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On Regular Factors

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In this paper we consider graphs which may have loops and multiple edges. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively. The degree of a vertex x of G is denote by $d_G(x)$. Let S,T be disjoint subsets of $V(G)$. $e_G(S,T)$ is the number of edges which join vertices of S and T, and $e_G(S)$ is the number of edges which join vertices of S. The other notations which are not explained explicitly are due to [3].

Let f be an integer-valued function defined on $V(G)$. A spanning subgraph H of G is called an f-factor of G if $d_H(x)=f(x)$ for any $x\in V(G)$. If f is the constant function taking the value k, the f-factor is said to be a k-factor or a k-regular-factor.

In this paper we shall give a sufficient condition for the existence of regular factors of regular graphs. This condition are related to the edge-connectivity of graphs. Such a research goes back to Petersen [4]. Various results are obtained by Petersen, Gallai, Plesnik et al., which are listed below. The main theorem in this paper is an extension of Proposition E (Plesnik [6]).

Proposition A. (Petersen [4]) *If G is $2k$-regular ($k \geqq 1$), G has an l-factor for any even integer l such that $0 \leqq l \leqq 2k$.* □

Proposition B. (Petersen [4]) *If G is 3-regular and 2-edge-connected, G has a 2-factor and a 1-factor.* □

Proposition C. (Berge [1], Plesnik [5]) *If G is r-regular, $(r-1)$-edge-connected and $|G|$ is even, then G has a 1-factor and an $(r-1)$-factor.* □

Proposition D. (Gallai [2], Plesnik [6]) *Let G be a $2k$-regular ($k \geqq 1$) and a-edge-connected graph. Then G has an l-factor for any odd integer l satisfying $\frac{1}{a}2k \leqq l \leqq \frac{a-1}{a}2k$.* □

Proposition E. (Plesnik [6]) *Let G be a $(2k+1)$-regular ($k \geqq 1$) and a-edge-connected graph. Then G has an l-factor for any even integer l satisfying $0 \leqq l \leqq \frac{a-1}{a}(2k+1)$.* □

Remark that the edge-connectivity of an even-regular graph is even. Hence when r is even, Proposition C is a special case of Proposition D. But if r is odd, Proposition C (and Proposition B) cannot be deduced from Proposition E.

Tutte obtained a criterion for the existence of an f-factor. We use it to obtain our theorems.

Lemma 1. (Tutte [7,8,9]) *A graph G has an f-factor if and only if*

$$h(S,T)+\sum_{t\in T}\{f(t)-d_{G-S}(t)\}\leqq\sum_{s\in S}f(s) \qquad (1)$$

for any disjoint subsets S,T of $V(G)$, where $h(S,T)$ is the number of components C of $G-(S\cup T)$ such that

$$\sum_{c\in V(C)}f(c)+e_G(V(C),T)\equiv 1 \pmod 2. \ \square$$

Here we use (1) in the following form.

$$h(S,T)+e_G(S,T)\leqq\sum_{s\in S}f(s)+\sum_{t\in T}d_G(t)-\sum_{t\in T}f(t). \qquad (1)'$$

First we improve the evaluation in Proposition E.

Theorem 2. *Let G be a $(2k+1)$-regular ($k\geqq 1$) and a-edge-connected graph. If a is even, G has an l-factor for any even integer l satisfying $0\leqq l\leqq \dfrac{a}{a+1}(2k+1)$.*

To prove theorem 2, we need some notations and lemmas.

Let G be a $(2k+1)$-regular graph. For $S,T\subset V(G)$ $(S\cap T=\phi)$ and an integer l, define $\delta(S,T;l)$ by:

$$\delta(S,T;l)=l|S|+(2k-l+1)|T|-h(S,T)-e_G(S,T).$$

Obviously G has an l-factor if and only if $\delta(S,T;l)\geqq 0$ for any $S,T\subset V(G)$ such that $S\cap T=\phi$. Let $H(S,T)$ be the set of components C of $G-(S\cup T)$ such that $e_G(V(C),T)\equiv 1 \pmod 2$. Suppose l be an even integer. Then $h(S,T)=|H(S,T)|$. Let V_0 be a set of vertices of $G-(S\cup T)$ which do not belong to components of $H(S,T)$. Define m_1,m_2,n_1,n_2 and N by $m_1=e_G(V_0,S)$, $m_2=e_G(S)$, $n_1=e_G(V_0,T)$, $n_2=e_G(T)$ and $N=e_G(S,T)$.

