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FUZZY論理関数の新しい標準形と数え上げへの応用

New Canonical Forms for and Their Application to

Fuzzy Switching Functions

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1. Introduction

Recently, fuzzy switching functions or fuzzy logic functions have been studied by many researchers because of their potential applicability to many fields such as computer science, system engineering and social science in an uncertain environment (see, for example, Kandel and Lee [1979] or Dubois and Prade[1980]). A fuzzy switching function is a function represented by a logic formula composed of variable x_i ($i=1,\cdots,n$) and logic operations AND(·), OR(+) and NOT($\bar{\ }$),where the variables x_i takes the value in the closed interval [0,1] and the logic operations AND, OR and NOT mean min, max and 1- , respectively.

It is easily shown that the number of distinct n-variable fuzzy switching functions is finite, but enumerating them exactly is very difficult. In fact, despite many attempts (Kameda and Sadeh[1977],Kandel[1978,1981], Mukaidono[1979]), the exact number of n-variable fuzzy switching functions has been reported only in the cases of n=1,2,3 (Mukaidono[1979]), and the upper and lower bounds of the number are not tight (Kameda and Sadeh[1977]) or are very complex (Kandel[1981]).

This paper proposes new canonical forms of fuzzy switching functions which are composed of the disjunctive and conjunctive parts. Using the new canonical forms, we could obtain the exact number of 4-variable fuzzy switching functions (Berman and Mukaidono[1981]) and an asymptotic bound for the number of n-variable fuzzy switching functions.

2. Fuzzy Switching Functions

A logic formula is a formula composed of each variable $x_i (i=1,\cdot\cdot,n)$ and logic operations AND($\cdot$), OR(+) and NOT($\overline{\ }$). If variables x_i takes a value in the closed interval [0,1] and the logic operations are defined as follow:

$$x_1 \cdot x_2 = \min(x_1,x_2),\quad x_1+x_2=\max(x_1,x_2),\quad \overline{x}_1 = 1-x_1,$$

then a logic formula represents a fuzzy switching function or fuzzy logic function which is a mapping from $[0,1]^n$ to [0,1].

[Example 1] A logic formula,

$$(x_1+x_2\cdot\overline{x}_3)\cdot(\overline{x}_1+\overline{x}_2+x_3+x_1\cdot x_2)+x_2\cdot\overline{x}_2\cdot x_3$$

represents a fuzzy switching function $F(x_1,x_2,x_3):[0,1]^3\to[0,1]$; for example, $F(0.1,0.2,0.3)=0.2\cdot 0.9+0.2=0.2$.

Hereafter, for simplicity, we will identify a logic formula with the fuzzy switching function represented by it and omit the symbol $\cdot$ in the logic formula so far as there is no confusion. Then, the above example fuzzy switching function is written such as

$$F=(x_1+x_2\overline{x}_3)(\overline{x}_1+\overline{x}_2+x_3+x_1x_2)+x_2\overline{x}_2x_3 \text{ -------------------------- (1)}$$

A literal is a variable x_i or $\overline{x}_i$, the negation of x_i. A phrase or product term is a conjunction of one or more literals and a clause or sum term is a disjunction of one or more literals. In the definition of the phrase or clause, it is assumed that any repeated literals are removed from it. There are two kinds of phrases: one is a contradictory phrase or complementary phrase, which contains a factor $x_i\bar{x}_i$ for at least one variable x_i; the other is a simple phrase, which does not contain the above. Similarly, there are two kinds of clauses: one is a tautological clause or complementary clause, which contains a factor $x_i+\bar{x}_i$ for at least one variable x_i; the other is a simple clause, which does not. If a contradictory phrase or tautological clause contains all variables as factors, then it is called complete.

It is known (Mukaidono[1972,1975], Davio and Thayse[1973]) that there are two kinds of canonical forms of a fuzzy switching function F: the canonical disjunctive form

$$F=F_{sp}+F_{cp} \quad \text{-------------------------------------(2)}$$

and the canonical conjunctive form

$$F=F_{sc}\cdot F_{cc} \quad \text{-------------------------------------(3)},$$

where F_{sp}(F_{cp}) is a disjunction of simple phrases (completed contradictory phrases) and F_{sc}(F_{cc}) is a conjunction of simple clauses (completed tautological clauses). These canonical forms are determined uniquely for any fuzzy switching function by ignoring the order in which phrases or clauses occur.

[Example 2] The following two formulas (4) and (5) are the canonical disjunctive form and canonical conjunctive form, respectively, of the fuzzy switching function F of Example 1;

$$F=\underbrace{x_1x_2+x_1x_3+x_1\bar{x}_2+\bar{x}_1x_2\bar{x}_3}_{F_{sp}}+\underbrace{\bar{x}_1\bar{x}_2x_2x_3}_{F_{cp}} \quad \text{------------------------(4)}$$

$$=\underbrace{(x_1+x_2)(x_1+\bar{x}_2+\bar{x}_3)}_{F_{sc}}\underbrace{(x_1+\bar{x}_1+\bar{x}_2+x_3)(\bar{x}_1+x_2+\bar{x}_2+x_3)}_{F_{cc}} \quad \text{----------(5)}.$$

Let V and V^n be a set of $\{0,1/2,1\}$ and the n-dimensional Cartesian product of it, respectively. A simple phrase corresponds to an element of V^n and vice versa as follows: a simple phrase α corresponds to an element $A=(a_1,\cdots,a_n)$ if $a_i=1$ iff x_i exists in α, $a_i=0$ iff $\bar{x}_i$ exists in α and $a_i=1/2$ iff x_i and $\bar{x}_i$ do not exist in α;for example, a simple phrase $x_1\bar{x}_3$ corresponds to an element $(1,1/2,0)$ of V^3 and so on in the case of $n=3$. Furthermore, there is a one-to-one correspondence between a set of completed contradictory phrases and V^n-B^n, where $B=\{0,1\}$; for example, a completed contradictory phrase $\bar{x}_1\bar{x}_2x_2x_3$ corresponds to an element $(0,1/2,1)$ of V^3-B^3 and vice versa. The number of elements of V^n is 3^n and that of V^n-B^n is 3^n-2^n. Therefore, it is evident that the number of distinct n-variable fuzzy switching functions is clearly smaller and their exact number is considerably smaller than $2^{3^n}\times 2^{3^n-2^n}=2^{2\times 3^n-2^n}$ because the absorption law holds for simple phrases and completed contradictory phrases. On the other, all distinct canonical dusjunctive forms (which are disjunctions of minterms) of binary switching functions represent always distinct fuzzy switching functions respectively; therefore, a self-evident lower bound of the number of n-variable fuzzy switching functions is 2^{2^n}. In the section 7, we will show better upper and lower bounds of it.

