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Quartic surfaces of elliptic ruled type

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§ 0. Introduction.

In this paper we shall investigate the structure of normal quartic surfaces in $\mathbb{P}^3$ whose resolutions are birationally equivalent to elliptic ruled surfaces. In this paper we call such a surface simply a quartic surface of elliptic ruled type.

Let X be a quartic surface of elliptic ruled type defined over an algebraically closed field k of characteristic $\neq 2, 3$, and let $\pi : \tilde{X} \longrightarrow X$ be the minimal resolution of X. We shll study X by studying the structure of $\tilde{X}$ and the linear system on $\tilde{X}$ which defines the morphism π.

Since the dualizing shaf ω_X of X is trivial, we can apply the results in [3]. Here we restate some of them (restricting to our present situation) which will play essential roles throughout this research. We use the terms and facts cited in § 1 of [3] without notice.

LEMMA 1. For any point P on X, the geometric genus $p_g(P)$ of P is not greater than 2.

Since $\omega_X \simeq O_X$, there exists a unique effective anti-canonical divisor on $\tilde{X}$ whose connected components correspond by π to singular points with $p_g \geq 1$

on X. We denote this divisor by $\tilde{D}$.

LEMMA 2. Let $\tilde{X} = X_n \xrightarrow{\mu_n} X_{n-1} \xrightarrow{\mu_{n-1}} \cdots \xrightarrow{\mu_1} X_0 = \bar{X}$ be a sequence of blow-downs obtaining a relatively minimal model $\bar{X}$ of $\tilde{X}$, and let $\tilde{H}$ be the pull back of a general hyperplane section of X to $\tilde{X}$ such that $\tilde{H}$ is irreducible and non-singular. Put

$$H_n = \tilde{H},\quad H_i = \mu_{i+1}\circ\mu_{i+2}\circ\cdots\circ\mu_n(H_n)\ (0 \leqq i \leqq n-1),\quad \bar{H} = H_0,$$

$$D_n = \tilde{D},\quad D_i = \mu_{i+1}\circ\mu_{i+2}\circ\cdots\circ\mu_n(D_n)\ (0 \leqq i \leqq n-1),\quad \bar{D} = D_0.$$

Then we have

i) $D_i \in |-K_{X_i}|$, $(0 \leqq i \leqq n)$,

ii) μ_i is a blow-up with center at a point on $\mathrm{supp}(D_{i-1}) \cap \mathrm{supp}(H_{i-1})$,

iii) $D_i = \mu_i^*(D_{i-1}) - E_i$ where E_i is the exceptional curve of the first kind for μ_i $(1 \leqq i \leqq n)$.

LEMMA 3. On the elliptic ruled surface $\bar{X} \xrightarrow{\varpi} C$, the effective anti-canonical divisor $\bar{D}$ is one of the following types:

i) $\bar{D} = C_0 + C_1$, where C_0 is a minimal section of ϖ and C_1 is another section disjoint from C_0.

ii) $\bar{D} = 2C_0 + \sum f_i$, where C_0 is as above and f_i's are fibres of ϖ.

First in § 1, we make a list of possible singularities on X with $p_g \geqq 1$. After that, restricting ourselves to the case of caracteristic 0, we study the structure of X in detail using the sequence of blow-ups as in Lemma 2.

We note that M.Kato and I.Naruki are studying the singularities on normal quartic surfaces too, and are getting similar results. However their method is to analyze polynomials of degree 4, and so is quite different from ours.

§ 1. Types of singularities on X.

In this section, we assume that X is a normal quartic surface of elliptic ruled type, and we shall list up the possibility of the types of singularities on X with positive geometric genus. It will be shown in §2 that every member in our list really appears as singularities on a normal quartic surface of elliptic ruled type in the case of characteristic 0.

LEMMA 4. The multiplicity of each singular point on X is equal to two.

PROOF. Let P be a sigular point on X, and let $p: X \dashrightarrow \mathbb{P}^2$ be the projection with center at P. If $\mathrm{mult}_P X = 4$, then X is a cone over a plane quartic curve, and hence minimal resolution of X is a ruled surface of genus 3. If $\mathrm{mult}_P X = 3$, then p turns out to be a birational map onto $\mathbb{P}^2$, and so X is rational. Therefore we have $\mathrm{mult}_P X = 2$. Q.E.D.

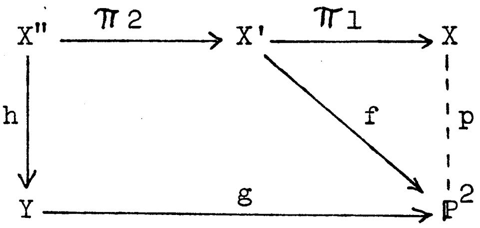
By Lemmas in § 0, the sigularities on X with $p_g \geqq 1$ are either two simple elliptic singularities or a singularity with $p_g = 2$. In what follows we use the notations in § 0.

PROPOSITION 1. If X has two simple elliptic singularities, then they are both of type $\widetilde{E}_7$ or of type $\widetilde{E}_8$.

