \{w=0\}$, $k \leq 15$ are thin and absolutely thin subsets of $V^*_k(6)$, $k \leq 15$. $V^*_{16}(6) \cap \{w=0\}$ is an irreducible component of $V^*_{16}(6)$. $V^*_k(6) \cap \{w=0\}$ is empty for $k \geq 17$.</u>

We may thus restrict attention to the complements $V^*_k(6) \setminus \{w=0\} = V^*_k(6) \setminus (\{s=0\} \cup \{w=0\})$, $k \geq 15$; that is, we assume:

$$w \neq 0 \quad . \tag{10.17}$$

Now, as a substitution scheme for annihilating $\Gamma_{13}(6)$, we introduce the following:

$$\begin{cases} k_4 = 8swu \\ r_1 = -8swu^2 \end{cases} \quad . \tag{10.18}$$

We have:

$$\Gamma_{14}(6) = s^6 w^2 (-768swu^4 + 32su^3 - 4au^2 + 384u^2w^2 - 192u^2w + 32uw - 8u - 1) \; . \tag{10.19}$$

If $u=0$, then this never vanishes by (10.6) and (10.17); so we obtain the three parameter family of D_{15}, which is, as a subset of $V^*_{15}(6)$, thin and absolutely thin. Thus we can assume:

$$u \neq 0 \quad . \tag{10.20}$$

We now annihilate (10.19) by setting:

(10.21) $$a = (-768swu^4+32su^3+384u^2w^2-192u^2w+32uw-8u-1) / 4u^3.$$

We need the following change of variables $(s,w,u) \to (s_1,w_1,u_1)$:

(10.22) $$\begin{cases} s = s_1 / 16u^3 \\ w = w_1 / 8u \\ u = u_1/2 -(s_1-2w_1+1) / 4 \end{cases}$$

We obtain:

(10.23) $$r_{15}(6) = s^6w^2(s_1w_1-u_1^2) / 4u^2 .$$

Since we assumed $sw \neq 0$, we also have $s_1w_1 \neq 0$. This allows us to set:

(10.24) $$\begin{cases} u_1 = s_1v \\ w_1 = s_1v^2 \end{cases}$$

to annihilate (10.23). We also obtain that $v \neq 0$.
We have:

(10.25) $$r_{16}(6) = s_1^2v^2(2s_1-1) / 64$$

We have to set:

(10.26) $$s_1 = 1/2$$

to annihilate (10.25), and we finally obtain

(10.27) $$r_{17}(6) = -v^3/4096$$

and this is not allowed to be zero, as we have remarked above.

Theorem 10.3. Let $V_k^*(6)$ $k=7,8,\ldots$ *be the descending chain of varieties introduced in the same way as in Section* 4, *for the family* (VI)' *of this section. Then, for* $k\le 11$, $13\le k\le 15$ *and* $17\le k\le 18$, *the variety* $V_k^*(6)$ *is irreducible and rational and* 19-k *dimensional. For* $k=12,16$, $V_k^*(6)$ *has two irreducible components both of which are rational and* 19-k *dimensional.* $V_{19}^*(6)$ *is empty. The moduli space* $\mathcal{M}_k(D)$ *is birationally equivalent to* $V_k^*(6)$ *for* $k\le 19$.

The last statement follows from the others, since we can prove, as in Section 5, that the natural map $V_k^*(6)\to\mathcal{M}_k(D)$ is generically injective, and since we obtain only absolutely thin parameter spaces for the special case of type (VI).

Remark. So far we have considered the complements $V_k^*(6) = V_k(6)\setminus V_\infty(6)$ and counted the number of their irreducible components. The variety $V_\infty(6)$, which is the intersection of all $V_k(6)$, has two irreducible components, one of which is $V'_\infty(6) := \{s=\ell_1=\ell_2=m_1=m_2=0\}$ and the other is $V''_\infty(6) := \{\ell_1=\ell_2=\ell_3=\ell_4=m_1=m_2=m_3=m_4=m_5=0\}$. Since $V'_\infty(6)$ is 6 dimensional, none of $V_k(6)$, $k\ge 13$ are irreducible. For $v\in V''_\infty(6)$, the singular locus of the associated surface S_v is a cubic curve in $\mathbb{CP}^3$; it is in general a Veronese cubic. On the other hand, for $v\in V'_\infty(6)\setminus V''_\infty(6)$, the singular locus of S_v is of degree 2 and lies in the plane $y=0$.

