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A note on reflexive algebras

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Introduction. Let A be an algebra over a field k. Then the dual coalgebra A^o is coreflexive when A is weakly reflexive. The converse of this proposition is not true in general. In fact, if A is left almost noetherian, then A^o is always coreflexive, and there exists a left almost noetherian algebra which is not weakly reflexive. In this paper, we consider the condition uncer which the conver is true.

Further, D. E. Radford showed that a left(or right) almost noetherian algebra with cofinite Jacobson radical is reflexive if and only if it is complete with respect to the radical-adic topology. We will characterize the reflexive algebras with cofinite Jacobson radical by this proposition.

We use the notations and terminology of [1], [4] and [5].

1. Topology of an algebra. Let A be an algebra over a fixed field k. Then the set of cofinite (two-sided) ideals forms a filterbase and we can define a uniform topology on A. In the following, we suppose that all algebras have the topologies of this type. In particular, k is a discrete space and so A^{o} is the set of continuous k-linear mappings of A to k. By Lemma 6.1.0 in [4], the kernel of the canonical mapping $A \to A^{**} \longrightarrow A^{o*}$ is the intersection of all cofinite ideals of A we get

Lemma 1. A is a proper algebra if and only if A is a Hausdorff space. Hence for an ideal I of A, A/I is proper if and only if I is closed.

We denotes the closur of a subset X of A by $\overline{X}$. For an ideal I of A $\overline{I}$ is the intersection of cofinite ideals containing I and so $\overline{I} = (I^{\perp} \cap A^{o})^{\perp}$.

Lemma 2. Let X and Y be subspaces of A. Then

$$(XY)^{\perp} \cap A^{o} = (X^{\perp} \cap A^{o}) \wedge (Y^{\perp} \cap A^{o}),$$

where $\wedge$ is the wedge of A°.

Proof. Let m be the multiplication mapping of A. Then

$$(X^{\perp} \cap A^{\circ}) \wedge (Y^{\perp} \cap A^{\circ})$$
$$= m^{*-1}((X^{\perp} \cap A^{\circ}) \otimes A^{\circ} + A^{\circ} \otimes (Y^{\perp} \cap A^{\circ}))$$
$$= m^{*-1}((X^{\perp} \otimes A^{*} + A^{*} \otimes Y^{\perp}) \cap (A^{\circ} \otimes A^{\circ}))$$
$$= m^{*-1}(X^{\perp} \otimes A^{*} + A^{*} \otimes Y^{\perp})$$
$$= m^{*-1}((X \otimes Y)^{\perp} \cap (A^{*} \otimes A^{*})$$
$$= (XY)^{\perp} \cap A^{\circ},$$

where the first equality follows by definition and second by 2.3.1. Lemma in [1] and third by Proposition 6.0.3 in [4]. q.e.d.

Corollary 3. $\overline{XY} = ((X^{\perp} \cap A^{\circ}) \wedge (Y^{\perp} \cap A^{\circ}))^{\perp}$.

Let J denote the Jacobson radical of A. Then we can prove the following lemma by the simillar way as in [3].

Lemma 4. (i) $\varinjlim (A/J^{n})^{\circ} = A^{\circ}$.

(ii) $A^{o*} \cong (\varinjlim (A/J^n)^o)^* \cong \varprojlim (A/J^n)^{o*}$.

(iii) The diagram

$$\begin{array}{ccc} \varprojlim A/\overline{J^n} & \longrightarrow & \varprojlim (A/\overline{J^n})^{o*} \\ \uparrow & \circlearrowright & \uparrow \\ A & \longrightarrow & A^{o*}, \end{array}$$

where the above mapping is one induced by the canonical ones $A/\overline{J^n} \longrightarrow (A/\overline{J^n})^{o*}$, the left hand side one is the isomorphism in (ii).

2. Main theorem. Let A be an algebra with cofinite Jacob Jacobson radical J. Then $\overline{J^n} = ((J^n)^{\perp} \cap A^o)^{\perp} = (\wedge^{n-1}(J^{\perp} \cap A^o))^{\perp}$ by Corollary 3. Hence if A is locally finite, then $\overline{J^n}$ is cofinite for all n. Since coreflexive coalgebras are locally finite weget

Proposition 5. Let A be an algebra with cofinite Jacobson radical. Suppose that the dual coalgebra A^o is coreflexive. Then A is reflexive if and only if the canonical mapping $A \longrightarrow \varprojlim A/\overline{J^n}$ is an isomorphism.

If J is finitely generated and cofinite, then J^n is also finitely generated and cofinite, and so $\overline{J^n} = J^n$, so that the above proposition implies that A is reflexive if and only if A is J-adically complete. This is the cosequence of 4.3.1. Proposition. We will show the converse.

An algebra is called left(or right) almost noetherian if every cofinite left(or right) ideal is finitely generated and is called almost noetherian if it is both left and righr noetherian. For example, if J is cofinite and finitely generated as a left(or right) ideal, then A is left almost noetherian. For details see [1].

Theorem. Let A be an algebra with cofinite Jacobson radical J. Then the following conditions are equivalent.

(i) A is reflexive.

(ii) A is left almost noetherian and is J-adically complete.

(iii) A is right almost noetherian and is J-adically complete.

(iv) A is almost noetherian and is J-adically complete.

(v) A° is coreflexive and the canonical mapping $A \longrightarrow \varprojlim A/\overline{J^n}$ is an isomorphism.

Proof. Suppose that A is reflexive. Then A can be considered as the dual space of A°, and so, the weak-* closure and the closure considered coincide with each other. Hence the sum of finitely many closed subspaces is closed and a finitely generated left(or right) ideal is closed. Since A° is coreflexive J^n is cofinite for all n. In particular, there exists a finitely generated left ideal L such that $J = L + \overline{J^2}$. We will show that $L = J$.

Since $\overline{J^2} = \overline{J(L + \overline{J^2})} = \overline{JL + J^3} = \overline{JL} + \overline{J^3} \subset L + \overline{J^3}$, $J = L + \overline{J^3}$. Inductively, we have $\overline{J^n} = \overline{J^{n-1}(L + J^2)} \subset L + \overline{J^{n+1}}$, and so, $J = L + \overline{J^n}$ for all n and $J = \bigcap_{n=1}^{\infty}(L + \overline{J^n})$. Since L is closed and A is linearly compact $J = L + \bigcap_{n=1}^{\infty} \overline{J^n}$ by [2]. We must show that $\bigcap_{n=1}^{\infty} \overline{J^n} = (0)$. Since A is proper the commutativity of the diagram in Lemma 4 (iii) implies this. Thus we get that $J = L$, and so A is left almost noetherian and Lemma 4 (iii) implies that A is J-adically complete.

Simillarly, if A is reflexive then J is finitely generaed as a right ideal, and so, A is almost noetherian.

3.3.3. Theorem and 3.1.2. Remark show that (ii)(or (iii) or (v)) implies (v). Proposition 5 shows that (v) implies (i). q.e.d.

References

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