PROOF. Let P and Q be the two simple elliptic singular points on X. We first show that the line ℓ through P and Q in $\mathbb{P}^3$ does not lie on X. Indeed, if ℓ was contained in X, let $\widetilde{\ell}$ be the proper transformation of ℓ

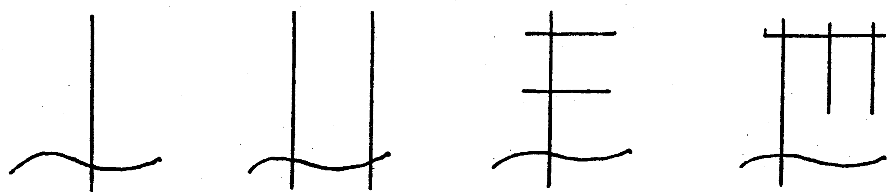
on $\tilde{X}$. Then $\tilde{\ell} \cap \tilde{C}_0 \neq \emptyset$, $\tilde{\ell} \cap \tilde{C}_1 \neq \emptyset$ and $\tilde{\ell}.\tilde{H} = 1$ where $\sim$ means the proper transformation of a curve into $\tilde{X}$. Since $\tilde{\ell}$ is isomorphic to $\mathbb{P}^1$, $\mu(\tilde{\ell})$ must be a fibre of $\bar{X}$ on which there is no center of the blow-ups $\mu : \tilde{X} \longrightarrow \bar{X}$. So $\bar{H}.f = 1$ for any fibre f of $\bar{X}$ and we get a contradiction because $\tilde{H}$ is a curve of genus 3.

To prove the Proposition, it is enough to show that if P is of type $\tilde{E}_7$, then so is Q, since a simple elliptic singularity of multiplicity 2 is either of type $\tilde{E}_7$ or $\tilde{E}_8$. Consider the following commutative diagram:



where p is the projection from X with center at P, $\pi_1 : X' \longrightarrow X$ is the blow-up with center at P, $\pi_2 : X'' \longrightarrow X'$ is the normalization of X' and $g : Y \longrightarrow \mathbb{P}^2$ is the finite morphism of degree 2 obtained by the Stein factorization of $f \circ \pi_2$. The degree of the branch locus of f is equal to 6 and X' is not normal since P is of type $\tilde{E}_7$ (Laufer [2]). Therefore the degree of the branch locus of g is less than or equal to 4. Since $\ell \not\subseteq X$, there is a neighbourhood U of Q in X such that U is transformed by π_1^{-1}, π_2^{-1} and h isomorphically into Y. So Y has a simple elliptic singularity which is isomorphic to Q. Thus by the classification of plane curves of degree less than or equal to 4 (cf. Hidaka-Watanabe [1]), we prove that Q is of type $\tilde{E}_7$. Q.E.D.

PROPOSITION 2. Suppose that X has a singular point P of geometric genus equal to two. Then the exceptional set for the minimal resolution of P is one of the following four types:



(i) $E1+A_1$ (ii) $E1+A_1+A_1$ (iii) $E1+D(3)$ (iv) $E1+D(4)$

where each curved line means a non-singular elliptic curve whose self-intersection number is equal to -1 in (i), -2 in (ii),(iii),(iv), and any straight line means a non-singular rational curve with self-intersection number equal to -2. (The symbols of these four types of singularities are due to S.S.-T. Yau.)

PROOF. Since $p_g(P) = 2$, and P is a Gorenstein singularity, we have $p_a(P) = 1$ (Hidaka-Watanabe [1]). So we can define the elliptic sequence $Z_{B_0}, \ldots, Z_{B_{\ell+1}}$ according to Yau [4]. By the definition, every Z_{B_i} is an effective divisor supported on the exceptional set $\mathrm{supp}(\tilde{D})$ on $\tilde{X}$. By Lemmas 2 and 3, $\mathrm{supp}(\tilde{D})$ is simple normal crossings and has the non-singular elliptic curve $\tilde{C}_0$ as a component. Therefore, by the property of elliptic sequence (Theorem 3.7 of [4], Proposition 2.1 of [5] and Corollary 2.3 of [6]), we have

1) $Z_{B_{\ell+1}} = \tilde{C}_0$,

2) $Z_{B_0}^2 \leqq \cdots \leqq Z_{B_{\ell+1}}^2 < 0$,

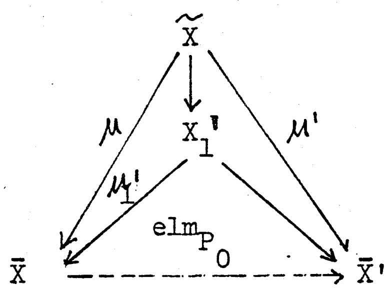
3) $\sum_{i=0}^{\ell+1} Z_{B_i}^2 = K_{\tilde{X}}^2 = -n$ (n being the number of blow-ups in $\mu : \tilde{X} \longrightarrow \bar{X}$),

4) $2 \geqq -\sum_{i=0}^{\ell} Z_{B_i}^2$.

(Although Yau's proof is in the case of $k = \mathbf{C}$, it is easy to check that 1)—4) remain true in our situation for any algebraically closed field k.)
From these we see that all possibility are as follows.

	ℓ	$Z_{B_0}^2$	$Z_{B_1}^2$	$\tilde{C}_0^2$	n
(a)	1	-1	-1	-1	3
(b)	0	-2		-1	3
(c)	0	-2		-2	4
(d)	0	-1		-1	2

Now we can choose a relatively minimal model $\bar{X}$ of $\tilde{X}$ so that $C_0^2 = \tilde{C}_0^2$. Indeed, if $C_0^2 > \tilde{C}_0^2$ for an $\bar{X}$, there exists a point P_0 on C_0 such that $\mu : \tilde{X} \longrightarrow \bar{X}$ factors through the blow-up $\mu_1' : X_1' \longrightarrow \bar{X}$ with center at P_0. Hence there is a morphism $\mu' : \tilde{X} \longrightarrow \bar{X}'$ where $\bar{X}'$ is the image of the elementary transformation of $\bar{X}$ with center at P_0, and so $\bar{X}'$ is another relatively minimal model of $\tilde{X}$ whose minimal section C_0' satisfies $C_0'^2 = C_0^2 - 1$.



