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One-step recurrent term in λ-calculus

Sachio HIROKAWA

Department of Information Science,
Faculty of Engineering, Shizuoka University,
Hamamatsu 432, Japan

Abstract.

The notion of a one-step recurrent term is introduced as a weakened notion of the recurrent term [3,4,6]. A recurrent term is a term in λ-β-calculus which is reducible to itself after any reduction. A term is one-step recurrent if and only if every its one-step reductum is reducible to itself. It is obvious that every recurrent term is one-step recurrent. In this paper we prove that the converse is equivalent to one of the conjectures by J.W.Klop [5]. And we also prove that the converse is true for the terms having at most two redexes, i.e., every one-step recurrent term having at most two redexes is recurrent. The proof uses the characterization of the recurrent terms by Böhm and Micali [3].

§1. Klop's conjecture.

We assume the reader acquainted with the basic theory given, for example, in [1], especially chapter 11, or [4] chapter 4. We write $M \rightarrow N$ for terms M and N when M is reducible to N by a one-step reduction, i.e., by a contraction of some redex in M. The symbol "$\twoheadrightarrow$" denotes the transitive reflexive closure of $\rightarrow$. We write $M \twoheadleftarrow\!\!\twoheadrightarrow N$ (cyclic equivalence) when $M \twoheadrightarrow N$ and $M \twoheadrightarrow N$. We call a cyclic equivalence class a plane. The plane containing a term M is represented by [M] . Let $N \in [M]$ be a term such that there is a irreversible step $N \rightarrow L$ for some L (i.e. $L \notin [M]$). Then we will say that we can leave the plane directly (in one step).

Now the original conjecture by J.W.Klop [5] is stated as follows:

(A) If a plane can be left somewhere, then it can be left at any point(Fig.1).

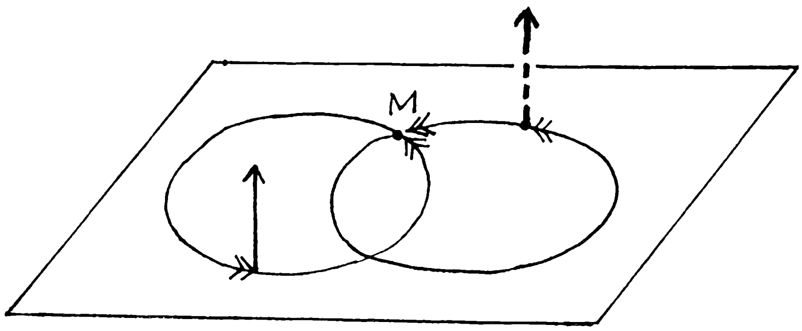


Fig. 1

A term M is recurrent if and only if $\forall N(M \twoheadrightarrow N \Longrightarrow N \twoheadrightarrow M)$, i.e., N may be reduced to M whenever M reduces to N. M is a one-step recurrent term if and only if $\forall N(M \rightarrow N \Longrightarrow N \twoheadrightarrow M)$, i.e., every one-step reductum of M is reducible to M. The conjecture (A) can be restated by using the notion of one-step recurrent terms as follows:

(B) If M and N are cyclically equivalent and M is not one-step recurrent, then N is not one-step recurrent.

The contraposotion of (B) is given by

(C) If M and N are cyclically equivalent and M is one-step recurrent, then N is also one-step recurrent.

In Theorem 1 below we prove still another equivalence (A)$\Leftrightarrow$(D):

(D) M is a recurrent term if and only if M is one-step recurrent.

Theorems 2 and 3 will later show that (D) is true under the condition that M has at most two redexes.

Theorem 1 : (A) is equivalent to (D).

Proof : We prove (C)$\Leftrightarrow$(D).

Let M be a one-step recurrent term and N be a term which is cyclically equivalent to M. Then by (D) M and N are recurrent terms. Therefore N is one-step recurrent, proving (D)$\Rightarrow$(C).