A partially ordered relation $\cent$ on V^n, which describes ambiguity and plays an important role in the theory of fuzzy switching functions, is defined as follows: for all a_i,a_j of V, $a_i \cent a_j$ iff either $1/2 \leq a_i \leq a_j$ or $1/2 \geq a_i \geq a_j$. We always have $1/2 \cent a_i$ for all a_i of V. Moreover, a_i of $(1/2,1]$ and a_j of $[0,1/2)$ cannot be compared with each other. $a_i \cent a_j$ means that a_i is more ambiguous than or equal to a_j. The relation $\cent$ is extensible to V^n as follows: for $A=(a_1,\cdots,a_n)$, $B=(b_1,\cdots,b_n)$, $A \cent B$ iff $a_i \cent b_i$ for all i.

3. B-equivalent Fuzzy Switching Functions

A fuzzy switching function F and a binary switching function f are said to be B-equivalent to each other if f(A)=F(A) holds for every element A of B^n. For example, we can interpret that the logical formula F_{sp}(F_{sc}) of the canonical disjunctive (conjunctive) form (2)((3)) of a fuzzy switching function F represents a binary switching function f. Then, the fuzzy switching function F and the binary switching function f are B-equivalent to each other. In general, there are many fuzzy switching functions each of which is B-equivalent to a binary switching function f, but their number is finite. The set of all fuzzy switching functions each of which is B-equivalent to f is written as B-eq(f). If two fuzzy switching functions F_1 and F_2 are equivalent when we interpret them as binary switching functions, that is, F_1 and F_2 are elements of B-eq(f) of a binary switching function f, then we say also that F_1 and F_2 are B-equivalent to each other.

[Example 3] If we interpret the formula (1) of Example 1 as a binary switching function f, then the Karnaugh map of f can be illustrated as in Figure 1. For example, a fuzzy switching function represented by

$$F'=x_2\bar{x}_3+x_1\bar{x}_2+x_1x_3 \quad \text{------------------------------------(6)}$$

is different from the fuzzy switching function represented by (1), but it satisfies the same Karnaugh map of Figure 1 as a binary switching function. That is, in this example, F, F' and f are B-equivalent to each other.

As mentioned above, the logic formulas F_{sp} of (2), a disjunction of simple phrases; $\alpha_1+\cdots+\alpha_m$ ($\alpha_i \not\subseteq \alpha_j$, $i\neq j$), -----------------(7)

and F_{sc} of (3), a conjunction of simple clauses;

$$\beta_1\cdot\cdots\cdot\beta_m \ (\beta_i \not\subseteq \beta_j,\ i\neq j) \quad \text{----------------(8)}$$

are considered to represent the same binary switching function f if we interpret them as binary switching functions. They are examples of the formulas representing f. Hereafter, we say the representation (7) of a binary switching function f to be a <u>disjunctive form (D-form)</u> of f and the representation (8) to be a <u>conjunctive form (C-form)</u> of f. There are many D-forms and C-forms representing f for, in general, any given binary switching function f, although their number is finite. That is, it is possible in the binary switching theory that different D-forms (C-forms) might represent the same binary switching function. But, in the fuzzy switching theory, different D-forms (C-forms) always represent different fuzzy switching functions, respectively.

For any fuzzy switching function F, let F_p be the fuzzy switching function represented by the disjunction (conjunction) of all prime implicants (prime implicates) of the f which is B-equivalent to F. It was shown(Mukaidono[1975b]) that, for any binary switching function f, the fuzzy switching function represented by the disjunction of all prime implicants of f is equal to such a function represented by the conjunction of all prime implicates of f. It is evident that if two fuzzy switching functions F' and F" are B-equivalent to each other, then F'_p and F''_p are the same fuzzy switching functions, which are also B-equivalent to F' and F".

<u>[Example 4]</u> For the example fuzzy switching function F of (1), F_{sp} and F_{sc} are given by

$$F_{sp}=x_1x_2+x_1x_3+x_1\bar{x}_2+\bar{x}_1x_2\bar{x}_3,$$

$$F_{sc}=(x_1+x_2)(x_1+\bar{x}_2+\bar{x}_3)$$

and F_p is given by

$$F_p=x_1+x_2\bar{x}_3=(x_1+x_2)(x_1+\bar{x}_3) \quad \text{------------------------------(9)}$$

4. New Canonical Forms of Fuzzy Switching Functions

In this section we introduce new canonical forms of fuzzy switching functions. The following lemmas hold for the F_{sp}, F_{sc}, F_{cp} and F_{cc} of the canonical forms (2) and (3) of a fuzzy switching function F. In the following, A* means a set of all elements of B^n obtained from A by replacing 1/2 occurring in A with 0 or 1, and F(A*) means a set of values F(A') for all elements A' of A*. For example, if A=(1/2,1,1), then A*={(0,1,1), (1,1,1)} and F(A*)={0,1} for the fuzzy switching function F of Example 1.

[Lemma 1](Mukaidono[1975])

For all A of V^n,

(1) $F(A^*)=F_{sp}(A^*)=F_{sc}(A^*)=F_p(A^*)$,

(2) $F(A)=1 \Rightarrow F(A^*)=\{1\} \Leftrightarrow F_{sc}(A)=1 \Leftrightarrow F_p(A)=1$,

(3) $F(A)=0 \Rightarrow F(A^*)=\{0\} \Leftrightarrow F_{sp}(A)=0 \Leftrightarrow F_p(A)=0$,

(4) $F(A)=1/2 \Leftarrow F(A^*)=\{0,1\} \Leftrightarrow F_p(A)=1/2$.

[Lemma 2](Mukaidono[1975b])

For all A's of V^n,

(1) $F(A^*)=\{1\} \Rightarrow F_{cp}(A)=0$,

(2) $F(A^*)=\{0\} \Rightarrow F_{cc}(A)=1$.

By the above lemmas, we can obtain the following theorem.