With the assumption of $C_0^2 = \tilde{C}_0^2$, and using Lemma2., we can list up the possibility of $\bar{D}$, μ and $\tilde{D}$ corresponding to each case in the table above:

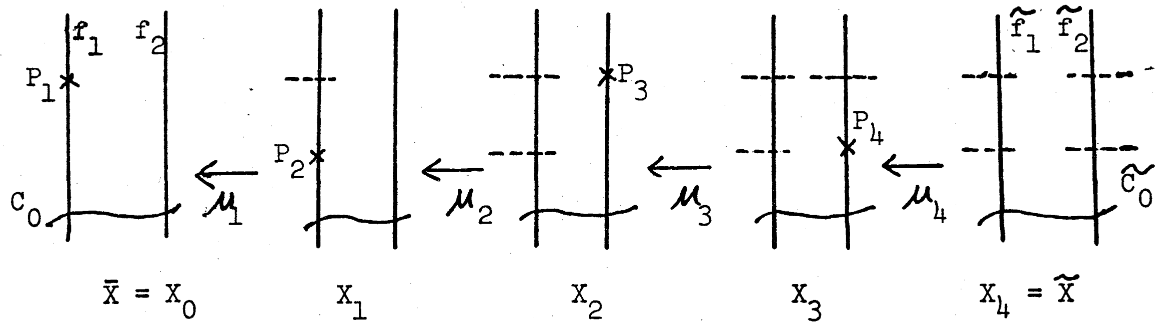
(a) and (b): $\bar{D} = 2C_0 + f_0$, f_0 is a fibre of $\varpi : \bar{X} \longrightarrow C$, and the centers of the blow-ups μ lie on $\mathrm{supp}(f_0) \smallsetminus \mathrm{supp}(C_0)$. Hence $\tilde{D} = 2\tilde{C}_0 + \tilde{f}_0$. ($\sim$ always means the proper transformation on $\tilde{X}$.)

(c-1)—(c-3): $\bar{D} = 2C_0 + f_1 + f_2$, f_1 and f_2 are disjoint fibres of ϖ.

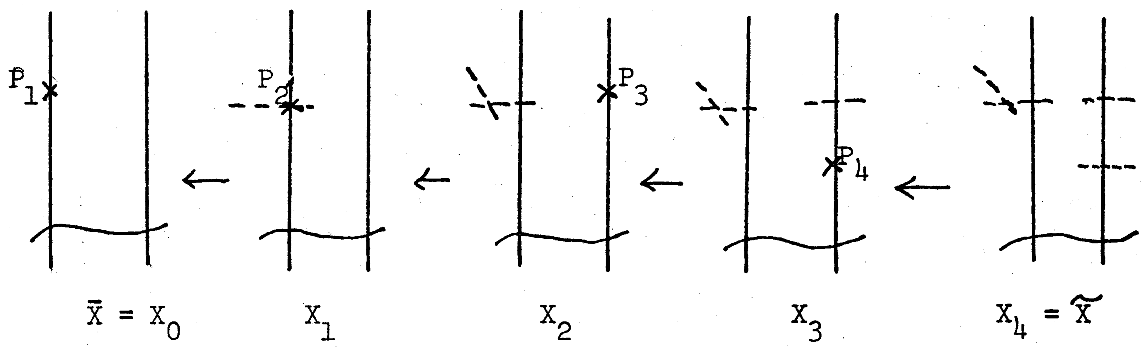
(c-4) and (c-5): $\bar{D} = 2C_0 + 2f_1$, f_1 is a fibre of ϖ.

In (c-1)—(c-5), μ and $\tilde{D}$ are as described in the following figures:

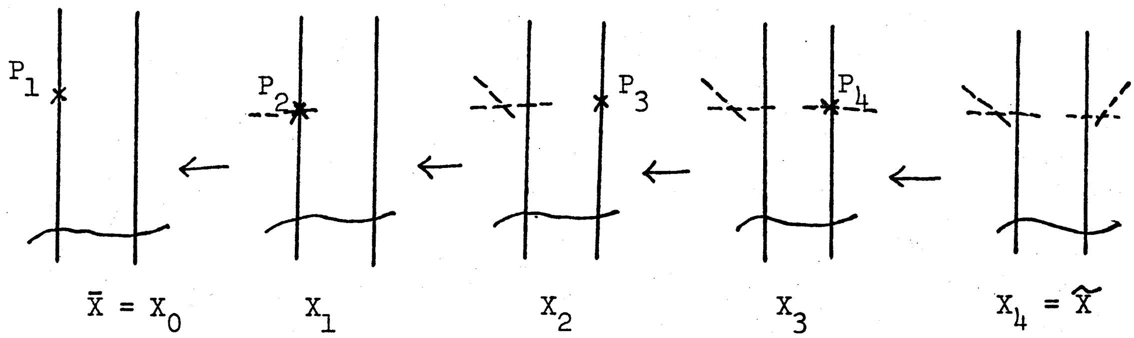
(c-1)


f_1 f_2
P_1
P_2
P_3
P_4
$\tilde{f}_1$ $\tilde{f}_2$
C_0
$\tilde{C}_0$
μ_1
μ_2
μ_3
μ_4
$\bar{X} = X_0$
X_1
X_2
X_3
$X_4 = \tilde{X}$

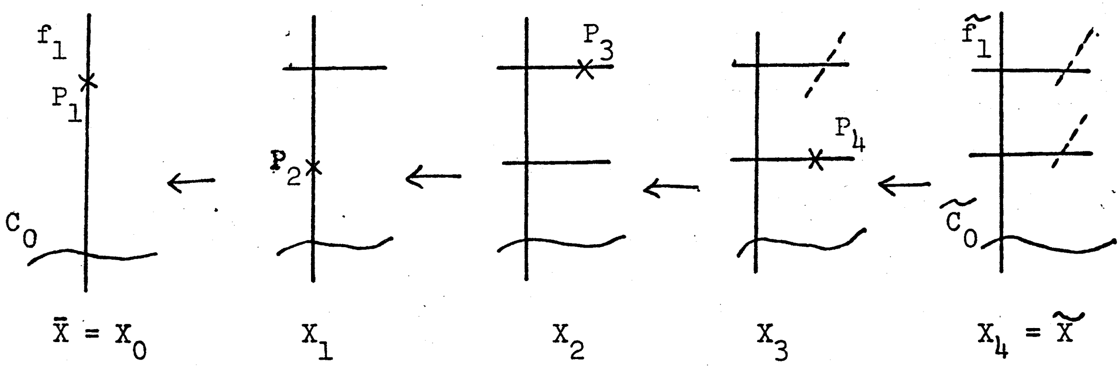
(c-2)