Remark. In the above one-parameter family of surfaces with D_{18} there is only one special member which has in addition to (0,0,0,1) one more singular point:

$$
\begin{aligned}
-512x^4+384x^3y-256x^3z-72x^2y^2+x^2z^2/4 \\
+x^2zw+x^2w^2+12xy^3+4xy^2z-31xyz^2 \\
+2xyzw+xz^3+2xz^2w+5y^4-4y^3z \\
+7y^2z^2/2+y^2zw+2yz^3+z^4 = 0
\end{aligned}
$$

The other singular point is (1,-8, 4,82), which is A_1.

Appendix

On rational double points on quartic surfaces

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In this appendix, we want to give a theoretical explanation to some results of the joint work of Kato and Naruki; namely we give a proof of the existence of rational double points A_k $(1 \leq k \leq 19)$, D_k $(4 \leq k < 18)$ and the non-existence of D_{19} on quartic surfaces, using the surjectivity of the period mapping for polarized K3 surfaces and the lattice theory due to Nikulin.

We would like to express here our hearty thanks to Professor Looijenga and Professor Namikawa. Actually this article grew out of the discussions with them.

1° Proposition 1. (1) There is no quartic surface in $\mathbb{P}^3$ with a rational double point of type A_k or D_ℓ with $k, \ell \geq 20$. (2) If there is a quartic surface in $\mathbb{P}^3$ with A_{19} or D_{19}, then the surface has no other singularities and the minimal resolution of the unique singular point is a K3 surface.

Proof. Let X be a quartic surface with singular points P_1, $\ldots, P_m$. There exists a smooth quartic surface Y which is a small deformation of X. We can choose mutually disjoint open sets U_1, ..., $U_m \subset Y$, each of which can be regarded as the Milnor fibre of the singular point $P_i \in X$. (Cf. Milnor[8].) Note that Y is a K3 surface and thus the intersection form on $H^2(Y, \mathbb{Z})$ has signature (3,19). Let n_i be the number of negative eigenvalues of the intersection form of the Milnor fibre of $P_i \in X$. Since there is a natural map $\oplus_{i=1}^{m} H_2(U_i, \mathbb{Z}) \rightarrow H_2(Y, \mathbb{Z})$, we can conclude that $\sum_{i=1}^{m} n_i \leq 19$. Our proposition follows from this inequality, since $n_i \geq 1$, and the fact that rational double points do not affect the condition of adjunction. (Cf. Durfee[6].) Q.E.D.

By Proposition 1, we can assume that X is a quartic surface in $\mathbb{P}^3$ with only rational double points in the sequel.

Let $\tilde{X}$ be the minimal resolution of singularities of X. $\tilde{X}$ is a K3 surface. Let $\mathcal{G} = \sum a_k A_k + \sum b_\ell D_\ell + \sum c_m E_m$ denote the configuration of singularities on X. We have inclusion relations

$$S = \mathbb{Z}\,\xi \oplus Q(\mathcal{G}) \hookrightarrow \mathrm{Pic}(\tilde{X}) \hookrightarrow H^2(\tilde{X}, \mathbb{Z})$$

where ξ is the class of the polarization of $\tilde{X}$ (thus $\xi^2 = 4$), and $Q(\mathcal{G})$ is the lattice generated by exceptional curves on $\tilde{X}$. Here $Q(\mathcal{G})$ is isomorphic to the root lattice of type $\mathcal{G}$. (Cf. Bourbaki[4].)

Thus we need in the first step of our proof the lattice embedding theory due to Nikulin[12], which gives an explicit criterion whether for a given $\mathcal{G}$, S can be realized as a sublattice of the even unimodular lattice L of signature (3, 19) which is isomorphic to $H^2(\tilde{X}, \mathbb{Z})$.

2° Let A be a finite abelian group. A mapping $q: A \longrightarrow \mathbb{Q}/2\mathbb{Z}$ is called a <u>finite quadratic form</u> if the following conditions are satisfied.

(1) $q(na) = n^2 q(a)$ for all $n \in \mathbb{Z}$ and $a \in A$.

(2) $q(a+a')-q(a)-q(a') \equiv 2b(a,a') \pmod{2\mathbb{Z}}$ for all $a,a' \in A$,

where $b: A \times A \longrightarrow \mathbb{Q}/\mathbb{Z}$ is a non-degenerated symmetric bilinear form on A.

For a given even lattice S, (i.e. free $\mathbb{Z}$-module of finite rank with a non-degenerated integer valued bilinear form $(\ ,\ )$ satistying $(x,x) \in 2\mathbb{Z}$ for all $x \in S$.) we can define a quadratic form q_S on $A_S = S^*/S$ where $S^* = \mathrm{Hom}(S.\mathbb{Z})$ by putting

$$q_S(t+S) = t^2 + 2\mathbb{Z} \quad \text{for} \quad t \in S^*.$$

We call q_S the <u>discriminant quadratic form</u> of the lattice S.