[Theorem 1] Let F be a fuzzy switching function and

$F=F_{sp}+F_{cp}$ and $F=F_{sc}\cdot F_{cc}$

be the canonical disjunctive and conjunctive forms, respectively. Then,

$$F_{sp}=F_p\cdot F_{cc} \quad \text{---(10)},$$

$$F_{sc}=F_p+F_{cp} \quad \text{---(11)}.$$

(Proof) It is sufficient if we prove that (10) and (11) are valid for all elements of V^n (Preparata and Yeh[1972],Mukaidono[1975]). For any A of V^n, $F(A^*)$ is equal to one of {0}, {1} and {0,1}. F, F_{sp},F_{sc} and F_p are B-equivalent to each other. Suppose $F(A^*)=\{0\}$, then, $F_{sp}(A)=F_p(A)=0$ (Lemma 1 (3)). Therefore, (10) holds. Next, suppose $F(A^*)=\{0,1\}$, then, $F(A)=1/2=F_{sp}(A)=F_p(A)$. Furthermore, $F_{cc}(A)\gtrsim 1/2$ always holds. Then, (10) is also valid. Lastly, suppose $F(A^*)=\{1\}$, then, $F_p(A)=1$ (Lemma 1 (2)) and we can show that the right side of (10) is equal to $F_{cc}(A)$. On the other hand $F_{cp}(A)=0$ holds in this case (Lemma 2). Therefore, we can obtain $F_{sp}(A)=F_{sc}(A)\cdot F_{cc}(A)$ because $F=F_{sp}+F_{cp}=F_{sc}\cdot F_{cc}$. Therefore, we can derive $F_{sp}(A)=F_{cc}(A)$ where the both sides of (10) are equal to each other. From the above, we can show that (10) always holds. In the similar manner, we can prove (11). (Q.E.D.)

By the above theorem, we can introduce the following new canonical forms, which are also determined uniquely for any given fuzzy switching function. These canonical forms of a fuzzy switching function F are composed of a D-form and C-form of the binary switching function which is B-equivalent to F.

[Theorem 2] Let F be a fuzzy switching function. Then,

$$F=F_{sp}+(1/2)\cdot F_{sc} \qquad (12)$$

$$F=F_{sc}\cdot(F_{sp}+1/2) \qquad (13).$$

(Proof) $F_{cc}\gtrsim 1/2$ and $F_{cp}\lesssim 1/2$ always holds. Therefore, (12) can be proved by Theorem 1 as follows: $F=F_{sp}+F_{cp}=F_p\cdot F_{cc}+F_{cp}=F_p\cdot(F_{cc}+1/2)+(1/2)\cdot F_{cp}=F_p\cdot F_{cc}+(1/2)\cdot(F_p+F_{cp})=F_{sp}+(1/2)\cdot F_{sc}$. We can also prove (13) in the similar manner. (Q.E.D.)

[Example 5] For the fuzzy switching function (4) or (5) of Example 2,

$x_1x_2+x_1x_3+x_1\bar{x}_2+\bar{x}_1x_2\bar{x}_3=(x_1+x_2)(x_1+\bar{x}_3)(x_1+\bar{x}_1+\bar{x}_2+x_3)(\bar{x}_1+x_2+\bar{x}_2+x_3)$

holds because of $F_{sp}=F_p \cdot F_{cc}$ (Theorem 1), and $(x_1+x_2)(x_1+\bar{x}_2+\bar{x}_3)=x_1+x_2\bar{x}_3+\bar{x}_1\bar{x}_2x_2x_3$ because of $F_{cc}=F_p+F_{cp}$ (Theorem 1). Also, the new canonical forms of that function are

$$F=x_1x_2+x_1x_3+x_1\bar{x}_2+\bar{x}_1x_2\bar{x}_3+(1/2)(x_1+x_2)(x_1+\bar{x}_2+\bar{x}_3),$$
$$=(x_1+x_2)(x_1+\bar{x}_2+\bar{x}_3)(1/2+x_1x_2+x_1x_3+x_1\bar{x}_2+\bar{x}_1x_2\bar{x}_3)$$

5. Enumerating D-forms of Binary Switching Functions

As described above, for any given binary switching function f, in general, there are many D-forms and C-forms representing f. Let DF(f) and CF(f) be the sets of all D-forms and all C-forms of f, respectively.

[Theorem 3] Let f be a binary switching function. The number of distinct fuzzy switching functions each of which is B-equivalent to f is obtained by the product of the numbers of distinct D-forms and C-forms of f; that is $|B\text{-eq}(f)|=|DF(f)|\times|CF(f)|$.

(Proof) Let $F_d=\alpha_1+\cdots+\alpha_m$ and $F_c=\beta_1\cdot\cdots\cdot\beta_n$ be a D-form and C-form of f respectively, where α_i is a simple phrase and β_i is a simple clause. Then, we can show that $F=F_d+(1/2)\cdot F_c$ ------------------------------------(14)
is a fuzzy switching function which is B-equivalent to f as follow: first, it is evident that (14) is B-equivalent to f because F_d and F_c represent the same binary switching function f and take the same value of B={0,1} simultaneously when an input to them takes the value of B^n. Next, we need to show that (14) is representable without the constant 1/2 for (14) to be a fuzzy switching function. From Theorem 1, F_d and F_c can be represented by using F_p of f as follows: $F_d=F_p\cdot F'_{cc}$, $F_c=F_p+F'_{cp}$, where F'_{cc} and F'_{cp} are a conjunction of completed tautological clauses and a disjunction of completed contradictory phrases, respectively. Then,

$F_d+(1/2)\cdot F_c=F_p\cdot F'_{cc}+(1/2)(F_p+F'_{cp})=F_p\cdot F'_{cc}+(1/2)F_p+(1/2)F'_{cp}=F_p(F'_{cc}+1/2)+$ $(1/2)F'_{cp}=F_p\cdot F'_{cc}+F'_{cp}=F_d+F'_{cp}$, that is, it can be represented without the constant 1/2. The above facts prove that (14) is a fuzzy switching function which is B-equivalent to f and that, if F_d or F_c is replaced with another D-form or C-form of f respectively, then (14) represents a different fuzzy switching function which is also B-equivalent to f. Conversely, any fuzzy switching function F can be represented by $F=F_{sd}+(1/2)F_{sc}$ from Theorem 2, where F_{sp} and F_{sc} are a D-form and C-form of f, respectively, each of which is B-equivalent to F. That is, any fuzzy switching function which is B-equivalent to f can be represented by the form of (14). In addition, if $F=F_{sp}+(1/2)F_{sc}$ and $F'=F'_{sp}+(1/2)F'_{sc}$ are two distinct fuzzy switching functions which are B-equivalent to f, then F_{sp} must be different from F'_{sp}, or F_{sc} different from F'_{sc}. Therefore, there is a one-to-one correspondence between the fuzzy switching functions which are B-equivalent to f and the fuzzy switching functions represented by the form of (14). The above fact proves the theorem. (Q.E.D.)

[Corollary 1] The number of distinct fuzzy switching functions which are B-equivalent to a binary switching function f is obtained by the product of the numbers of distinct D-forms of f and $\overline{f}$; that is,

$$|B\text{-}eq(f)|=|DF(f)|\times|DF(\overline{f})|.$$

(Proof) Let $f=\beta_1\cdot\cdots\cdot\beta_m$ be a C-form of f. Then, there is a corresponding D-form $\overline{\beta}_1+\cdots+\overline{\beta}_m$ of $\overline{f}$, where $\overline{\beta}_i$ can be transformed into a simple phrase by applying De Morgan's laws to β_i. That is, $\overline{\beta}_1+\cdots+\overline{\beta}_m$ corresponds to a D-form of $\overline{f}$ in a one-to-one correspondence. Therefore, the number of distinct C-forms of f is equal to the number of distinct D-forms of $\overline{f}$. This fact leads to the theorem by Theorem 3. (Q.E.D.)