P_1
P_2
P_3
P_4
$\bar{X} = X_0$
X_1
X_2
X_3
$X_4 = \tilde{X}$

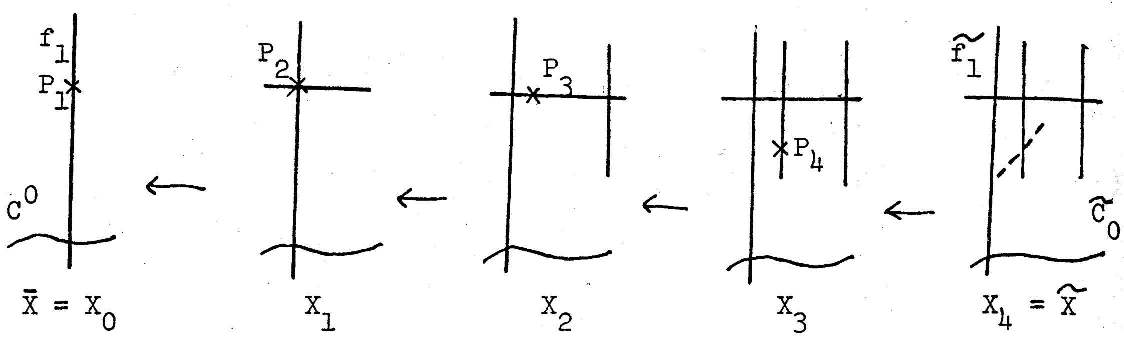
(c-3)


P_1
P_2
P_3
P_4
$\bar{X} = X_0$
X_1
X_2
X_3
$X_4 = \tilde{X}$

(c-4)


f_1
P_1
C_0
P_2
P_3
P_4
$\tilde{f}_1$
$\tilde{C}_0$
$\bar{X} = X_0$
X_1
X_2
X_3
$X_4 = \tilde{X}$

(c-5)



where P_i is the center of μ_i (the order of the blow-ups may be changed), the bold lines mean the support of the effective anti-canonical divisor D_i on each step, and the dotted lines mean the exceptional curves of the blow-ups which do not appear in D_i.

It remains to show that the cases (a) and (b) do not occur. For (a), since $\ell = 1$, $\tilde{D}$ must have at least three irreducible components by a property of the elliptic sequence; $\mathrm{supp}(\tilde{C}_0) = \mathrm{supp}(Z_{B_2}) \subsetneqq \mathrm{supp}(Z_{B_1}) \subsetneqq \mathrm{supp}(Z_{B_0}) = \mathrm{supp}(\tilde{D})$. Hence a contradiction. For (b), let r_i be the multiplicity of H_i at P_i ($1 \leqq i \leqq 3$) and let $\bar{H} = mC_0 + mf$ (f denotes a fibre and $m \in \mathbb{Z}$). Then we have

$$4 = \tilde{H}^2 = \bar{H}^2 - \sum r_i^2 = m^2 - \sum r_i^2$$

and

$$0 = \tilde{H}.\tilde{f}_0 = \bar{H}.f_0 - \sum r_i = m - \sum r_i .$$

Hence $\quad 4 = (\sum r_i)^2 - \sum r_i^2 = 2(r_1r_2 + r_2r_3 + r_3r_1).$

Since $r_i \geqq 1$ ($1 \leqq i \leqq 3$), we get a contradiction. Q.E.D.

(d): $\bar{D} = 2C_0 + f_0$, f_0 is a fibre of ϖ, the centers of μ lie on $\mathrm{supp}(f_0) \setminus \mathrm{supp}(C_0)$, and hence $\tilde{D} = 2\tilde{C}_0 + \tilde{f}_0$.

§ 2. Statements of the main results.

In this section, we assume that the ground field k is algebraically closed and of characteristic 0.

Let $Y_n \xrightarrow{\mu_n} Y_{n-1} \xrightarrow{\mu_{n-1}} \cdots \xrightarrow{\mu_1} Y_0 = Y$ be a sequence of blow-ups where Y is a non-singular surface. Let $P_i \in Y_{i-1}$ denote the center of μ_i and $E_i \subset Y_i$ the exceptional curve for μ_i $(1 \leqq i \leqq n)$. Then we call $(P_1,\ldots,P_n)$ a sequence of points on Y admitting infinitely near points, and the sequence above of blow-ups is called the blow-up of $(P_1,\ldots,P_n)$. For a divisor D on Y and non-negative integers m_i $(1 \leqq i \leqq n)$, we denote by $|D - m_1P_1 - \ldots - m_nP_n|$ the linear subsystem of $|D|$ consisting of elements $D' \in |D|$ such that $\mu_n^*(\mu_{n-1}^*(\ldots(\mu_1^*(D') - m_1E_1)\ldots) - m_{n-1}E_{n-1}) - m_nE_n$ remain effective on Y_n. When there is no danger of confusion, we denote also by P_i the image on Y of the point $P_i \in Y_{i-1}$.