It is easy to see that any finite quadratic form $q: A \longrightarrow \mathbb{Q}/2\mathbb{Z}$ is decomposed into an orthogonal sum

$$\oplus_p q_p : \oplus_p A_p \longrightarrow \mathbb{Q}/2\mathbb{Z}$$

with respect to the associated finite bilinear form b. Here p denotes a prime mumber, A_p is a maximal p-subgroup of A and $q_p = q|A_p$.

For every prime number p we can replace $\mathbb{Z}$ by the ring of p-adic integers $\mathbb{Z}_p$, $\mathbb{Q}$ by $\mathbb{Q}_p$. Analogously we can define the discriminant quadratic form $q_K: A_K \longrightarrow \mathbb{Q}_p/2\mathbb{Z}_p$ for any even lattice K over $\mathbb{Z}_p$. Note that we can regard $\mathbb{Q}_p/2\mathbb{Z}_p$ as a subgroup of $\mathbb{Q}/2\mathbb{Z}$ and any finite quadratic form $q_p: A_p \longrightarrow \mathbb{Q}/2\mathbb{Z}$ over an abelian p-group is factored to $A_p \longrightarrow \mathbb{Q}_p/2\mathbb{Z}_p \hookrightarrow \mathbb{Q}/2\mathbb{Z}$. Thus we identify q_p with the induced map $A_p \longrightarrow \mathbb{Q}_p/2\mathbb{Z}_p$ in the sequel.

Proposition 2 and 3 are known. (Nikulin[12].)

Proposition 2. Any finite quadratic form $q_p: A_p \longrightarrow \mathbb{Q}/2\mathbb{Z}$ over an abelian p-group A_p is a direct sum of the following generators, $q_\theta^{(p)}(p^k)$, $u_+^{(2)}(2^k)$ and $v_+^{(2)}(2^k)$.

$K_\theta^{(p)}(p^k)$: the 1-dimensional p-adic lattice determined by the matrix (θp^k), where $k \geq 1$ and $\theta \in \mathbb{Z}_p^*$ (taken mod $(\mathbb{Z}_p^*)^2$)

$U^{(2)}(2^k)$ and $V^{(2)}(2^k)$: the 2-dimensional 2-adic lattices determined by the matrices

$$\begin{pmatrix} 0 & 2^k \\ 2^k & 0 \end{pmatrix} \qquad \begin{pmatrix} 2^{k+1} & 2^k \\ 2^k & 2^{k+1} \end{pmatrix}$$

respectively, where $k \geq 1$.

$q_\theta^{(p)}(p^k)$, $u_+^{(2)}(2^k)$ and $v_+^{(2)}(2^k)$: the discriminant quadratic form of $K_\theta^{(p)}(p^k)$, $U^{(2)}(2^k)$ and $V^{(2)}(2^k)$, respectively.

We denote by $\ell(A)$ the minimum number of generators of the finite abelian group A and by $|A|$ the order of A.

Proposition 3. Let $q_p: A_p \longrightarrow \mathbb{Q}/2\mathbb{Z}$ be a finite quadratic form over an abelian p-group A_p. There exists a unique p-adic lattice $K(q_p)$ of rank $\ell(A_p)$ whose discriminant form is isomorphic to q_p, except in the case when $p = 2$ and q_2 is $q_\theta^{(2)}(2) \oplus q_2'$ for some θ.

By Proposition 3, the discriminant discr $K(q_p)$ of $K(q_p)$ is defined by q_p uniquely modulo $(\mathbb{Z}_p^*)^2$ unless $p = 2$ and

$q_2 \neq q_\theta^{(2)}(2) \oplus q_2'$. An embedding $S \hookrightarrow L$ of lattices is called <u>primitive</u> if L/S is free.

<u>Theorem 4.</u> (Nikulin[11].) <u>The following properties are equivalent:</u>

(A) <u>There exists a primitive embedding of an even lattice S with signature (t_+, t_-), into some even unimodular lattice of signature (ℓ_+, ℓ_-).</u>

(B) <u>All of the following conditions are simultaneously satisfied:</u>

1) $\ell_+ - \ell_- \equiv 0 \pmod 8$

2) $\ell_- - t_- \geq 0$, $\ell_+ - t_+ \geq 0$, $\ell_+ + \ell_- - t_+ - t_- \geq \ell(A_S)$

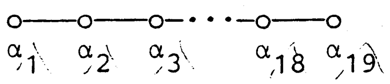
3) $(-1)^{\ell_+ - t_+}|A_S| \equiv \operatorname{discr} K((q_S)_p) \pmod{(\mathbb{Z}_p^*)^2}$

<u>for all the odd primes</u> P <u>such that</u> $\ell_+ + \ell_- - t_+ - t_- = \ell((A_S)_p)$

4) $|A_S| \equiv \pm \operatorname{discr} K((q_S)_2) \pmod{(\mathbb{Z}_2^*)^2}$

if $\ell_+ + \ell_- - t_+ - t_- = \ell((A_S)_2)$ <u>and</u> $(q_S)_2 \neq q_\theta^{(2)}(2) \oplus q_2'$.