For showing the converse (C)$\Rightarrow$(D), let M be a one-step recurrent term and N be any reductum of M. We now prove the existence of a reduction $N \twoheadrightarrow M$ by induction on the length of the reduction $M \twoheadrightarrow N$. Let the reduction $M \twoheadrightarrow N$ be of length n, say,

$$M \equiv M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_{n-1} \rightarrow M_n \equiv N.$$

Since M_0 is one-step recurrent, M_1 is reducible to M_0. Thus M and N are cyclically equivalent. Therefore, M_1 is one-step recurrent by (C). This argument can be continued until M_{n-1}, and all M_i's are one-step recurrent. So there exists the reduction

$$M \equiv M_0 \twoheadleftarrow M_1 \twoheadleftarrow M_2 \twoheadleftarrow \cdots \twoheadleftarrow M_{n-1} \twoheadleftarrow M_n \equiv N. \qquad \text{Q.E.D.}$$

§2. One-step recurrent term and recurrent term.

To show that (D) is true under the condition that M has at most two redexes, we need to state a result from Böhm and Micali [3]. Given a set $\mathcal{F}$ of redexes in a term M, a development of $(M,\mathcal{F})$ is a reduction path σ in which all redexes contracted in σ are residuals of $\mathcal{F}$, so that the prolongation of the reduction as $(M,\mathcal{F})$-development is not possible. Let G(M) denote the term obtained by a complete development of $(M,\mathcal{F})$, where $\mathcal{F}$ is the set of all redexes in M.

Lemma (Böhm and Micali [3]) : M is a recurrent term if and only if G(M) reduces to M.

Now we have the following results :

Theorem 2 : Any one-step recurrent term having only one redex is recurrent.

Proof : Let M be one -step recurrent term which has only one redex. Since M has only one redex in it , its one-step reductum and G(M) agree precisely. M is one-step recurrent, therefore G(M) reduces to M. Thus M is a recurrent term by the lemma.

Q.E.D.

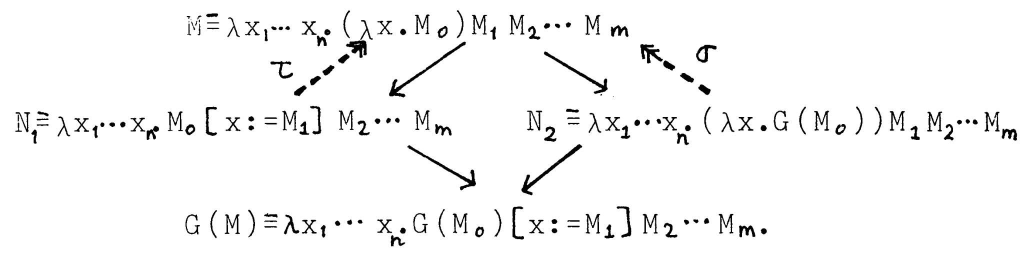
Theorem 3 : Any one-step recurrent term having exactly two redexes is recurrent.

Proof : Let M be an arbitrary one-step recurrent term which has exactly two redexes in it. Applying the lemma , it suffices to show that G(M) reduces to M. We prove it by induction on the structure of M.

Case 1 : M is in head normal form , i.e. , $M=\lambda x_1 \cdots x_n . x_i M_1 M_2 \cdots M_m$. Then every M_i is one-step recurrent and has at most two redexes. So it is recurrent from Theorem 2 and induction hypothesis. Thus M is recurrent.

Case 2 : M is not in head normal form , i.e. , $M \equiv \lambda x_1 \cdots x_n (\lambda x . M_0) M_1 M_2 \cdots M_m (m \geqslant 1)$.