Let C be a non-singular curve and let $\mathcal{E}$ be a vector bundle of rank 2 on C. Assume that $\mathcal{E}$ is the direct sum of two line bundles on C. Then the ruled surface $\mathbb{P}(\mathcal{E}) \longrightarrow C$ has a section which is disjoint from a minimal section. We denote by C_1 such a section, and by C_0, as before, a minimal section. In general these sections are not uniquely determined, but we fix a pair (C_0,C_1) once and for all on $\mathbb{P}(\mathcal{E})$ unless otherwise mentioned.

THEOREM 1. Let C be a non-singular elliptic curve and let L be an invertible sheaf of degree 2 on C. Let $\bar{X} = \mathbb{P}(O_C \oplus L) \xrightarrow{\varpi} C$ be the induced ruled surface. Fix an effective divisor $\bar{D} \in |-K_{\bar{X}}|$ and take a sequence of points (P_1, P_2, P_3) on $\bar{X}$ admitting infinitely near points such that:

if $\bar{D} = C_0 + C_1'$ where C_1' is a section of ϖ, then P_i is infinitely near a point on C_1' (for each i); and if $\bar{D} = 2C_0 + f_1 + f_2$ where f_i's are fibres, then the position of P_1, P_2, P_3 is one of the types (c-1)—(c-5) in § 1. And in each cases, (*) there is no section C' of ϖ such that $C' \sim C_1$, $C' \nleq \bar{D}$ and that $C' \ni P_i, P_j$ for some i, j $(1 \leqq i < j \leqq 3)$.

Then

1) there exists a unique point P_4 infinitely near a point on $\bar{X}$ such that $|2C_1 - P_1 - P_2 - P_3| = |2C_1 - P_1 - P_2 - P_3 - P_4|$.

Moreover let $\mu : \tilde{X} \to \bar{X}$ denote the blow-up of $(P_1, \ldots, P_4)$ and let $\tilde{H}$ be the proper transformation by μ of a general member in $|2C_1 - P_1 - \ldots - P_4|$.

Then we have

2) $\mathrm{Bs}\,|\tilde{H}| = \emptyset$,

3) $\dim |\tilde{H}| = 3$, $\tilde{H}^2 = 4$,

4) $\tilde{H}$ is a non-hyperelliptic curve.

Therefore $\tilde{H}$ defines a birational morphism from $\tilde{X}$ to a normal quartic surface $X \subset \mathbb{P}^3$ with singularities of type $2\tilde{E}_7$, $E1+A_1+A_1$, $E1+D(3)$ or $E1+D(4)$.

Conversely, any quartic surface of elliptic ruled type with at least one of these four types of singularities is obtained by the construction above.

REMARK. Unless the condition (*), 1)—3) of the Theorem 1 hold true. But in this case, $\tilde{H}$ turnes out to be a hyperelliptic curve so that $\tilde{H}$ defines a morphism of degree 2 onto a normal quadric surface in $\mathbb{P}^3$. (cf. Lemmas 6 and 7 in § 3.)

THEOREM 2. Let C be a non-singular elliptic curve and let L be an invertible sheaf of degree 1 on C. Let $\bar{X} = \mathbb{P}(O_C \oplus L) \xrightarrow{\varpi} C$ be the induced ruled surface. Fix an effective divisor $\bar{D} \in |-K_{\bar{X}}|$ and take a point P_1 on $\bar{X}$ such that:

if $\bar{D} = C_0 + C_1'$ where C_1' is a section of ϖ, then $P_1 \in \operatorname{supp}(C_1')$ with $O_{C_1'}(2P_1) \not\cong L^2$ (identifying C_1' with C via ϖ); and if $\bar{D} = 2C_0 + f_0$ where f_0 is a fibre of ϖ, then $P_1 \in \operatorname{supp}(f_0) \setminus \operatorname{supp}(C_0 + C_1)$.

Then

1) there exists a unique point P_2 infinitely near a point on $\bar{X}$ such that $|3C_1 - 2P_1| = |3C_1 - 2P_1 - P_2|$.

Moreover let $\mu : \tilde{X} \longrightarrow \bar{X}$ denote the blow-up of (P_1, P_2) and let $\tilde{H}$ be the proper transformation by μ of a general member in $|3C_1 - 2P_1 - P_2|$.

Then we have

2) $\operatorname{Bs} |\tilde{H}| = \emptyset$,

3) $\dim |\tilde{H}| = 3$, $\tilde{H}^2 = 4$,

4) H is a non-hyperelliptic curve.

Therefore $\tilde{H}$ defines a birational morphism from $\tilde{X}$ to a normal quartic surface $X \subset \mathbb{P}^3$ with singularities of type $2\tilde{E}_8$ or $E1+A_1$.

Conversely, any quartic surface of elliptic ruled type with at least one of these two types of singularities is obtained by the construction above.

COROLLARY. Let X be a quartic surface of elliptic ruled type. Then the singular set of X is one of the following types, and each of them really occurs.

$\{2\tilde{E}_7$, a subgraph of $A_3\}$, $\{E1+A_1+A_1$, a subgraph of $A_1\}$,
$\{E1+D(3)\}$, $\{E1+D(4)\}$, $\{2\tilde{E}_8\}$, $\{E1+A_1\}$.

§ 3 Proof of Theorem 1.

We start with $\bar{X}$, $\bar{D}$, P_1, P_2, and P_3 as in the Theorem, but at first we do not assume the condition (*).