<u>Example</u> 5.(1) We consider $S = \mathbb{Z}\xi \oplus Q(A_{19})$, where $\xi^2 = 4$ and $Q(A_{19})$ denotes the root lattice of type A_{19}. Let L be an even unimodular lattice with signature (3, 19), which is unique up to on isomorphism . (Cf. Serre[15].) Then, S has <u>no</u> primitive embedding into L. Indeed, let $\alpha_1, \ldots, \alpha_{19}$ be the basis of $Q(A_{19})$ associated with the Dynkin diagram (Cf. Bourbaki [4].)



Let $\omega_1, \ldots, \omega_{19} \in Q(A_{19})^*$ be the dual basis of $\alpha_1, \ldots, \alpha_{19}$. We have

$$A_S = S^*/S = \langle f \rangle \oplus \langle g \rangle \cong \mathbb{Z}/4 \oplus \mathbb{Z}/20$$

$$f \equiv \tfrac{1}{4}\xi \pmod S, \qquad g \equiv \omega_1 = -\tfrac{1}{20}(19\alpha_1 + 18\alpha_2 + \ldots + \alpha_{19}) \pmod S$$

Let $\omega \in H^0(\tilde{X}, K_{\tilde{X}})$ be a non-zero holomorphic 2-form on $\tilde{X}$. We can regard ω as a cohomology class via the Hodge decomposition $H^2(\tilde{X}, \mathbb{C}) = H^2(\mathcal{O}_{\tilde{X}}) \oplus H^1(\Omega^1_{\tilde{X}}) \oplus H^0(K_{\tilde{X}})$. Since ω is unique up to a non-zero constant multiple, the image of ω by $\psi \otimes \mathbb{C}: H^2(\tilde{X}, \mathbb{C}) \longrightarrow L \otimes \mathbb{C}$ defines a unique element $[\omega]$ in $\mathbb{P}(L \otimes \mathbb{C})$ depending on the triplet, which is called the <u>period</u> of $(\tilde{X}, \tilde{\mathcal{H}}, \psi)$. Since $\omega \cdot \tilde{\mathcal{H}} = 0$, $\omega^2 = 0$, and $\omega \cdot \bar{\omega} > 0$, the period $[\omega]$ belongs to the period domain

$$D = \{ [\chi] \in \mathbb{P}(L \otimes \mathbb{C}) \mid \chi \cdot \tilde{\xi} = 0,\ \chi^2 = 0 \text{ and } \chi \cdot \bar{\chi} > 0 \}.$$

<u>Theorem 8</u>. (Kulikov, Todorov) (Cf. Namikawa[9].)

<u>For every point</u> $[\chi] \in D$, <u>there exists a polarized marked</u> K3 <u>surface of degree</u> 4 $(\tilde{X}, \tilde{\mathcal{H}}, \psi)$ <u>whose period coincides with</u> $[\chi]$.

The next lemma is well-known and easy to prove.

<u>Lemma 9</u>. <u>The Chern class map gives an isomorphism</u>

$$\mathrm{Pic}(\tilde{X}) \cong \{ a \in H^2(\tilde{X}, \mathbb{Z}) \mid a \cdot \omega = 0 \}.$$

Thus in what follows we regard $\mathrm{Pic}(\tilde{X})$ as a subgroup of $H^2(\tilde{X}, \mathbb{Z})$ via the Chern class map.

<u>Example</u> 5.(3). Recall that $S = \mathbb{Z}\xi \oplus Q(A_{19})$ has an embedding $S \hookrightarrow L$ and that it can be factored to the composition of an embedding $S \hookrightarrow S'$ into an overlattice S' and a primitive embedding $S' \hookrightarrow L$.

The element $\tilde{\xi}$ with $\tilde{\xi}^2 = 4$ is unique up to automorphisms of L. Thus we can assume $\tilde{\xi} = \xi$ by composing an automorphism with the

$q_{S}(f) \equiv \frac{1}{4}, \quad q_{S}(g) \equiv -\frac{19}{20} \pmod{2\mathbb{Z}}$

We have only to check the condition 4) in Theorem 4, (B).