Remark that n_1 is even. We shall write $H(S,T)=\{C_1,\ldots,C_r\}$ $(r=h(S,T))$ and set $s_i=e_G(V(C_i),S)$ and $t_i=e_G(V(C_i),T)$.

Lemma 3.

$$(2k+1)\delta(S,T;l)=lm_1+2lm_2+(2k-l+1)n_1+2(2k-l+1)n_2$$
$$+\sum_{i=1}^{r}(ls_i+(2k-l+1)t_i-(2k+1)).$$

Proof. Considering the sum of degrees of the vertices in S,

$$(2k+1)|S|=\sum_{i=1}^{r}s_i+m_1+2m_2+N.$$

Similarly,

$$(2k+1)|T|=\sum_{i=1}^{r}t_i+n_1+2n_2+N.$$

From these two equations and the definition of $\delta(S,T;l)$, we obtain Lemma 3.

□

Proof of Theorem 2.

If $0\leqq l\leqq \frac{a}{a+1}(2k+1)$, $lm_1+2lm_2+(2k-l+1)n_1+2(2k-l+1)n_2\geqq 0$. In case $S\cup T=\phi$, it is obvious that $(2k+1)\delta(S,T;l)\geqq 0$. We shall show that $\psi(s_i,t_i)\geqq 0$ if $S\cup T\neq\phi$, where $\psi(s_i,t_i)=ls_i+(2k-l+1)t_i-(2k+1)$. From the assumptions, we obtain the following.

$$s_i\geqq 0. \tag{2}$$

$$t_i\equiv 1 \pmod 2,\ \text{especially}\ t_i\geqq 1. \tag{3}$$

$$s_i+t_i\geqq a. \tag{4}$$

If $s_i\geqq 1$, $\psi(s_i,t_i)\geqq l+(2k-l+1)-(2k+1)=0$, since $t_i\geqq 1$. Hence we may confine ourselves to the case $s_i=0$. Assume $(2k-l+1)t_i-(2k+1)<0$, then

$$t_i < \frac{2k+1}{2k-l+1} \leqq a+1,$$

since $0 \leqq l \leqq \frac{a}{a+1}(2k+1)$. Hence $t_i \leqq a$. But since t_i is odd and a is even, $t_i \leqq a-1$. This contradicts (4). Therefore we obtain $\delta(S, T; l) \geqq 0$ and G has an l-factor.

□

Next we assure that the evaluation of theorem 2 is the best.

Theorem 4. *For any integer k and even integer a satisfying $0 \leqq a \leqq 2k+1$, there exists a $(2k+1)$-regular and a-edge-connected graph which does not have an l-factor for any even integer l satisfying $l > \frac{a}{a+1}(2k+1)$.*

Proof. We shall construct a graph $G(a, k)$ which has the required property. $G(a, k)$ has subgraphs $H_i (1 \leqq i \leqq 2k+1)$ and $J_j (1 \leqq j \leqq a+1)$. First we define H_i and J_j and then construct $G(a, k)$.

(i) H_i $(1 \leqq i \leqq 2k+1)$.

H_i has $(2k+3)$ vertices, say

$$V(H_i) = \{x_{i,1}, \ldots, x_{i,2k+3}\}.$$

Let $\tilde{H}_i$ be a complete graph with the vertex set $V(H_i)$ and P_i be a path defined by $P_i = x_{i,1} x_{i,2} \cdots x_{i,a+2}$. We define $E(H_i)$ as follows.

$$E(H_i) = E(\tilde{H}_i) - E(P_i) - x_{i,a+3}x_{i,a+4} - x_{i,a+5}x_{i,a+6} - \cdots - x_{i,2k+1}x_{i,2k+2} - x_{i,2k+3}x_{i,1}.$$

H_i has the following properties.

$$d_{H_i}(x_{i,j}) = \begin{cases} 2k & 1 \leqq j \leqq a+1 \\ 2k+1 & j \geqq a+2 \end{cases}$$

H_i is $2k$-edge-connected.

(ii) $J_j (1 \leqq j \leqq a+1)$

J_j is a complete bipartite graph with partite sets

$$Y_j = \{y_{j,1}, \ldots, y_{j,2k+1}\} \text{ and } Z_j = \{z_{j,1}, \ldots, z_{j,2k}\}.$$

Obviously $d_{J_j}(y_{j,l}) = 2k$, $d_{J_j}(z_{j,l}) = 2k+1$ and J_j is $2k$-edge-connected.