Proof of 1), 2) and 3): We will prove only in the case of $\bar{D} = C_0 + C_1'$ where C_1' is a section, since the proofs in other cases are similar. By $C_1' \sim C_1$, we may assumu that $\bar{D} = C_0 + C_1$. Since $2C_1.C_1 = 4$ and since C_1 is an elliptic curve, we see $\dim |2C_1 - P_1 - P_2 - P_3| \,|\, C_1 = 0$. Hence we can define a unique point P_4 lying on the proper transformation of C_1 on the blow-up of (P_1, P_2, P_3) of $\bar{X}$ such that $|2C_1 - P_1 - P_2 - P_3 - P_4| = |2C_1 - P_1 - P_2 - P_3|$. Set $\Lambda = |2C_1 - P_1 - P_2 - P_3 - P_4|$ and let f_i denote the fibre on $\bar{X}$ through P_i $(1 \leq i \leq 4)$. Since $O_{C_1}(P_1 + P_2 + P_3 + P_4) = O_{C_1}(2C_1) = L^2$ identifying C_1 and C via ϖ, we find the following four elements of Λ: $2C_0 + \sum_{i=1}^{4} f_i$, $C_0 + f_5 + f_6 + C_1$, $C_0 + f_7 + f_8 + C_1$ and $2C_1$, where f_j's $(5 \leq j \leq 8)$ are distinct fibres of ϖ such that $f_5 + f_6$, $f_7 + f_8 \in |\varpi^* L|$. Let $s_1, \ldots, s_4 \in H^0(\bar{X}, O_{\bar{X}}(2C_1))$ denote the defining equations of above four divisors. Obviously s_i's are linearly independent. Since $\Lambda \subsetneqq |2C_1 - P_1 - P_2| \subsetneqq |2C_1 - P_1| \subsetneqq |2C_1|$ as is shown easily and since $\dim |2C_1| = 6$, s_i's form a basis of Λ. Therefore, letting $\bar{H}$ be a general member of Λ and letting us define $\mu : \tilde{X} \longrightarrow \bar{X}$ and $\tilde{H}$ as in the Theorem, we deduce that:

the base points of Λ are exactly $P_1, \ldots, P_4$,

$\mathrm{Bs}\, |\tilde{H}| = \emptyset$,

$\dim |\tilde{H}| = \dim \Lambda = 3$ and $\bar{H}$ is non-sigular at $P_1, \ldots, P_4$.

Thus 1), 2) and 3) follow.

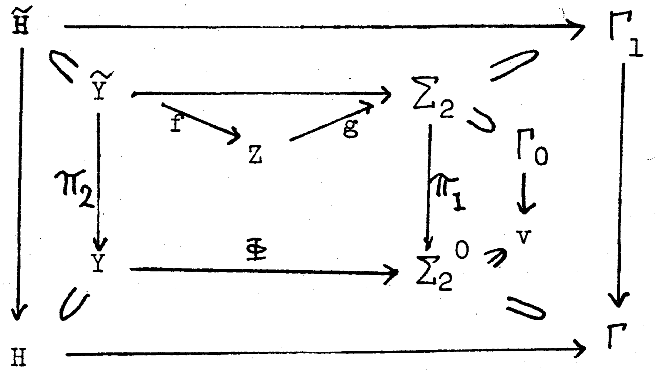
Let X denote the image of $\tilde{X}$ under the morphism $\Phi_{|\tilde{H}|}$ defined by $\tilde{H}$.

Then X is a hypersurface in $\mathbb{P}^3$ of degree 4 or 2 by 3), and X is normal because the pull back of a general hyperplane section of X is a non-singular curve $\tilde{H}$. It is clear by our construction that there is a unique effective anti-canonical divisor $\tilde{D}$ on $\tilde{X}$, and that the configuration of $\tilde{D}$ is $2\tilde{E}_7$, $E1+A_1+A_1$, $E1+D(3)$ or $E1+D(4)$ according to our choice of $\bar{D}$, P_1, P_2 and P_3. Moreover, by a direct computation, we see that $\mathrm{supp}(\tilde{D})$ is one of the connected components of the exceptional set for $\Phi_{|\tilde{H}|}$. By $\tilde{H}.K_{\tilde{X}} = 0$, the restriction of $\Phi_{|\tilde{H}|}$ to $\tilde{H}$ is the canonical map of $\tilde{H}$, therefore X is a quartic surface if and only if $\tilde{H}$ is non-hyperelliptic. We will show a criterion for this property in a little generalized situation.

Let Σ_0 and Σ_2^0 denote a non-singular quadric surface in $\mathbb{P}^3$ and a cone over a non-singular conic in $\mathbb{P}^3$ respectively, and let v denote the vertex of Σ_2^0. Let Y be a non-singular surface birationally equivalent to an elliptic ruled surface. Suppose that there is an effective anti-canonical divisor D of Y such that every irreducible component of D has self-intersection number greater than -4. Moreover suppose that there exists on Y a non-singular curve H of genus 3 such that $\mathrm{Bs}\,|H| = \emptyset$, $\dim|H| = 3$ and that $H.D = 0$ so that we have a morphism $\Phi = \Phi_{|H|}$ from Y onto a normal quartic surface in $\mathbb{P}^3$ or a Σ_0 or a Σ_2^0. Since $H.D = 0$, $\Phi(D)$ is finite points.

LEMMA 5. Assume that $\Phi(Y) = \Sigma_2^0$. Then $v \notin \Phi(D)$.

PROOF. Let $\pi_1 : \Sigma_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) \longrightarrow \Sigma_2^0$ be the blow-up with center at v, and let Γ_0 be the exceptional rational curve for π_1. Let Γ be a general hyperplane section of Σ_2^0 and let Γ_1 denote the proper transformation of Γ on Σ_2. We may assume that $\Phi(H) = \Gamma$. Consider the following commutative diagram:



where $\pi_2 : \tilde{Y} \longrightarrow Y$ is a minimal sequence of blow-ups such that the induced map $\tilde{Y} \longrightarrow \Sigma_2$ becomes a morphism, Z is the Stein factorization of $\tilde{Y} \longrightarrow \Sigma_2$, and $\tilde{H}$ is the proper transformation of H. Let B denote the branch locus of g. Then $B \sim 2(n\Gamma_0 + mf)$ for some $n, m \in \mathbb{Z}$ where f denotes a fibre of the ruling on Σ_2. Since $\tilde{H} \longrightarrow \Gamma_1$ is a morphism of degree 2 between non-singular curves of genus 3 and 0 respectively, we have $B.\Gamma_1 = 8$ and hence $B \sim 2n\Gamma_0 + 8f$. Furthermore, since B is reduced,

$$-2 = \Gamma_0^2 \leqq \Gamma_0.B = -4n + 8.$$

If $n = 0$, then Z is a ruled surface of genus 3, and hence a contradiction. Therefore we have $n = 1$ or 2.

