16

A coupled KdV equation is one case of the four-reduction of the KP hierarchy

Junkichi SATSUMA (Miyazaki Medical College)
薩摩　順吉
Ryogo HIROTA (Hiroshima University)
広田　良吾

It is shown that the coupled KdV equation introduced by the present authors is a special case of the four-reduced KP hierarchy which is included in the general theory of τ functions. From the fact it is also shown that the soliton solutions can be derived from those of the KP equation.

1. Four-reduction of the KP hierarchy

Recently Sato et al. developed the general theory of τ functions.[1,2] Their theory starts with a system of linear equations for an eigenfunction $\psi(x,\lambda)$,

$$L(x,\partial)\psi(x,\lambda) = \lambda\psi(x,\lambda), \tag{1}$$

$$\frac{\partial}{\partial x_n}\psi(x,\lambda) = B_n(x,\partial)\psi(x,\lambda), \quad n = 1,2,\cdots. \tag{2}$$

The pseudo-differential operator L is defined by

$$L(x,\partial) = \partial + u_2(x)\partial^{-1} + u_3(x)\partial^{-2} + u_4(x)\partial^{-3} + \cdots, \tag{3}$$

where $u_n(x)$ are functions in infinitely many time variables $x = (x_1,x_2,x_3,x_4,\cdots)$ and ∂ denotes $\partial/\partial x_1$. The operators B_n are defined as the differential part of $L(x,\partial)^n$. For example,

$$B_1 = \partial, \tag{4a}$$

$$B_2 = \partial^2 + 2u_2, \tag{4b}$$

$$B_3 = \partial^3 + 3u_2\partial + 3u_3 + 3u_{2,x}, \tag{4c}$$

$$B_4 = \partial^4 + 4u_2\partial^2 + (4u_3 + 6u_{2,x})\partial + 4u_4 + 6u_{3,x}$$

$$+ 4u_{2,xx} + 6u_2^2, \tag{4d}$$

where subscript x denotes partial differentiation.

From the compatibility condition of (1) and (2), we have a series of nonlinear partial differential equations,

$$\frac{\partial L}{\partial x_n} = [B_n, L] \equiv B_n L - LB_n, \tag{5}$$

or equivalently,

$$\frac{\partial B_m}{\partial x_n} - \frac{\partial B_n}{\partial x_m} = [B_n, B_m]. \tag{6}$$

For $n = 2$ and $m = 3$, as a nontrivial simple case, (6) gives the Kadomtsev-Petviashvili (KP) equation,

$$\frac{\partial}{\partial x_1} \left(\frac{\partial u_2}{\partial x_3} - \frac{1}{4} \frac{\partial^3 u_2}{\partial x_1^3} - 3u_2 \frac{\partial u_2}{\partial x_1} \right) - \frac{3}{4} \frac{\partial^2 u_2}{\partial x_2^2} = 0. \tag{7}$$

Hence the series of equations is called the KP hierarchy.

If we demand that u_n ($n = 1,2,3,\cdots$) do not depend on $x_\ell, x_{2\ell}, x_{3\ell}, \cdots$ for a positive integer ℓ, the system of equations is called the ℓ-reduction of the KP hierarchy. In this case, $\psi(x,\lambda)$ satisfies

$$\frac{\partial}{\partial x_\ell} \psi(x,\lambda) = \lambda^\ell \psi(x,\lambda), \tag{8}$$

or

$$B_\ell(x,\partial)\psi(x,\lambda) = \lambda^\ell \psi(x,\lambda). \tag{9}$$

18

Here we study a simple case of the four-reduction. Let us consider the following system of linear equations;

$$\frac{\partial}{\partial y}\,\psi = B_2\psi, \tag{10}$$

$$\frac{\partial}{\partial t}\,\psi = B_3\psi, \tag{11}$$

$$B_4\psi = \lambda^4\psi, \tag{12}$$

where we have changed the variables as $x_1 = x$, $x_2 = y$ and $x_3 = t$ for convenience. The compatibility condition of (10) and (11) gives the KP equation. From (11) and (12) we have