(iii) Joining $x_{i,j}$ in $V(H_i)$ with $y_{j,i}$ in Y_j $(1 \leqq i \leqq 2k+1, 1 \leqq j \leqq a+1)$, we obtain $G(a,k)$. Obviously $G(a,k)$ is $(2k+1)$-regular and $\min\{a+1, 2k\}$-edge-connected. Hence G is a-edge-connected (since $0 \leqq a \leqq 2k+1$ and a is even).

Let $S = \bigcup_{j=1}^{a+1} Z_j$, $T = \bigcup_{j=1}^{a+1} Y_j$. We shall show $\delta(S,T;l) < 0$ if $l > \frac{a}{a+1}(2k+1)$ and l is even. In this case,

$$H(S,T) = \{H_1, \ldots, H_{2k+1}\}$$
$$m_1 = m_2 = n_1 = n_2 = 0, s_i = 0, t_i = a+1$$

So by Lemma 9, $\delta(S,T;l) = (2k-l+1)(a+1) - (2k+1)$. It is easy to show that $(2k-l+1)(a+1) - (2k+1) < 0$ if $l > \frac{a}{a+1}(2k+1)$. Therefore G does not have an l-factor. □

If a is odd, the bound of Proposition E is the best.

Theorem 5. *For any integer k and odd integer a satisfying $1 \leqq a \leqq 2k+1$, there exists a $(2k+1)$-regular and a-edge-connected graph which does not have an l-factor for any even integer l satisfying $l > \frac{a-1}{a}(2k+1)$.*

Proof. If $a = 2k+1$, then $l > \frac{a-1}{a}(2k+1) = 2k$ and the theorem follows. We consider the case $a \leqq 2k-1$. Since a is odd, $G(a-1,k)$, defined in the proof of theorem 9, is $2k$-regular, a-edge-connected and does not have an l-factor for any even integer l satisfying $l > \frac{a-1}{a}(2k+1)$. □

When the regularity of graphs is even and the edge-connectivity is odd, Proposition D gives the best bound.

Theorem 6. *For any integer k and any even integer a satisfying $1 \leqq a \leqq 2k$, there exists a $2k$-regular and a-edge-connected graph G with even number of vertices which does not have an l-factor for any odd integer l such that $l < \frac{1}{a}2k$ or $l > \frac{a-1}{a}2k$.*

Proof. If a $2k$-regular graph G has an l-factor H for some l such that $l > \frac{a-1}{a}2k$, $G-H$ is a $(2k-l)(< \frac{1}{a}2k)$-factor of G. Hence it suffices to show that some graph G does not have an l-factor for any l such that $l < \frac{1}{a}2k$. Clearly the result follows if $a=2k$. If $a \leqq 2k-2$, we construct a graph $G'(a,k)$ as follows. $G'(a,k)$ has $H'_i(a \leqq i \leqq 2k)$ and $J'_j(1 \leqq j \leqq a)$ as subgraphs.

(i) H'_i $(1 \leqq i \leqq 2k)$

$V(H'_i) = \{x_{i,1}, \ldots, x_{i,2k+1}\}$. Let $\tilde{H}'_i$ be a complete graph with the vertex set $V(H'_i)$. Then we define $E(H'_i)$ by

$$E(H'_i) = E(\tilde{H}'_i) - x_{i,1}x_{i,2} - x_{1,3}x_{1,4} - \cdots - x_{1,a-1}x_{i,a}.$$

(ii) $J'_j(1 \leqq j \leqq a)$

J'_j is a complete bipartite graph with partite sets $Y'_j = \{y_{j,1}, \ldots, y_{j,2k}\}$ and $Z'_j = \{z_{j,1}, \ldots, z_{j,2k-1}\}$.