In fact, $(q_S)_2 = q_1^{(2)}(2^2) \oplus q_1^{(2)}(2^2)$ and thus discr $K((q_S)_2) = 2^4$.

On the other hand $|A_S| = 5\times 2^4 \not\equiv \pm 2^4 \pmod{(\mathbb{Z}_2^*)^2}$ (since $5 \not\equiv \pm 1 \pmod 8$) This implies that the condition 4) does <u>not</u> hold.

<u>3°</u> Of course, we need to discuss the non-primitive embeddings, too. We say that an even lattice S' containing a given even lattice S, is an <u>overlattice</u> of S if S'/S is finite. Let $H_{S'} = S'/S$. We have a chain of embeddings $S \hookrightarrow S' \hookrightarrow S'^* \hookrightarrow S^*$. Hence $H_{S'} \subset S'^*/S \subset S^*/S = A_S$ and $(S'^*/S)/H_{S'} = A_{S'}$.

A subgroup $H \subset A_S$ is called <u>isotropic</u> if $q_S|H = 0$.

<u>Proposition 6</u>. (A) <u>The correspondence $S' \mapsto H_{S'}$ determines a bijection between even overlattices of S and isotropic subgroups of A_S.</u>

(B) <u>Denoting</u> $(H_{S'})^{\perp} = \{ x\in A_S \mid b_S(x,y) = 0$ for all $y\in H_{S'}\}$, <u>we have</u> $(H_{S'})^{\perp} = S'^*/S \subset A_S$, $A_{S'} = H_{S'}^{\perp}/H_{S'}$ <u>and</u> $q_{S'} = (q_S|H_{S'}^{\perp})/H_{S'}$.

<u>Example</u> 5. (2) Let us count up all the even overlattices of $S = \mathbb{Z}\xi \oplus Q(A_{19})$. It is easy to see that for integers $m,n\in\mathbb{Z}$ with $0\leq m<3$, and $0\leq n<19$, $q_S(mf+ng) = 0$ implies $(m, n) = (0, 0)$ or $(2, 10)$. Therefore $H = \langle 2f+10g\rangle$ is the unique non-trivial isotropy subgroup of A_S and $S' = S+\mathbb{Z}(\frac{1}{2}\xi+10\omega_1)$ is the unique non-trivial even overlattice of S. We can show that S' has a primitive embedding into L. Let $\bar{f}$ and $\bar{g}$ be the images of $f, g\in A_S$ in

A_{S}/H. We have

$$A_{S'} = H^{\perp}/H = \langle \bar{f}+5\bar{g}\rangle \oplus \langle \bar{f}+15\bar{g}\rangle \oplus \langle 4\bar{g}\rangle \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/5,$$

$$(q_{S'})_2 = q_1^{(2)}(2) \oplus q_1^{(2)}(2) \quad \text{and}$$

signature of S' = signature of S = (1, 19).

By Theorem 4, we know that there exists a primitive embedding $S' \longrightarrow L$

Remark. It is easy to see that

(1) There are no elemets $e \in S'$ with $e \cdot \xi = 2$ and $e^2 = 0$.

(2) $\{ s \in S' \mid s \cdot \xi = 0,\ s^2 = -2 \} = \{ r \in Q(A_{19}) \mid r^2 = -2 \}$.

4° In order to show the existence of a quartic surface with A_{19} we also need the theory of period mappings, which was developed only a few years ago.

We fix an even unimodular lattice L with signature (3, 19) and an element $\xi \in L$ with $\xi^2 = 4$. It is known that the pair (L, ξ) is unique up to isomorphisms.

Definition 7. We call the following triplet $(\mathcal{X}, \mathcal{H}, \psi)$ a polarized marked K3 surface of degree 4. (Script)

(1) $\mathcal{X}$ is a smooth K3 surface.

(2) $\mathcal{H}$ is a line bundle on $\mathcal{X}$ which is numerically effective. (i.e. for every curve C on $\mathcal{X}$, the intersection $C \cdot \mathcal{H}$ is non-negative.)

(3) $\psi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} L$ is an isomorphism of lattices such that $\psi(c(\mathcal{H})) = \xi$, where $c(\mathcal{H})$ is the first Chern class of the line bundle $\mathcal{H}$.

embedding if necessary. We identify S with the image of the embedding.