The above corollary shows that the problem of obtaining the number of fuzzy switching functions which are B-equivalent to f is equal to the problem of enumerating D-forms of f and $\overline{f}$. Therefore, we will consider, hereafter in this section, the problem of enumerating distinct D-forms of any binary switching function.

Let $V^n(f)$ be a sub-set of V^n, which is composed of elements corresponding to simple phrases included (be subsumed) by a binary switching function f; that is,

$V^n(f)=\{A_\alpha \mid A_\alpha$ is the element corresponding to a simple phrase α such as $\alpha \subseteq f\}$.

$V^n(f)$ is a partially ordered set with regard to the relation ¢ (the relation ¢ was defined in the section 2 and is the same as the inclusion relation between simple phrases).

<u>[Example 6]</u> The binary switching function f (Figure 1) of our example gives $V^3(f)=\{(1,1/2,1/2),(1,0,1/2),(1,1,1/2),(1,1/2,0),(1,1/2,1),(1/2,1,0),$ $(1,0,0),(1,0,1),(1,1,0),(1,1,1),(0,1,0)\}$, which is illustrated as a partially ordered set in Figure 2.

The maximum elements of the partially ordered set $V^n(f)$ correspond to the prime implicants of f. The set $V^n(f)$ is determined uniquely by the maximum elements. Especially, the set V^n is a partially ordered set called an upper semi-lattice with the maximal element $(1/2, 1/2,\cdots,1/2)$. V^n_i is defined to be a set of each element A of V^n such that the number of 1/2's of A is i. Of course, V^n_0 is equal to B^n. The set $V^n_i(f)$ is defined in the similar manner. $V^n_{\overline{0}}(f)$ means $V^n(f)-V^n_0(f)$, that is, $V^n_{\overline{0}}(f)$ is a partially ordered set obtained from $V^n(f)$ ignoring the elements of $V_0^n(f)$.

[Example 7] For our example f,

$V_0^3(f)=\{(1,0,1),(1,1,1),(1,0,0),(1,1,0),(0,1,0)\}$,

$V_1^3(f)=\{(1,0,1/2),(1,1/2,1),(1,1,1/2),(1,1/2,0),(1/2,1,0)\}$,

$V_2^3(f)=\{(1,1/2,1/2)\}$, $V_3^3(f)=\phi$,

$V^3(f)=\{(1,0,1/2),(1,1/2,1),(1,1,1/2),(1,1/2,0),(1/2,1,0),(1,1/2,1/2)\}$.

A set $\{A_1,\cdots,A_s\}$, where A_i $(i=1,\cdots,s)$ is an element of $V^n(f)$, is called an anti-chain of $V^n(f)$ if they cannot be compared with each other, that is, $A_i \not\subseteq A_j$ $(i\neq j)$ for all i and j. We consider, for convenience, here, that the empty set is an anti-chain.

[Theorem 4] A D-form of f corresponds to an anti-chain $\{A_1,\cdots,A_s\}$ of $V^n(f)$ by

$$A_1^* \cup \cdots \cup A_s^* = V_0^n(f) \qquad (15)$$

and vice versa.

(Proof) Let $\alpha_1+\cdots+\alpha_s$ be a D-form of f. Then, $\{A_1,\cdots,A_s\}$ is an anti-chain of $V^n(f)$, where A_i is an element of V^n corresponding to α_i. Furthermore, (15) holds because $\alpha_1+\cdots+\alpha_s$ represents the binary switching function f. We can prove the reverse in the similar manner. (Q.E.D.)

We can conclude from the above theorem that the problem of obtaining DF(f) is equal to the problem of obtaining anti-chains $\{A_1,\cdots,A_s\}$ of $V^n(f)$ such that $\alpha_1+\cdots+\alpha_s$ (α_i corresponds to A_i) represents the binary switching function f. In addition, we can improve the above result as follows:

[Theorem 5] The number of D-forms of a binary switching function f is equal to the number of anti-chains of $V_{\bar{0}}^n(f)$.

(Proof) An anti-chain $\{A_1,\cdots,A_s\}$ of $V^n(f)$ satisfying $A_1^* \cup \cdots \cup A_s^* = V_0^n(f)$

corresponds uniquely to an anti-chain of $V_{\bar{0}}^{n}(f)$ which is a subset of and obtained from $\{A_1,\cdots,A_s\}$ by ignoring the elements of V_0^n. Conversely, an anti-chain $\{A_1',\cdots,A_t'\}$ of $V_{\bar{0}}^{n}(f)$ corresponds uniquely to an anti-chain of $V^n(f)$ by adding elements of V_0^n to that anti-chain in such a way to satisfy (15). Therefore, the number of anti-chains of $V_{\bar{0}}^{n}(f)$ is equal to the number of anti-chains of $V^n(f)$ satisfying (15). This shows that the number of anti-chains of $V_{\bar{0}}^{n}(f)$ is equal to that of D-forms of f from Theorem 4. (Q.E.D.)

[Example 8] We will enumerate the D-forms of example f. In order to do so, we have to enumerate anti-chains of $V_{\bar{0}}^{3}(f)$. $V_{\bar{0}}^{3}(f)$ is obtained, as in Figure 3, from Figure 2 by ignoring the elements of V_0^3. Here, two elements (1,1/2,1/2) and (1/2,1,0) correspond to the prime implicants x_1 and $x_2\bar{x}_3$ of f, respectively. The number of the anti-chains can be enumerated as follows: we can chose (1,1/2,1/2) and (1/2,1,0) independently as factors of anti-chains because they are not related each other. There are two choices over (1/2,1,0): we can choose or not choose it. As for (1,1/2,1/2), if we choose it, no other choices remain, but if we do not, we have $2^4=16$ choices because we can choose independently four elememts of V_1^3 covered by (1,1/2,1/2). In total, there are 1+16=17 ways regarding (1,1/2,1/2). Therefore, the number of anti-chains, which is equal to the number of D-forms, is 2x17=34.

We see that the number of D-forms of f is equal to the number of anti-chains of $V_{\bar{0}}^{n}(f)$. The set $V_{\bar{0}}^{n}(f)$ is easily obtained from the set of all prime implicants of f as follows: First, determine $V^n(f)$ which is a set of elements of V^n covered by the prime implicants and, next, ignore the elements of V_0^n from the $V^n(f)$. At present, we have no good algorithm to enumerate anti-chains of $V_{\overline{0}}^{n}(f)$, but in the cases of small n we can enumerate them by a computer in the similar manner described in Example 8.