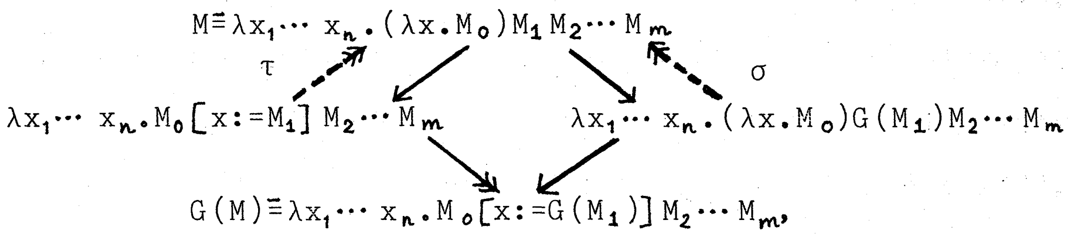
Case 2a : M_0 has one redex. Then all M_i's are in normal form, so that $G(M) \equiv \lambda x_1 \cdots x_n . G(M_0)[x:=M_1] M_2 \cdots M_m$, where $G(M_0)[x:=M_1]$ represents the term obtained from $G(M_0)$ by substituting every free occurrence of x by M_1. Since M is one-step recurrent, one-step reductums, $N_1 \equiv \lambda x_1 \cdots x_n M_0 [x:=M_1] M_2 \cdots M_m$ and $N_2 \equiv \lambda x_1 \cdots x_n (\lambda x . G(M_0)) M_1 M_2 \cdots M_m$, of M have a reduction path which ends at M. Let σ and τ be such reductions :



If the residual of the left most redex $(\lambda x.G(M_0))M_1$ in N_2 is contracte in σ, then we can construct from σ another reduction path from N_2 to M which goes through G(M). Thus G(M) reduces to M. On the other hand, if σ does not contract the residual of $(\lambda x.G(M_0))M_1$, then there exists a "subreduction" in σ which reduces $G(M_0)$ to M_0. Hence we have the reduction :

$$G(M) \equiv \lambda x_1 \cdots x_n.G(M_0)[x:=M_1]\, M_2 \cdots M_m \twoheadrightarrow \lambda x_1 \cdots x_n.M_0[x:=M_1]\, M_2 \cdots M_m \overset{\tau}{\twoheadrightarrow} M.$$

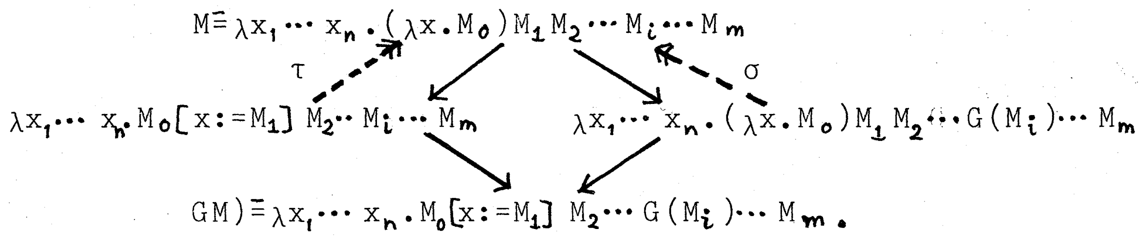
<u>Case 2b</u> : M_1 has one redex. Then we have the diagram :



where the dotted arrows have an existential meaning. If σ contracts the residual of $(\lambda x.M_0)G(M_1)$, then there is a reduction which is strongly equivalent to σ and goes through G(M). Therefore G(M) reduces to M. On the other hand, if σ does not contract the residual of $(\lambda x.M_0)G(M_1)$, then σ contains a subreduction which reduces $G(M_1)$ to M_1. Thus we have the reduction :

$$G(M) \equiv \lambda x_1 \cdots x_n.M_0[x:=G(M_1)]\, M_2 \cdots M_m \twoheadrightarrow \lambda x_1 \cdots x_n.M_0[x:=M_1]M_2 \cdots M_m \overset{\tau}{\twoheadrightarrow} M.$$

<u>Case 2c</u> : M_i has one redex for some i ($2 \leq i \leq m$). Then we have the diagram :



We can prove the existence of a reduction $G(M) \twoheadrightarrow M$ in the same manner as in Case 2b. Q.E.D.

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