Let $S^\perp$ denote the orthogonal complement of S in L, i.e. $S^\perp = \{ x \in L \mid x \cdot s = 0 \text{ for all } s \in S \}$. $M_{\hat{S}} = \mathbb{P}(S^\perp \otimes \mathbb{C})$ can be regarded as a subspace of $\mathbb{P}(L \otimes \mathbb{C})$. Then we have $M_{\hat{S}} \cap D \neq \phi$ since the intersection form on $S^\perp$ has signature (2, 0). Pick a polarized marked K3 surface $(\tilde{X}_{\hat{0}}, \mathcal{H}_{\hat{0}}, \psi_{\hat{0}})$ whose period belongs to $M_{\hat{S}} \cap D$. We would like to show that for this triplet the image $X_{\hat{0}}$ of the map $\phi_{\hat{\mathcal{H}}_{\hat{0}}}$ associated with the linear system $|\mathcal{H}_{\hat{0}}|$ is a quartic surface with A_{19}.

Claim 10. $\psi(\mathrm{Pic}(\tilde{X}_{\hat{0}})) = S'$.

Proof. By the choice of $(\tilde{X}_{\hat{0}}, \mathcal{H}_{\hat{0}}, \psi_{\hat{0}})$, we have $S \subset \psi(\mathrm{Pic}(\tilde{X}_{\hat{0}}))$. On the other hand $\mathrm{Pic}(\tilde{X}_{\hat{0}})$ is primitive in $H^2(\tilde{X}_{\hat{0}}, \mathbb{Z})$ by Lemma 9. Thus we get Claim 10. Q.E.D.

5° Now we quote geometric results from Saint-Donat[14].

Let $\tilde{X}$ be a K3 surface and let $\mathcal{H}$ be a numerically effective line bundle with $\mathcal{H}^2 = c(\mathcal{H})^2 = 4$.

Proposition 11. If the linear system $|\mathcal{H}|$ has base points, we have $|\mathcal{H}| = |3E+\Gamma|$, where E is a non-singular elliptic curve, Γ is a non-singular rational curve, $E \cdot \Gamma = 1$ and Γ is the fixed part of $|\mathcal{H}|$.

Proposition 12. If the linear system $|\mathcal{H}|$ has no base points, one of the following cases takes place.

(1) The map $\phi_{\mathcal{H}}$ associated with the linear system $|\mathcal{H}|$ is a birational morphism and its image is a quartic surface in $\mathbb{P}^3$ with only rational double points as singularities.

(2) $\phi_{\mathcal{H}}$ is of degree 2 and its image is a non-singular quardratic surface.

(3) $\phi_{\mathcal{H}}$ is of degree 2 and its image is a quadratic surface with a unique singular point λ. In this case (3) the inverse image of λ by $\phi_{\mathcal{H}}$ is either

(A) two disjoint smooth rational curves, or

(B) a configuration of smooth rational curves whose dual graph is of type A_3 or D_ℓ with $\ell \geqq 4$.

From Proposition 11, 12 and the Riemann-Roch theorem we can easily deduce the next lemma.

Lemma 13. The following two conditions are equivalent.

(i) $\phi_{\mathcal{H}}$ is not a birational morphism.

(ii) There exist a line bundle $\mathcal{E}$ with $\mathcal{E}^2 = 0$ and $\mathcal{E} \cdot \mathcal{H} = 0$.

Script style

Indeed, for example, in case (2), let G be a general member of one of the two ruling $\mathbb{P}^1$-families of the image in $\mathbb{P}^3$. The inverse image $E = \phi_{\mathcal{H}}^{-1}(G)$ is a smooth elliptic curve and the line bundle associated with E satisfies the conditions in (ii).

Example 5. (4). By the remark just following Example 5, (2), Claim 1(and Lemma 13, we know that $\phi_{\mathcal{H}_0}$ is a birational morphism.

Lemma 14. For every class $\beta \in \mathrm{Pic}(\tilde{X})$ with $\beta^2 \geq -2$, either β or $-\beta$ is effective. i.e. $H^0(\beta) \neq 0$ or $H^0(-\beta) \neq 0$.

Proof. Since the canonical bundle $K_{\tilde{X}}$ is trivial by the Riemann-Roch theorem we have $\dim H^0(\beta) + \dim H^0(-\beta) \geq \beta^2/2+2$, which implies the lemma. Q.E.D.

Proposition 15. Assume that $\phi_{\mathcal{H}}$ is a morphism and let A denote the set of points where $\phi_{\mathcal{H}}$ fails to be finite.

(1) Each irreducible component C of A is a non-singular rational curve with self-intersection -2 and with $C \cdot \mathcal{H} = 0$.