6. An Algorithm Enumerating Fuzzy Switching Functions

Based on the results described in the previous section, we can enumerate fuzzy switching functions which are B-equivalent to any binary switching function f. Therefore, by summing up the number of all B-equivalent fuzzy switching functions corresponding to the binary switching functions of 2^{2^n}, we can obtain the number of n-variable fuzzy switching functions. Here, we have the following theorem:

[Theorem 6] If binary switching functions f_1 and f_2 are elements of the same NPN equivalent class (Muroga[1979]), that is, f_1 is obtained from f_2 by negating or permutating the variables or negating the function, then the numbers of fuzzy switching functions which are B-equivalent to f_1 and f_2 are identical, that is, $|B\text{-eq}(f_1)|=|B\text{-eq}(f_2)|$.

(Proof) The partially ordered set $V^n(f)$ of a binary switching function f is isomorphic even if the variables are negated or permutated. Furthermore, the number of fuzzy switching functions which are B-equivalent to f is obtained by $|DF(f)|\times|DF(\overline{f})|$ (Lemma 1). Therefore, the number is unchanged even if the function is negated. (Q.E.D.)

The following is an algorithm enumerating n-variable fuzzy switching functions:

(1) list the NPN equivalence classes of n-variable binary switching functions; $\sigma_1,\cdots,\sigma_s$,

(2) for each representative function f_i of σ_i, determine the number of D-forms of f and $\overline{f}$; $|DF(f)|, |DF(\overline{f})|$,

(3) $\psi(n)=\sum_{i=1}^{s}|\sigma_i| \times |DF(f_i)| \times |DF(\overline{f}_i)|$ is the number of n-variable fuzzy switching functions.

Here, for example, the number of NPN equivalence classes in the cases of n=1,2,3,4 are 2, 4, 14, 222, respectively (Harrison[1961]).

[Example 9] Let us enumerate the fuzzy switching functions which are B-equivalent to our example f. As stated in Example 8, $|DF(f)|=34$. $V_{\overline{0}}^{\underline{3}}(\overline{f})$ is illustrated in Figure 4. In this case, it is evident that the number of the anti-chains is $2^2=4$. Therefore, the number of the fuzzy switching functions which are B-equivalent to f is 34x4=136.

7. The Number of Fuzzy Switching Functions in Four or Fewer Variables

According to the algorithm of the previous section, the number of four-or-less-variable fuzzy switching functions is enumerated as in Tables 1 -- 4. In the tables, the first column indicates the label of the NPN equivalence class (we used Harrison's labels for n=3 and 4), the second column is the number of elements of each equivalence class, the third are the minterms of the representative binary switching function f of each equivalence class, the fourth is the number of D-forms of f, the fifth is the number of D-forms of $\overline{f}$, the sixth is the number of fuzzy switching functions which are B-equivalent to f, and the last column is the total number of the fuzzy switching functions which are B-equivalent to one of the NPN equivalence class. The last line shows the number of n-variable fuzzy switching functions. We enumerate them by hand for n=1, 2, 3 and by computer in the case of n=4.

[Example 10] The label of our example f can be found to be 6 as shown in Table 3 by negating x_1 and x_2 and the function.

8. Upper and Lower Bounds of the Number of n-Variable Fuzzy Switching Functions

In this section, $\psi(n)$ denotes the number of n-variable fuzzy switching functions.

[Theorem 7] $\psi(n)<2^{3^n}$.

(Proof) We can show from Corollary 1 that $|B\text{-eq}(f)|<2^{|V_{\bar{0}}^{n}(f)|}\times 2^{|V_{\bar{0}}^{n}(\bar{f})|}\leq 2^{|V^n-V_0^n|}=2^{3^n-2^n}$. Therefore, $\psi(n)<2^{2^n}\times 2^{3^n-2^n}=2^{3^n}$. (Q.E.D.)

This theorem shows that Kameda's result(Kameda and Sadeh[1977]) is incorrect, because their result was $\psi(n)\geq 2^{3^n}$.

[Theorem 8] $\psi(n)>2^{\frac{3}{2\sqrt{\pi}}\left(\frac{3^n}{K}\right)}$, where $K=\sqrt{n}\left(1+\frac{3}{8n}\right)$

(Proof) This result can be obtained by applying Stirling's approximation to the fact that the number of the factors of the longest anti-chains is $2^{2^m\binom{n}{m}}$, where $m=\left\lceil\frac{2n-1}{3}\right\rceil$. (Q.E.D.)

[Theorem 9] $\lim_{n\to\infty}\sqrt[n]{\log_2\psi(n)}=3$.

(Proof) It is easily shown from Theorem 7 and 8. (Q.E.D.)

It can be shown (Berman and Mukaidono[1981a]) that we can improve the upper and lower bounds described above (Theorem 7 and 8).

9. Conclusion

It has been shown that the problem of enumerating n-variable fuzzy switching functions can be reduced to the problem of enumerating disjunctive forms of any binary switching function f, which can be solved by enumerating anti-chains of the partially ordered set composed of simple phrases covered by f. Using the above finding, we are able to enumerate the exact number of four-variable fuzzy switching functions, which is 160,297,985,276.

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x_3 \ $x_1 x_2$	0 0	0 1	1 1	1 0
0	0	1	1	1
1	0	0	1	1

Figure 1: Karnaugh map of the example binary switching function f.

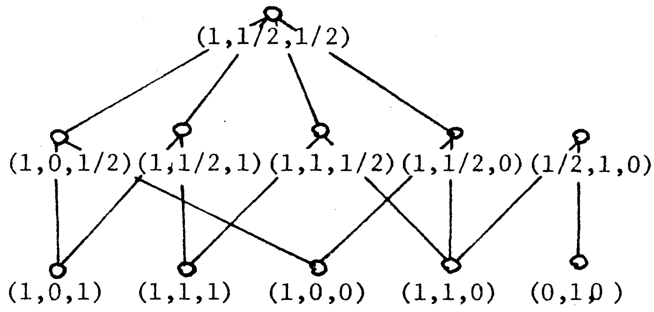

Figure 2: A partially ordered set $V^3(f)$ of the example binary switching function f.

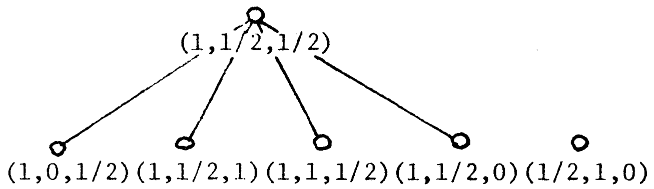

Figure 3: The set $V_0^3(f)$ of the example binary switching function f.