(iii) Combining $x_{i,j}$ of $V(H_i)$ with $y_{j,i}$ of Y'_j $(1 \leqq i \leqq 2k, 1 \leqq j \leqq a)$, we obtain $G'(a,k)$. Clearly $|G|$ is even and it is easy to show that G is $2k$-regular and a-edge-connected. Set

$$\delta'(S,T;l) = l|S| + (2k-l)|T| - h(S,T) - e_G(S,T)$$

for $S, T \subset V(G)$ $(S \cap T = \phi)$. If $\delta'(S,T;l) < 0$ for some disjoint subsets S, T of $V(G)$, then G does not have an l-factor. If we set $S = \bigcup_{j=1}^{a} Y'_j$ and $T = \bigcup_{j=1}^{a} Z'_j$, we obtain $\delta(S,T;l) = al - 2k$ in the same way as in the proof of theorem 9, and $al - 2k < 0$ if $l < \frac{1}{a}2k$. Hence G does not have an l-factor for any even integer l such that $l < \frac{1}{a}2k$ or $l > \frac{a-1}{a}2k$. □

From Propositions A,D,E, Theorems 2,4,5,6, the best possible bounds of regularity of regular factors in regular graphs are obtained. In these theorems, only the regularity and edge-connectivity of graphs are given. If more properties about graphs are assumed, better bounds can be obtained. For example, graphs which are not 2-edge-connected can have 1-factors and 2-factors.

Proposition F. (Berge [1]) *If G is 3-regular and all of bridges of G are on the same elementary path, then G has a 1-factor and 2-factor.*

We extend the above theorem.

Theorem 7. *Let G be a $(2k+1)$-regular and $(a-1)$-edge-connected graph $(a \geqq 1)$. Suppose there are disjoint subsets $S_1, \ldots, S_r$ of $V(G)$ satisfying:*

$$V(G) = \bigcup_{i=1}^{r} S_i$$

$$<S_i>_G \text{ is } a-edge-connected,$$

$$e_G(S_i, S_{i+1}) = a-1 \quad (1 \leqq i \leqq r-1),$$
$$e_G(S_i, S_j) = 0 \text{ if } |i-j|>1.$$

If a is even, then G has an l-factor for any even integer l satisfying $0 \leqq l \leqq \frac{a}{a+1}(2k+1)$ (cf. Theorem 2).

Proof.

Suppose G does not have an l-factor for some even integer l such that $0 \leqq l \leqq \frac{a}{a+1}(2k+1)$. Then there exist some $S, T \subset V(G)$, $S \cap T = \phi$, such that $\delta(S, T; l) < 0$. Hence by Lemma 3, $\sum_{i=1}^{r} \psi(s_i, t_i) < 0$. From the assumptions of this theorem at most two pairs (s_i, t_i) satisfy the condition $s_i + t_i = a-1$ and $s_j + t_j \geqq a$ for the other pairs (s_j, t_j). If $s_j + t_j \geqq a$, $\psi(s_j, t_j) \geqq 0$. (See the proof of Theorem 7.). Hence we may assume $\psi(s_j, t_j) \geqq 0$ $(1 \leqq j \leqq r-2)$, $s_r + t_r = a-1$ and $\psi(s_r, t_r) < 0$. Then $s_r = 0$ and $t_r = a-1$. If $\psi(s_{r-1}, t_{r-1}) \geqq 0$,

$$\begin{aligned} 0 > (2k+1)\delta(S, T; l) \geqq \psi(s_r, t_r) &= (2k-l+1)(a-1)-(2k+1) \\ &> 2k+1, \end{aligned}$$

since $2k-l+1, a-1>0$. This is a contradiction, because $\delta(S, T; l)$ is an integer. Thus $\psi(s_{r-1}, t_{r-1}) < 0$, $s_{r-1} + t_{r-1} = a-1$ and $\psi(s_{r-1}, t_{r-1}) = \psi(s_r, t_r) = (2k-l+1)(a-1)-(2k+1)$. Therefore

$$0 > (2k+1)\delta(S, T; l) > -2(2k+1)$$

and $\delta(S, T; l) = -1$. On the other hand, $\psi(s_i, t_i) \equiv 0 \pmod 2$, since l is even and t_i is odd. Hence $\delta(S, T; l) \equiv n_1 \pmod 2$. But by the Definition of n_1, n_1 is an even integer. This is a contradiction. □

By the same argument the following theorem can be obtained.

Theorem 8. *Let G be a $(2k+1)$-regular and $(a-1)$-edge-connected graph $(a \geqq 1)$. $\{S_1, \ldots, S_r\}$ is the partition of $V(G)$ which has the same properties as in the previous theorem. If a is odd, then G has an l-factor for any even integer l satisfying* $0 \leqq l \leqq \frac{a-1}{a}(2k+1)$ *(cf. Proposition E).* □

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