(2) The root system $R_{\tilde{X}}$ in $\mathrm{Pic}(\tilde{X})$ generated by the classes associated to the components of A coincides with the set $B = \{\beta \in \mathrm{Pic}(\tilde{X}) \mid \beta \cdot c(\mathcal{H}) = 0,\ \beta^2 = -2\}$.

Proof. (1) Obviously $C \cdot \mathcal{H} = 0$. Then by the Hodge index theorem $C^2 < 0$ and we have $p_a(C) = C^2/2+1 \leq 0$. Since the arithmetic genus $p_a(C)$ is always a non-negative integer, we have $p_a(C) = 0$ and $C^2 = -2$. It is well-known that if $p_a(C) = 0$, C is a non-singular rational curve.

(2) Obviously $R_{\tilde{X}} \subset B$. Pick $\beta \in B$. By Lemma 14, β or $-\beta$ is effective. If β is effective, β is the class of a curve Y on $\tilde{X}$. Let $Y = \sum a_i Y_i$ $(0 < a_i \in \mathbb{Z},\ Y_i \neq Y_j$ if $i \neq j)$ be the decomposition into irreducible components. We have $\sum a_i Y_i \cdot \mathcal{H} = 0$ and $Y_i \cdot \mathcal{H} \geq 0$ since $\mathcal{H}$ is numerically effective, which implies $Y_i \cdot \mathcal{H} = 0$ for all i. Thus Y_i is an irreducible component of A and we have $\beta \in R_{\tilde{X}}$. If $-\beta$ is effective, by the same reasoning we get $-\beta \in R_{\tilde{X}}$ and

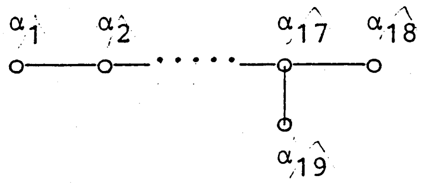
thus $\beta \in R_{\tilde{X}}$. Q.E.D.

Example 5. (5). Let A_0 be the union of the exceptional curves of the morphism $\phi_{\mathcal{H}_0}$. By the remark following Example 5, (2), Claim 10 and Proposition 15 ,(2), we know that the root system in $\mathrm{Pic}(\tilde{X}_0)$ generated by the classes of irreducible components of A_0 is of type A_{19}. By Saint-Donat[13], $\phi_{\mathcal{H}_0} | \tilde{X}_0 - A_0$ is an embedding. Therefore $\phi_{\mathcal{H}_0}(A_0)$ is the single singular point of type A_{19}, which is just what we want.

6° Example 16. Consider the case D_{19}. Set $S = \mathbb{Z}\xi \oplus Q(D_{19})$. By a calculation with the quadratic form, it is easy to show the following.

(1) S has no primitive embedding into L.

(2) S has a unique overlattice S'. Let $\alpha_1, \ldots, \alpha_{19}$ be the base of $Q(D_{19})$ associated with the Dynkin diagram

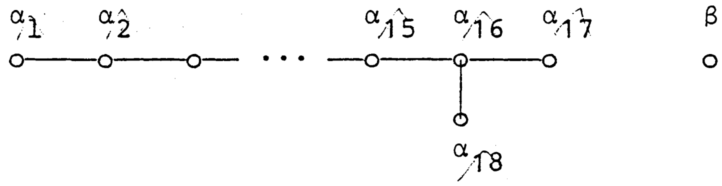


Then, $S' = S + \mathbb{Z}\left(\frac{\xi + \alpha_{18} - \alpha_{19}}{2}\right)$.

Assume that there is a quartic surface X with D_{19}. The minimal resolution $\tilde{X}$ is a K3 surface and there is an embedding $S \hookrightarrow \mathrm{Pic}(\tilde{X})$ such that the image of ξ coincides with the class $\mathcal{H}$ of the pull-back of the hyperplane line bundle. By (1), (2) above and by the same reasoning as in Claim 10, we have $S' \cong \mathrm{Pic}(\tilde{X})$. Let $\mathcal{E} \in \mathrm{Pic}(\tilde{X})$ be the class corresponding to $(\xi + \alpha_{18} - \alpha_{19})/2$. We have $\mathcal{E}^2 = 0$ and

$\mathcal{E}\cdot\mathcal{H} = 2$, which contradicts Lemma 13. Thus there is no quartic surface with D_{19}.

Example 17. $D_{18}+A_1$. Set $S = \mathbb{Z}\,\xi\oplus Q(D_{18}+A_1)$. We can show that the following overlattice S' of S has a primitive embedding into L. Let $\alpha_1, \ldots, \alpha_{18}, \beta$ be the basis associated with the Dynkin diagram



Then $S' = S+\mathbb{Z}\,\rho$ with $\rho = \frac{1}{2}\xi+\frac{1}{2}\sum_{i=1}^{9}\alpha_{2i-1}+\frac{1}{2}\beta$.