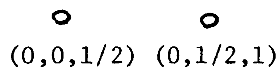

Figure 4: The set $V_0^3(\overline{f})$ of the example binary switching function f.

label	#1: multiplicity	minterm	$\lvert DF(f)\rvert$	$\lvert DF(\bar{f})\rvert$	#2: $\lvert DF(f)\rvert \times \lvert DF(\bar{f})\rvert$	#1×#2
1	2	*	1	2	2	4
2	2	1	1	1	1	2
Total	4					6

Table 1. The Number of One-Variable Fuzzy Switching Functions

label	#1: multiplicity	minterm	$\lvert DF(f)\rvert$	$\lvert DF(\bar{f})\rvert$	#2: $\lvert DF(f)\rvert \times \lvert DF(\bar{f})\rvert$	#1×#2
1	2	* *	1	17	17	34
2	8	3 *	1	4	4	32
3	4	2 3	2	2	4	16
4	2	1 2	1	1	1	2
Total	16					84

Table 2. The Number of Two-Variablr Fuzzy Switching Functions

label	#1: multiplicity	minterm	$\lvert DF(f)\rvert$	$\lvert DF(\bar{f})\rvert$	#2: $\lvert DF(f)\rvert \times \lvert DF(\bar{f})\rvert$	#1×#2
1	2	* * * *	1	6211	6211	12422
2	16	7 * * *	1	621	621	9936
3	24	6 7 * *	2	145	290	6960
4	24	4 7 * *	1	68	68	1632
5	8	0 7 * *	1	64	64	512
6	48	5 6 7 *	4	34	136	6528
7	48	1 6 7 *	2	16	32	1536
8	16	3 5 6 *	1	8	8	128
9	6	4 5 6 7	17	17	289	1734
10	6	2 3 4 5	4	4	16	96
11	2	1 2 4 7	1	1	1	2
12	8	3 5 6 7	8	8	64	512
13	24	3 4 6 7	8	8	64	1536
14	24	0 5 5 7	4	4	16	384
Total	256					43918

Table 3. The Number of Three-Variable Fuzzy Switching Functions

NO.	X	STANDARD SUM	F=1	F=0	F1*F0	F1*F0*X
1	2	** ** ** ** ** ** ** **	1	31901034831	31901034831	63802069662
2	32	15 ** ** ** ** ** ** **	1	1087646789	1087646789	34804697248
3	64	14 15 ** ** ** ** ** **	2	103893164	207786328	13298324992
4	96	12 15 ** ** ** ** ** **	1	43722077	43722077	4197319392
5	64	8 15 ** ** ** ** ** **	1	39739369	39739369	2543319616
6	16	0 15 ** ** ** ** ** **	1	38462300	38462300	615396800
7	192	13 14 15 ** ** ** ** **	4	10083670	40334680	7744258560
8	384	9 14 15 ** ** ** ** **	2	4256361	8512722	3268885248
9	128	11 13 14 ** ** ** ** **	1	1984626	1984626	254032128
10	128	1 14 15 ** ** ** ** **	2	3846230	7692460	984634880
11	96	3 12 15 ** ** ** ** **	1	1787807	1787807	171629472
12	192	3 13 14 ** ** ** ** **	1	1681300	1681300	322809600
13	48	12 13 14 15 ** ** ** **	17	2325923	39540691	1897953168
14	96	8 9 14 15 ** ** ** **	4	456349	1825396	175238016
15	32	9 10 12 15 ** ** ** **	1	99376	99376	3180032
16	32	0 1 14 15 ** ** ** **	4	385513	1542052	49345664
17	24	0 3 12 15 ** ** ** **	1	83521	83521	2004504
18	48	0 3 13 14 ** ** ** **	1	73984	73984	3551232
19	128	11 13 14 15 ** ** ** **	8	992313	7938504	1016128512
20	384	10 13 14 15 ** ** ** **	8	987846	7902768	3034662912
21	384	8 13 14 15 ** ** ** **	4	458566	1834264	704357376
22	192	3 13 14 15 ** ** ** **	4	420325	1681300	322809600
23	32	7 11 13 14 ** ** ** **	1	98825	98825	3162400
24	384	2 13 14 15 ** ** ** **	4	416977	1667908	640476672
25	384	6 10 13 15 ** ** ** **	2	196082	392164	150590976
26	192	0 13 14 15 ** ** ** **	4	389028	1556112	298773504
27	192	4 11 12 15 ** ** ** **	4	416810	1667240	320110080
28	384	4 9 14 15 ** ** ** **	2	182296	364592	140003328
29	192	7 8 13 14 ** ** ** **	1	90824	90824	17438208
30	384	0 9 14 15 ** ** ** **	2	180868	361736	138906624
31	128	0 11 13 14 ** ** ** **	1	79488	79488	10174464
32	384	11 12 13 14 15 ** ** **	34	229283	7795622	2993518848
33	384	10 11 12 13 15 ** ** **	16	106476	1703616	654188544
34	128	8 11 13 14 15 ** ** **	8	49688	397504	50880512
35	192	3 12 13 14 15 ** ** **	17	97257	1653369	317446848
36	384	7 8 9 14 15 ** ** **	8	45217	361736	138906624
37	384	7 10 11 12 13 ** ** **	4	21268	85072	32667648
38	128	7 9 10 12 15 ** ** **	2	9936	19872	2543616
39	128	7 8 11 13 14 ** ** **	1	4968	4968	635904
40	384	2 3 12 13 15 ** ** **	8	42212	337696	129675264
41	192	1 2 13 14 15 ** ** **	4	19652	78608	15092736
42	192	0 3 13 14 15 ** ** **	4	18496	73984	14204928
43	384	3 5 8 14 15 ** ** **	2	8704	17408	6684672
44	96	5 6 9 10 15 ** ** **	1	4624	4624	443904
45	32	7 11 13 14 15 ** ** **	16	98825	1581200	50598400
46	384	6 11 13 14 15 ** ** **	16	98041	1568656	602363904
47	384	5 10 13 14 15 ** ** **	16	97282	1556512	597700608
48	384	4 11 13 14 15 ** ** **	8	45412	363296	139505664
49	768	5 11 12 14 15 ** ** **	8	45574	364592	280006656
50	192	7 11 12 13 14 ** ** **	4	22869	91476	17563392
51	128	0 11 13 14 15 ** ** **	8	39744	317952	40697856
52	768	1 11 12 14 15 ** ** **	8	42392	339136	260456448
53	384	3 11 12 13 14 ** ** **	8	45394	363152	139450368
54	768	3 10 12 13 15 ** ** **	4	21196	84784	65114112
55	384	3 8 13 14 15 ** ** **	4	19872	79488	30523392
56	384	6 7 9 11 12 ** ** **	4	19792	79168	30400512

Table 4. The number of four-variable fuzzy switching functions (continued)