$$u_{2,t} = u_{2,xxx} + 3u_{3,xx} + 3u_{4,x} + 6u_2 u_{2,x}, \tag{13a}$$

$$u_{3,t} = -\frac{3}{4}\,u_{2,xxxx} - 2u_{3,xxx} - \frac{3}{2}\,u_{4,xx} - \frac{9}{2}\,u_2 u_{2,xx}$$
$$- \frac{9}{2}\,u_{2,x}{}^2 - 3u_2 u_{3,x} - 3u_{2,x}u_3, \tag{13b}$$

$$u_{4,t} = \frac{3}{8}\,u_{2,xxxxx} + \frac{3}{4}\,u_{3,xxxx} + \frac{1}{4}\,u_{4,xxx} + \frac{3}{4}\,u_2 u_{2,xxx}$$
$$+ \frac{27}{4}\,u_{2,x}u_{2,xx} - 3u_2 u_{3,xx} + \frac{9}{2}\,u_{2,x}u_{3,x}$$
$$+ \frac{3}{2}\,u_{2,xx}u_3 - 6u_2 u_{4,x} - 3u_3 u_{3,x} - 9u_2^2 u_{2,x}. \tag{13c}$$

Introducing dependent variables u, ϕ, ω as

$$u_2 = u, \tag{14a}$$

$$u_3 = -\frac{1}{2}\,u_x + \phi_x, \tag{14b}$$

$$u_4 = \frac{1}{4}\,u_{xx} - \frac{1}{2}\,u^2 - \phi_{xx} - \phi^2 + \omega, \tag{14c}$$

(13) may be rewritten as

$$u_t = \frac{1}{4}\,u_{xxx} + 3uu_x + 3(-\phi^2 + \omega)_x, \tag{15a}$$

$$\phi_t = -\frac{1}{2}\phi_{xxx} - 3u\phi_x, \qquad\qquad (15b)$$

$$\omega_t = -\frac{1}{2}\omega_{xxx} - 3u\omega_x. \qquad\qquad (15c)$$

If we take $\omega = 0$ in (15), it reduces to a coupled KdV equation,

$$u_t = \frac{1}{4}u_{xxx} + 3uu_x - 6\phi\phi_x, \qquad\qquad (16a)$$

$$\phi_t = -\frac{1}{2}\phi_{xxx} - 3u\phi_x, \qquad\qquad (16b)$$

on which we have shown the existence of the three soliton solution and five nontrivial conserved quantities.[3] This equation describes an interaction of two long waves with different dispersion relations.

From the viewpoint of inverse scattering transform, the system of linear equations (11) and (12) might be enough to solve an initial value problem of the coupled KdV equation. For the purpose of obtaining its soliton solutions, however, the general theory of τ functions is very powerful, since the solutions are given as a specialization of the soliton solutions of the KP equation.

In order to investigate the relationship between solutions of the coupled KdV and the KP equations, we may employ the compatibility condition of (10) and (12). After substituting (14), we find from (10) and (12),

$$u_y = 2\phi_{xx}, \qquad\qquad (17a)$$

$$\phi_y = -2\phi^2 + \omega^2, \qquad\qquad (17b)$$

$$\phi_{yy} + \phi_{xxxx} + 8u\phi_{xx} + 4u_x\phi_x = 0. \qquad\qquad (17c)$$

It is easily shown that (17c) is compatible with (15), (17a) and (17b). Substituting (17a) and (17b) into (15a) yields the KP equation,

$$u_{xt} - \frac{1}{4} u_{xxxx} - 3(uu_x)_x - \frac{3}{4} u_{yy} = 0. \tag{18}$$

Thus we can conclude that the coupled KdV equation is a special case of the four-reduction of the KP equation and its solutions may be derived from those of the KP equation.

2. Soliton solutions of the coupled KdV equation

Here we derive several soliton solutions of the coupled KdV equation. For the purpose, it is most convenient to rewrite the equation in the bilinear form.