Fixing an embedding $S \hookrightarrow S' \hookrightarrow L$ such that $S' \hookrightarrow L$ is primitive and such that the image of ξ coincides with $\tilde{\xi}$, we can choose a K3 surface $(\tilde{X}_1, \mathcal{H}_1, \psi_1)$ whose period belongs to $D\cap\mathbb{P}(S^{\perp}\otimes\mathbb{C})$. We have an isomorphism $S' \cong \mathrm{Pic}(\tilde{X})$ such that ξ corresponds to the class $\mathcal{H}$ of the polarization.

Suppose, $|\mathcal{H}_1|$ has base points. By Proposition 11, $|\mathcal{H}_1| = |3E+\Gamma|$ for some curves E, Γ on $\tilde{X}$ such that $E\cdot\Gamma = 1$ and $E^2 = 0$. Particularly $E\cdot\mathcal{H}_1 = 1$. However, for every element $s\in S'$, $s\cdot\xi$ is an even integer, which is a contradiction. Thus $|\mathcal{H}_1|$ has no base poits.

Suppose, $\phi_{\mathcal{H}_1}$ is of degree 2. If the image of $\phi = \phi_{\mathcal{H}_1}$ is a smooth quadratic surface Λ, let G be a general member of one of the ruling $\mathbb{P}^1$-families. If the image Λ is singular, let G be a general generatrix of Λ. Then, the strict inverse image $\phi^{-1}(G)$ is a smooth elliptic curve and it defines a class $e\in S'$ such that $e^2 = 0$, $e\cdot\xi = 2$ and $e\cdot\beta = 0$. If $e\in S$, $e\cdot\xi\in 4\mathbb{Z}$, which is absurd. Thus we can write $e = \rho + n\beta + e'$ with $e' \in \mathbb{Z}\xi \oplus Q(D_{18})$ and $n \in \mathbb{Z}$. Then $0 = \rho\cdot\beta + n\beta^2 + e'\cdot\beta = -1-2n^2+0$ and we have $1 = -2n^2$, which is also absurd.

Therefore we can conclude that $\phi_{\mathcal{H}_1}$ is a birational morphism. Since it is easy to show that

$$\{\, s \in S' \mid s\cdot\xi = 0,\ s^2 = -2 \,\} = \{\, r \in Q(D_{18}+A_1) \mid r^2 = -2 \,\},$$

$X_1 = \phi_{\mathcal{H}_1}(\tilde{X}_1)$ is a quartic surface with D_{18} and A_1.

<u>Example</u> 18. A_k, $1 \leqq k \leqq 18$.

In this case $S = \mathbb{Z}\xi \oplus Q(A_k)$ has a primitive embedding into L. Let $(\tilde{X}, \mathcal{H}, \psi)$ be a marked polarized K3 surface of degree 4 whose period is a general point of $D \cap \mathbb{P}(S^{\perp} \otimes \mathbb{C})$. Then we have $S \cong \mathrm{Pic}(\tilde{X})$ and it is quite easy to check that

(1) There is no element $\mathcal{E} \in \mathrm{Pic}(\tilde{X})$ with $\mathcal{E}\cdot\mathcal{H} = 2$ and $\mathcal{E}^2 = 0$.

(2) $\{\, \beta \in \mathrm{Pic}(\tilde{X}) \mid \beta\cdot\mathcal{H} = 0,\ \beta^2 = -2 \,\} \cong \{\, r \in Q(A_k) \mid r^2 = -2 \,\}$.

Thus we can conclude that $X = \phi_{\mathcal{H}}(\tilde{X})$ is a quartic surface with an A_k singular point.

<u>Example</u> 19. D_k, $1 \leqq k \leqq 17$.

In this case also $S = \mathbb{Z}\xi \oplus Q(D_k)$ has a primitive embedding into L. Thus there exists a quartic surface with a D_k singular point for $1 \leqq k \leqq 18$.

<u>Theorem 20</u>. (1) <u>There exists a quartic surface in $\mathbb{P}^3$ with a rational double point of type A_k (resp. D_ℓ) if</u> $1 \leqq k \leqq 19$. (<u>resp.</u> $1 \leqq \ell \leqq 18$.)

(2) There is no quartic surface in $\mathbb{P}^3$ with a rational double point of type A_k (resp. D_ℓ) is $k \geq 20$. (resp. $\ell \geq 19$.)

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