NO.	X	STANDARD SUM	F=1	F=0	F1*F0	F1*F0*X
57	384	6 7 8 11 13 ** ** **	2	9280	18560	7127040
58	32	0 7 11 13 14 ** ** **	1	4096	4096	131072
59	192	10 11 12 13 14 15 ** **	145	53238	7719510	1482145920
60	192	8 11 12 13 14 15 ** **	68	24844	1689392	324363264
61	64	9 10 11 12 13 14 ** **	64	24844	1590016	101761024
62	192	2 3 12 13 14 15 ** **	34	10553	358802	68889984
63	64	6 7 10 11 12 13 ** **	8	2482	19856	1270784
64	96	1 2 12 13 14 15 ** **	17	4624	78608	7546368
65	96	4 7 8 11 12 15 ** **	16	4913	78608	7546368
66	192	4 7 9 10 13 14 ** **	4	1088	4352	835584
67	96	5 6 9 10 12 15 ** **	1	272	272	26112
68	192	1 6 8 9 14 15 ** **	16	4624	73984	14204928
69	192	1 6 10 11 12 13 ** **	4	1024	4096	786432
70	96	1 2 3 12 13 14 ** **	16	4624	73984	7102464
71	16	3 5 6 9 10 12 ** **	1	256	256	4096
72	192	7 11 12 13 14 15 ** **	68	22869	1555092	298577664
73	384	6 11 12 13 14 15 ** **	68	22787	1549516	595014144
74	192	4 11 12 13 14 15 ** **	68	22706	1544008	296449536
75	384	3 11 12 13 14 15 ** **	68	22697	1543396	592664064
76	384	7 10 11 12 13 15 ** **	32	10634	340288	130670592
77	128	7 8 11 13 14 15 ** **	16	4968	79488	10174464
78	768	2 11 12 13 14 15 ** **	34	10598	360332	276734976
79	768	7 8 10 13 14 15 ** **	32	10598	339136	260456448
80	384	6 10 11 12 13 15 ** **	16	5317	85072	32667648
81	384	6 8 11 13 14 15 ** **	16	4968	79488	30523392
82	384	0 11 12 13 14 15 ** **	34	9936	337824	129724416
83	768	4 10 11 12 13 15 ** **	32	10598	339136	260456448
84	384	4 9 10 13 14 15 ** **	16	4968	79488	30523392
85	384	4 8 11 13 14 15 ** **	8	2484	19872	7630848
86	384	0 10 11 12 13 15 ** **	16	4968	79488	30523392
87	128	0 8 11 13 14 15 ** **	16	4968	79488	10174464
88	384	5 7 8 10 14 15 ** **	32	9896	316672	121602048
89	768	5 7 10 11 12 14 ** **	16	4948	79168	60801024
90	384	0 6 11 13 14 15 ** **	16	4352	69632	26738688
91	384	2 4 11 13 14 15 ** **	8	2312	18496	7102464
92	384	3 4 10 13 14 15 ** **	8	2176	17408	6684672
93	384	2 7 11 12 13 14 ** **	4	1160	4640	1781760
94	384	0 4 11 13 14 15 ** **	16	4640	74240	28508160
95	384	6 7 8 9 12 15 ** **	16	4948	79168	30400512
96	384	0 5 10 13 14 15 ** **	16	4624	73984	28409856
97	768	6 7 8 11 12 13 ** **	8	2320	18560	14254080
98	768	1 4 10 13 14 15 ** **	8	2176	17408	13369344
99	192	3 4 8 13 14 15 ** **	4	1024	4096	786432
100	384	0 1 10 13 14 15 ** **	16	4640	74240	28508160
101	384	0 3 8 13 14 15 ** **	8	2176	17408	6684672
102	384	1 2 8 13 14 15 ** **	4	1088	4352	1671168
103	128	0 7 9 10 12 15 ** **	2	512	1024	131072
104	192	2 3 4 8 13 15 ** **	4	1024	4096	786432
105	32	0 7 11 13 14 15 ** **	16	4096	65536	2097152
106	384	1 6 11 13 14 15 ** **	16	4640	74240	28508160
107	384	3 5 10 13 14 15 ** **	16	4948	79168	30400512
108	96	3 7 11 12 13 14 ** **	16	5297	84752	8136192
109	128	9 10 11 12 13 14 15 **	621	12422	7714062	987399936
110	192	1 2 3 12 13 14 15 **	68	1156	78608	15092736
111	128	1 6 7 10 11 12 13 **	8	128	1024	131072
112	32	3 5 6 9 10 12 15 **	1	16	16	512

Table 4. The number of four-variable fuzzy switching functions (continued)