Introducing the dependent variable transformation,

$$u = (\log f)_{xx}, \tag{19a}$$

$$\phi = g/f, \tag{19b}$$

$$\omega = h/f, \tag{19c}$$

(15) may be reduced to their bilinear forms,

$$(D_x D_t - \frac{1}{4} D_x^4) f \cdot f = 6(-g^2 + fh), \tag{20a}$$

$$(D_t + \frac{1}{2} D_x^3) f \cdot g = 0, \tag{20b}$$

$$(D_t + \frac{1}{2} D_x^3) f \cdot h = 0, \tag{20c}$$

where we have used the binary operators defined by

$$D_i^n D_j^m a \cdot b \equiv (\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i'})^n (\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j'})^m a(x) b(x') \Big|_{x=x'}. \tag{21}$$

If h is taken to be zero, (20a) and (20b) becomes essentially

equivalent with the bilinear form of the coupled KdV equation in Ref.3. The subsidiary conditions (17a) and (17b) are simply written as

$$f_y = 2g, \tag{22a}$$

$$g_y = 2h. \tag{22b}$$

Substituting (22) into (20) yields

$$(D_x D_t - \frac{1}{4} D_x^4 - \frac{3}{4} D_y^2) f \cdot f = 0, \tag{23a}$$

$$D_y (D_t + \frac{1}{2} D_x^3) f \cdot f = 0, \tag{23b}$$

$$\frac{\partial}{\partial y} [D_y (D_t + \frac{1}{2} D_x^3) f \cdot f] = 0, \tag{23c}$$

the first of which is the KP equation itself and the second is one of the KP hierarchy.

Let us consider the soliton solutions of the KP equation. The N-soliton solution is written as

$$f_N = 1 + \sum_{j=1}^{N} a_j e^{\xi_j} + \sum_{1 \le i < j \le N} c_{ij} a_i a_j e^{\xi_i + \xi_j} +$$

$$+ \cdots + (\prod_{1 \le i < j \le N} c_{ij}) \prod_{j=1}^{N} a_j e^{\xi_1 + \xi_2 + \cdots + \xi_N}, \tag{24}$$

where

$$\xi_j = (p_j - q_j)x + (p_j^2 - q_j^2)y + (p_j^3 - q_j^3)t \tag{25}$$

and

$$c_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i - q_j)(q_i - p_j)} . \tag{26}$$

Although the N-soliton solution from the general theory contains infinitely many time variables , we include only the

terms of x, y and t in (25) since we are interested in the solutions of two of the four-reduced KP hierarchy, i.e. (23a) and (23b). The four-reduction requires

$$p_j^4 = q_j^4 \qquad\qquad (27)$$

in the solution.

There are three nontrivial cases in choosing the form of the phase factor ξ_j;

1) for $p_j = -q_j$, $\xi_j = P_j x + \frac{1}{4} P_j^3 t$ $(P_j \equiv 2p_j)$, $\qquad$ (28a)

2) for $p_j = iq_j$, $\xi_j = P_j x - iP_j^2 y - \frac{1}{2} P_j^3 t$ $(P_j \equiv (1+i)p_j)$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (28b)$$

3) for $p_j = -iq_j$, $\xi_j = P_j x + iP_j^2 y - \frac{1}{2} P_j^3 t$ $(P_j \equiv (1-i)p_j)$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (28c)$$

Consider the one-soliton solution,

$$f = 1 + a_1 \exp \xi_1. \qquad\qquad (29)$$

If we choose the case 1), (29) reduces to

$$f = 1 + a_1 \exp(P_1 x + \frac{1}{4} P_1^3 t). \qquad\qquad (30)$$

From the conditions (22), we have g = h = 0, i.e. $\phi = \omega = 0$, in this case. Hence this solution corresponds to the KdV soliton of the system of equations (15), which is explicitly written as

$$u = \frac{P^2}{4} \text{ sech}^2 \frac{1}{2}(P_1 x + \frac{1}{4} P_1^3 t) \qquad\qquad (31)$$

for $a_1 = 1$. Secondly, if we choose the case 2) or 3), we obtain

$$f = 1 + a_1 \exp \eta_1, \qquad\qquad (32a)$$

$$g = \frac{1}{2} ia_1 P_1^2 \exp \eta_1, \tag{32b}$$

$$h = -\frac{1}{4} a_1 P_1^4 \exp \eta_1, \tag{32c}$$

where $\eta_1 = P_1 x \mp i P_1^2 y - \frac{1}{2} P_1^3 t$. By taking $y = 0$, this solution gives a steady solution of (15). However, since we can not choose $h = 0$ with $g \neq 0$, this does not give a solution of the coupled KdV equation (16).