Assume that $\Phi(D) \ni v$. Then there exists a non-singular elliptic curve $C \leqq D$ such that $\Phi(C) = v$ by Lemmas 2 and 3. Let $\tilde{C}$ denote the proper transformation of C on $\tilde{Y}$. If $f(\tilde{C})$ is a curve, then clearly $\tilde{Y} = Y$, and $g^{-1}(\Gamma_0) = f(\tilde{C}) \cong \tilde{C} = C$ (consider the configuration of D). Hence we get a contradiction because $C^2 = 2 \cdot \Gamma_0^2 = -4$. Thus $f(\tilde{C})$ is a point, and so Z has a singular point on $g^{-1}(\Gamma_0)$ which is not a rational double point.

Suppose $n = 1$. Then $B.f = 2$. From this we see easily that Z has at most rational bouble points. Suppose $n = 2$, then $B \sim 4\Gamma_0 + 8f$. If $\Gamma_0 \nsubseteq B$, then Z is smooth over Γ_0 since $\Gamma_0.B = 0$. Hence $B = \Gamma_0 + B_0$ for some reduced effective divisor B_0 not containing Γ_0. But since $B_0.\Gamma_0 = 2$, also in this case Z has at most rational bouble double points over Γ_0. Hence a contradiction. Q.E.D.

LEMMA. 6. $\Phi(Y)$ is a normal quartic surface if and only if there is no curve C on Y satisfying:

(**) C is non-singular , elliptic,

$C.H = 2$, and $\mathrm{supp}(C) \cap \mathrm{supp}(D) = \emptyset$.

FROOF. Suppose that $\Phi(Y)$ is a quartic surface and that there were a curve C on Y which satisfies (**). Then $\Phi(C)$ is a curve and is birationally equivalent to C, because $C.H > 0$ and Φ is a birational morphism. But by $\Phi(C).h = C.H = 2$ for a hyperplane section h of $\Phi(Y)$, $\Phi(C)$ must be a rational curve, and hence a contradiction. Next suppose $\Phi(Y) = \Sigma_0$. Let us take a general fibre ℓ of one of the two rulings on Σ_0.Then it is easy to verify that the proper transformation of ℓ on Y satisfies (**). Finally suppose $\Phi(Y) = \Sigma_2^0$ and let ℓ be a general generating line of Σ_2^0. Then, by Lemma 5, ℓ and $\Phi(D)$ are disjoint. It folllows that the proper transformation of ℓ on Y satisfies (**). Q.E.D.

Now returning to the proof of Theorem 1, we only have to show the following Lemma, which is proved by a direct computation (omitted).

LEMMA 7. The non-existence of cueves satisfying (**) is equivalent to the condition (*).

In the end, we prove the second part of the Theorem. So let X be a quartic surface of elliptic ruled type with at least $2\widetilde{E}_7$, $E1+A_1+A_1$, $E1+D(3)$, or $E1+D(4)$ as singularities. Let $\pi:\widetilde{X} \longrightarrow X$ be the minimal resolution of X. Then from the proof of Proposition 2, there is a ruled surface $\bar{X} = \mathbb{P}(E) \xrightarrow{\varpi} C$ over an elliptic curve C such that $C_0^2 = -2$, in particular E splits (the notations are the same as before), and that there exists a sequence of points (P_1, P_2, P_3, P_4) on $\bar{X}$ admitting infinitely near points such that by the blowing-up of them we obtain $\widetilde{X}$. Note that this map $\widetilde{X} \xrightarrow{\mu} \bar{X}$ is a sequence of Lemma 2. We have already shown that (P_1, P_2, P_3, P_4) must satisfy the conditions i)—iii) in the Theorem, and hence we only have to show the following

LEMMA 8. Let $\bar{H} \subset \bar{X}$ be as in Lemma 2. Then $\bar{H} \in |2C_1|$.

PROOF. We use the notations in Lemma2. We first prove that $\bar{H} \equiv 2C_1$. Since $\widetilde{H}.\widetilde{C}_0 = 0$, wehave $\bar{H}.C_0 = 0$. Hence $\bar{H} = mC_0 + 2mf$ for some $m \in \mathbb{Z}$, and we want to show that $m = 2$.

Case 1. X has a singularity of $p_g = 2$: We may assume that the order of P_1, P_2, P_3, P_4 is the same as one of the figures (c-1)—(c-5) in § 2, and we use the notations there. Put $r_i = \text{mult}_{P_i} H_i$ for $1 \leqq i \leqq 4$. Then we have

$$4 = \widetilde{H}^2 = \bar{H}^2 - \sum r_i^2 = 2m^2 - \sum r_i^2.$$

On the other hand , since the proper transformation of $\bar{H}$ and f_1 must be disjoint after blowing-up (P_1, P_2), we get

$$r_1 + r_2 = \bar{H}.f_1 = m.$$

Replacing $\{P_1, P_2, f_1\}$ with $\{P_3, P_4, f_2\}$ if necessary in the case of (c-1)—(c-3), we can assume that

$$r_1^2 + r_2^2 \geqq r_3^2 + r_4^2.$$

Therefore we have

$$4 = 2(r_1 + r_2)^2 - \sum r_i^2$$
$$=(r_1^2 + r_2^2 - r_3^2 - r_4^2) + 4r_1r_2 \geqq 4r_1r_2.$$

It follows that $r_1 = r_2 = 1$ and so $m = 2$.