NO.	X	STANDARD SUM	F=1	F=0	F1*F0	F1*F0*X
113	384	7 10 11 12 13 14 15 **	290	5317	1541930	592101120
114	768	5 10 11 12 13 14 15 **	290	5299	1536710	1180193280
115	384	7 8 11 12 13 14 15 **	136	2484	337824	129724416
116	384	7 9 10 12 13 14 15 **	136	2484	337824	129724416
117	384	1 10 11 12 13 14 15 **	145	2484	360180	138309120
118	384	3 8 11 12 13 14 15 **	136	2484	337824	129724416
119	384	7 8 9 10 13 14 15 **	128	2484	317952	122093568
120	384	3 9 10 12 13 14 15 **	68	1242	84456	32431104
121	128	7 9 10 11 12 13 14 **	64	1242	79488	10174464
122	128	6 7 10 11 12 13 15 **	64	1241	79424	10166272
123	768	4 5 10 11 13 14 15 **	136	2474	336464	258404352
124	192	5 6 9 10 13 14 15 **	64	1156	73984	14204928
125	384	4 7 9 10 13 14 15 **	32	544	17408	6684672
126	192	4 7 8 11 13 14 15 **	16	272	4352	835584
127	768	2 3 9 12 13 14 15 **	68	1160	78880	60579840
128	384	0 3 11 12 13 14 15 **	68	1088	73984	28409856
129	384	3 5 10 11 12 13 14 **	64	1156	73984	28409856
130	768	2 5 10 11 12 13 15 **	64	1088	69632	53477376
131	384	1 2 11 12 13 14 15 **	34	544	18496	7102464
132	384	6 7 9 10 11 12 13 **	32	580	18560	7127040
133	384	4 7 9 10 11 12 15 **	32	544	17408	6684672
134	384	0 7 10 11 12 13 15 **	32	512	16384	6291456
135	384	1 7 10 11 12 13 14 **	16	272	4352	1671168
136	384	1 6 10 11 12 13 15 **	16	256	4096	1572864
137	384	5 6 9 10 11 12 15 **	8	136	1088	417792
138	384	4 5 7 8 10 11 15 **	64	1088	69632	26738688
139	384	4 5 6 9 10 11 15 **	32	544	17408	6684672
140	768	4 5 7 9 10 11 14 **	32	544	17408	13369344
141	192	5 6 7 9 10 11 12 **	16	272	4352	835584
142	768	2 3 4 9 13 14 15 **	16	256	4096	3145728
143	384	2 3 5 9 12 14 15 **	8	128	1024	393216
144	128	1 2 4 11 13 14 15 **	8	128	1024	131072
145	192	1 2 7 11 12 13 14 **	4	64	256	49152
146	96	3 7 11 12 13 14 15 **	289	5297	1530833	146959968
147	768	3 7 10 12 13 14 15 **	136	2474	336464	258404352
148	384	2 7 11 12 13 14 15 **	68	1160	78880	30289920
149	384	3 7 8 12 13 14 15 **	136	2320	315520	121159680
150	384	3 6 9 12 13 14 15 **	68	1156	78608	30185472
151	768	2 7 9 12 13 14 15 **	68	1088	73984	56819712
152	192	0 7 11 12 13 14 15 **	68	1024	69632	13369344
153	384	6 7 9 11 12 13 14 **	64	1237	79168	30400512
154	768	6 7 8 11 12 13 15 **	64	1160	74240	57016320
155	768	6 7 9 10 12 13 15 **	32	580	18560	14254080
156	768	6 7 8 9 11 12 15 **	64	1160	74240	57016320
157	384	6 7 8 9 11 13 14 **	32	580	18560	7127040
158	768	4 7 8 10 11 13 15 **	32	544	17408	13369344
159	128	0 7 8 11 13 14 15 **	32	512	16384	2097152
160	768	5 6 8 9 11 14 15 **	32	544	17408	13369344
161	384	1 6 8 11 13 14 15 **	16	256	4096	1572864
162	384	0 1 6 11 13 14 15 **	32	512	16384	6291456
163	384	0 3 5 10 13 14 15 **	16	256	4096	1572864
164	96	0 3 7 11 12 13 14 **	16	256	4096	393216
165	8	8 9 10 11 12 13 14 15	6211	6211	38576521	308612168
166	12	4 5 6 7 8 9 10 11	289	289	83521	1002252
167	8	2 3 4 5 8 9 14 15	16	16	256	2048
168	2	1 2 4 7 8 11 13 14	1	1	1	2

Table 4. The number of four-variable fuzzy switching functions (continued)

NO.	X	STANDARD SUM	F=1	F=0	F1*F0	F1*F0*X
169	64	7 9 10 11 12 13 14 15	1242	1242	1542564	98724096
170	192	6 9 10 11 12 13 14 15	1242	1242	1542564	296172288
171	192	4 9 10 11 12 13 14 15	1242	1242	1542564	296172288
172	64	0 9 10 11 12 13 14 15	621	621	385641	24681024
173	64	1 6 7 10 11 12 13 15	64	64	4096	262144
174	192	2 4 6 9 11 13 14 15	272	272	73984	14204928
175	64	1 6 7 10 11 12 13 14	64	64	4096	262144
176	64	3 5 6 9 10 12 14 15	16	16	256	16384
177	384	1 4 5 10 11 12 14 15	272	272	73984	28409856
178	192	3 4 7 9 10 12 13 14	64	64	4096	786432
179	192	1 2 3 8 12 13 14 15	136	136	18496	3551232
180	192	1 6 7 8 10 11 12 13	32	32	1024	196608
181	64	3 5 6 8 9 10 12 15	8	8	64	4096
182	32	6 7 10 11 12 13 14 15	1241	1241	1540081	49282592
183	96	6 7 8 9 12 13 14 15	1237	1237	1530169	146896224
184	48	5 6 9 10 12 13 14 15	272	272	73984	3551232
185	48	5 6 8 11 12 13 14 15	272	272	73984	3551232
186	96	0 1 10 11 12 13 14 15	290	290	84100	8073600
187	96	0 3 8 11 12 13 14 15	272	272	73984	7102464
188	96	0 7 8 9 10 13 14 15	256	256	65536	6291456
189	96	0 3 9 10 12 13 14 15	68	68	4624	443904
190	32	0 7 9 10 11 12 13 14	64	64	4096	131072
191	24	0 1 2 5 10 13 14 15	256	256	65536	1572864
192	32	0 1 2 4 11 13 14 15	64	64	4096	131072
193	96	0 1 2 5 11 12 14 15	64	64	4096	393216
194	48	0 1 2 7 11 12 13 14	16	16	256	12288
195	192	5 7 10 11 12 13 14 15	1237	1237	1530169	293792448
196	384	4 7 10 11 12 13 14 15	580	580	336400	129177600
197B	384	6 7 8 9 11 13 14 15	578	580	335240	128732160
198B	384	6 7 9 10 11 12 13 15	272	290	78880	30289920
199	384	2 6 9 11 12 13 14 15	580	580	336400	129177600
200B	384	6 7 8 9 11 12 14 15	544	580	315520	121159680
201B	384	6 7 9 10 11 12 13 14	256	290	74240	28508160
202	384	3 6 8 11 12 13 14 15	272	272	73984	28409856
203	384	2 7 8 11 12 13 14 15	136	136	18496	7102464
204B	768	6 7 8 9 11 12 13 14	272	290	78880	60579840
205B	384	5 6 9 10 11 12 13 14	256	272	69632	26738688
206B	384	4 7 8 9 10 13 14 15	128	136	17408	6684672
207	384	2 3 7 9 12 13 14 15	272	272	73984	28409856
208B	96	5 6 7 9 10 11 13 14	256	289	73984	7102464
209B	192	5 6 7 9 10 11 12 15	64	68	4352	835584
210	384	2 3 7 8 12 13 14 15	272	272	73984	28409856
211B	384	5 6 7 9 10 11 12 14	128	136	17408	6684672
212B	768	4 6 7 9 10 11 13 14	128	136	17408	13369344
213	384	2 5 7 8 11 12 14 15	64	64	4096	1572864
214B	192	5 6 7 8 9 10 13 14	256	272	69632	13369344
215B	384	5 6 7 8 10 11 12 13	128	136	17408	6684672
216	384	0 6 7 8 11 12 13 15	128	128	16384	6291456
217B	192	5 6 7 8 9 10 12 15	64	68	4352	835584
218B	192	3 5 6 9 10 12 13 14	64	64	4096	786432
219	192	0 6 7 9 11 12 13 14	64	64	4096	786432
220	384	2 4 5 8 11 13 14 15	32	32	1024	393216
221	384	0 6 7 8 9 10 13 15	128	128	16384	6291456
222B	384	3 5 6 8 10 11 12 13	64	64	4096	1572864
Total						160297985276

Table 4. The number of four-variable fuzzy switching functions