Next we consider the two-soliton solution of the KP equation,

$$f = 1 + a_1 \exp \xi_1 + a_2 \exp \xi_2 + a_1 a_2 c_{12} \exp(\xi_1 + \xi_2). \tag{33}$$

Choosing ξ_1 and ξ_2 as (28a), we have

$$f = 1 + a_1 e^{\eta_1} + a_2 e^{\eta_2} + \left(\frac{P_1 - P_2}{P_1 + P_2}\right)^2 a_1 a_2 e^{\eta_1 + \eta_2}, \tag{34a}$$

$$g = h = 0, \tag{34b}$$

$$\eta_j = P_j x + \frac{1}{4} P_j^3 t, \tag{34c}$$

which correspond to the KdV two-soliton solution. By choosing ξ_1 as (28b) and ξ_2 as (28c), and taking $P_1 = P_2$, $a_1 = -a_2 = \sqrt{2}i$, (33) reduces to

$$f = 1 + 2\sqrt{2} \sin P_1^2 y \, e^{\eta_1} + e^{2\eta_1}, \tag{35a}$$

$$\eta_1 = P_1 x - \frac{1}{2} P_1^3 t, \tag{35b}$$

which yields

$$g = \sqrt{2} P_1^2 \cos P_1^2 y \, e^{\eta_1}, \tag{36a}$$

$$h = -\frac{\sqrt{2}}{2} P_1^4 \sin P_1^2 y \, e^{\eta_1}. \tag{36b}$$

For y = 0, h becomes zero and (35a) and (36a) give another soliton (cKdV soliton) solution of (16),

$$u = \frac{P^2}{2} \operatorname{sech}^2(P_1 x - \frac{1}{2} P_1^3 t), \tag{37a}$$

$$\phi = \frac{\sqrt{2}}{2} P_1^2 \operatorname{sech}(P_1 x - \frac{1}{2} P_1^3 t). \tag{37b}$$

It is noted that, if we take $a_1 = a_2 = 1$ and $y = 0$ in the above procedure, we obtain the one-soliton solution of a coupled equation,

$$u_t = \frac{1}{4} u_{xxx} + 3uu_x + 3\omega_x, \tag{38a}$$

$$\omega_t = -\frac{1}{2} \omega_{xxx} - 3u\omega_x. \tag{38b}$$

These procedure in obtaining solutions of (15) can be extended to the N-soliton state. From (24) we recover the KdV N-soliton solution by choosing ξ_j (j=1,2,$\cdots$,N) as (28a). If we take ξ_j as (28b) and ξ_{N+j} as (28c) for j = 1,2,$\cdots$,N, put $P_j = P_{j+N}$ and choose a_j properly in the KP 2N-soliton solution, we obtain an N-soliton solution describing a multiple collision of the cKdV solitons. Moreover, it is also possible to get solutions showing interactions of the KdV solitons, (31), and the cKdV solitons, (37). For example, in the three-soliton solution of the KP equation, taking ξ_1 as (28a), ξ_2 as (28b) and ξ_3 as (28c), putting $P_2 = P_3^*$, $a_2 = -a_3 = i$, and then choosing y = 0, we obtain a solution describing a head-on collision of the KdV and the cKdV solitons.

References

1) M.Sato and Y.Sato (Mori): RIMS Kokyuroku 388 (1980)183; 414 (1981)181.
2) E.Date, M.Jimbo, M.Kashiwara and T.Miwa: RIMS preprint 394 (1982).
3) R.Hirota and J.Satsuma: Phys. Lett. 85A (1981) 407.