Case 2. X has $2\tilde{E}_7$: Let $r_i = \mathrm{mult}_{P_i} H_i$ $(1 \leqq i \leqq 4)$, then we have

$$4 = \tilde{H}^2 = \bar{H}^2 - \sum r_i^2 = 2m^2 - \sum r_i^2,$$
$$\sum r_i = \bar{H}.C_1 = 2m.$$

Therefore we get $m \neq 3$, because the above equations have no positive integral solution (r_1, r_2, r_3, r_4) if $m = 3$. Next, since $P_1,\dots,P_4$ lie on $\mathrm{supp}(C_1)$, we can find another relatively minimal model $\tilde{X} \xrightarrow{\mu'} \bar{X}'$ such that $\bar{X}' = \mathbb{P}(O_C \oplus L')$ where L' is an invertible sheaf of degree 0 on C. Then there are two disjoint sections C_0', C_1' of $\varpi' : \bar{X}' \longrightarrow C$ and four points $P_1', P_2' \in C_0'$, $P_3', P_4' \in C_1'$ admitting infinitely near points such that μ' is the blow-up of (P_1', P_2', P_3', P_4') and that $\mu'(\tilde{C}_0 + \tilde{C}_1) = C_0' + C_1'$. We may assume that $O_{\bar{X}'}(C_1')|_{C_1'} \cong L'$ and $O_{\bar{X}'}(C_0')|_{C_0'} \cong L'^{-1}$. Set $\bar{H}' = \mu'(\tilde{H})$. Then $\bar{H}' \equiv mC_0' + nf$ for some $n \in \mathbb{Z}$ where f is a fibre of ϖ'. By $\tilde{H}.\tilde{C}_0 = \tilde{H}.\tilde{C}_1 = 0$, we get

$$r_1' + r_2' = \bar{H}'.C_0' = n, r_3' + r_4' = \bar{H}'.C_1' = n.$$

We may assume that $r_1' \geqq r_3' \geqq r_4' \geqq r_2'$ so that $r_1'^2 + r_2'^2 \geqq r_3'^2 + r_4'^2$ and equality holds if and only if $r_1' = r_3'$. Then we obtain

$$4 = \tilde{H}^2 = \bar{H}'^2 - \sum r_i'^2 \geqq 2m(r_1' + r_2') - 2(r_1'^2 + r_2'^2)$$

i.e. $$0 \geqq r_1'(m - r_1') + r_2'(m - r_2') - 2 \qquad (***)$$

and equality holds if and only if $r_1' = r_3'$. Since $m \geqq \bar{H}'.f \geqq r_1' \geqq r_2' \geqq 1$, the right hand side of (***) is either i) -2, ii) -1 or iii) 0. In the case of i), we have $m - r_1' = m - r_2' = 0$ and so $r_1' = r_3'$ which implies a contradiction. In case ii), we get $m - r_1' = 0$, $r_2' = m - r_2' = 1$ and hence $m = 2$. (But it follows that $r_1' = r_3'$ because $r_1' = 2$ and $r_2' = 1$, hence this case is impossible.) In iii), there are three possibilities: a) $r_1' = m - r_1' = r_2' = m - r_2' = 1$, b) $m - r_1' = 0$, $r_2' = 1$, $m - r_2' = 2$, and c) $m - r_1' = 0$, $r_2' = 2$, $m - r_2' = 1$. In a) we get $m = 2$, and so we have done. In b) and c), we get $m = 3$, but this value has been already excluded.

Therefore we conclude that $\bar{H} \equiv 2C_1$, and hence $\bar{H} \sim 2C_1 + \varpi^*(D)$ for some divisor D on C of degree 0. Since $\bar{X} = \mathbb{P}(O_C \oplus L)$ for some invertible sheaf L of degree 2 on C, we have

$$\dim H^0(\bar{X}, O_{\bar{X}}(\bar{H})) = \dim H^0(C, (O_C \oplus L \oplus L^2) \otimes O_C(D))$$

$$= \begin{cases} 7 & \text{if } D \sim 0 \\ 6 & \text{if } D \not\sim 0 \end{cases}$$

and

$$\dim H^0(\bar{X}, O_{\bar{X}}(\bar{H} - C_0)) = \dim H^0(C, (L \oplus L^2) \otimes O_C(D)) = 6.$$

Since $\bar{H}$ is irreducible, $\bar{H}$ has no fixed component, hence we prove that $D \sim 0$, i.e. $\bar{H} \sim 2C_1$. Q.E.D.

§ 4. Proof of Theorem 2.

We proceed in several steps which is almost parallel in the proof of Theorem 1. The detail is omitted here.

§ 5. Proof of Corollary.

Let $\tilde{X} \longrightarrow X$ denote the minimal resolution of X. Looking at the construction of $\tilde{X}$ described in Theorems 1 and 2, we can list up the configuration of curves on $\tilde{X}$ which is disjoint from $\tilde{H}$ (notations are those in Theorems 1 and 2) as follows:

$\{2\tilde{E}_7$, a subgraph of $A_3\}$, $\{E1+A_1+A_1$, a subgraph of $A_1+A_1\}$,

$\{E1+D(3)\}$, $\{E1+D(4)\}$, $\{2\tilde{E}_8\}$, $\{E1+A_1\}$.

The possibilty of $\{E1+A_1+A_1,\ A_1,\ A_1\}$ arises when we take P_1, P_2, P_3 as (c-3) in Theorem 1. But it is easy to show that in this case the section of ϖ through P_1 must pass through P_3. From this the Corollary follows. Q.E.D.

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