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ON THE STABILITY OF SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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1. Introduction

In this paper I will report on some new methods and results in the stability theory for the Navier-Stokes equations. These results have been obtained jointly with Rolf Rannacher for applications in numerical analysis [1]. Our study begins phenomenologically, by introducing various concepts of stability suggested by experimental observations. In particular, the concepts of stability we deal with are thought to describe the stability of such phenomena as Taylor cells and von-Kármán vortex shedding. We will also define a new notion of "contractive stability to a tolerance", which is useful in numerical analysis, and is thought to describe the partial stability observed in some flows exhibiting slight or incipient turbulence. Most of the results which will be described in the present paper concern either the principle of linearized stability or the limiting behaviour of stable solutions as $t \to \infty$. Concerning our applications to the numerical analysis of the Navier-Stokes problem, presented in [1] , we mention only that the results are of two general types. First, that if the solution of the initial boundary value problem is stable, then the error in its discrete approximation remains small uniformly in time, as $t \to \infty$. Second, that from the stability of a discrete solution, for a single sufficiently small choice of the mesh size, one can infer the global existence of a closely neighboring smooth solution.

2. Notation

We consider the nonstationary Navier-Stokes problem

$$
\begin{aligned}
&u_t - \Delta u + u\cdot\nabla u + \nabla p = f \ , \\
(1) \qquad &\nabla\cdot u = 0 \ , \quad \text{for} \quad (x,t) \in \Omega \times (0,\infty) \ , \\
&u|_{t=0} = a \ , \qquad u|_{\partial\Omega} = b \ ,
\end{aligned}
$$

in a bounded two or three-dimensional domain Ω . Here u represents the velocity of a viscous incompressible fluid, p the pressure, f the prescribed external force, a the prescribed initial velocity, and b the prescribed boundary values. The fluid's density and viscosity have been normalized, as is always possible, by changing the scales of space and time.

As usual, $L^p(\Omega)$, or simply L^p , denotes the space of functions defined and p^{th}-power summable in Ω , and $\|\cdot\|_{L^p}$ its norm. We denote the inner product in L^2 by $(\cdot,\cdot)$ and let $\|\cdot\| = \|\cdot\|_{L^2}$.

C^∞ is the space of functions continuously differentiable any number of times in Ω, and C_o^∞ consists of those members of C^∞ with compact support in Ω. The Sobolev space H^m is obtained by the completion in the norm

$$\|u\|_m = \Big(\sum_{0\le|\alpha|\le m} \|D^\alpha u\|^2\Big)^{1/2} ,$$

expressed in multi-index notation, of those members of C^∞ for which the norm is finite. H_o^1 is the closure of C_o^∞ in H^1. Spaces of R^n-valued functions will be denoted with boldface type. We use

$$(\nabla u,\nabla v) = \sum_{1\le i,j\le n} (\partial_j u_i,\partial_j v_i) , \quad \|\nabla u\| = (\nabla u,\nabla u)^{1/2} ,$$

as inner product and norm for H_o^1. Finally, we need the spaces

$$\mathbf{J} = \{\phi \in \mathbf{L}^2 : \nabla\cdot\phi = 0 \text{ in } \Omega \text{ and } \phi\cdot n_{|\partial\Omega} = 0 \text{ , weakly}\} ,$$

$$\mathbf{J}_1 = \{\phi \in \mathbf{H}_o^1 : \nabla\cdot\phi = 0\}$$

of solenoidal functions.

Denoting the orthogonal projection of $\mathbf{L}^2$ onto $\mathbf{J}$ by P, we introduce the "Stokes operator" $\tilde{\Delta} = P\Delta$. The mapping $\tilde{\Delta} : \mathbf{J}_1 \cap \mathbf{H}^2 \to \mathbf{J}$ is one-to-one and onto, and

$$\|v\|_2 \le c\|\tilde{\Delta} v\|$$

holds for all $v \in \mathbf{J}_1 \cap \mathbf{H}^2$, provided the boundary $\partial\Omega$ is sufficiently regular. We assume this ($\partial\Omega \in C^2$ suffices) as well as some regularity of the prescribed data; namely that $a \in \mathbf{J}_1$ and $f \in L^\infty(0,\infty;\mathbf{L}^2)$. For the sake of simplicity in our presentation, we assume that the boundary values

$b = 0$. All of our results remain valid in the case of inhomogeneous boundary conditions if one assumes an appropriate degree of smoothness of the boundary values, as well as the same conditions of spatial and temporal invariance as may be required of f . Finally, we assume that the strong solution u, p of problem (1) exists globally and satisfies

$$\sup_{[0,\infty)} \|\nabla u\| \le M ,$$

for some constant M .

3. Stability and Exponential Stability

Questions about the stability of u concern the behaviour of "perturbed solutions", by which we mean any solution v of the Navier-Stokes problem

$$v_t - \Delta v + v\cdot\nabla v + \nabla q = f ,$$

$$\nabla\cdot v = 0 \quad \text{for} \quad (x,t) \in \Omega \times (t_o,\infty) , \tag{2}$$

$$v|_{t_o} = v_o , \quad v|_{\partial\Omega} = 0 ,$$

starting at an initial time $t_o \ge 0$, with an initial value v_o near $u(t_o)$. We refer to $w = v - u$ as a "perturbation" of u , and to t_o and $w_o = v_o - u(t_o)$ as the "initial time" and "initial value" of the perturbation w . To avoid any doubt about the global existence of v (and hence of w) one may consider it as a "weak solution" in the sense of Hopf; however, it turns out one can prove that small perturbations of a stable solution are always "strong", i.e., smooth.

The ordinary, simplest, notion of stability is the following.

Definition 1. The solution u of problem (1) is said to be stable if, for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that every perturbation w, with $w_o \in J$ and $\|w_o\| < \delta$, satisfies $\sup_{[t_o,\infty)} \|w\| < \varepsilon$.

Here, in speaking of "every perturbation", it should be understood that we are refering to every perturbation, starting at every initial time $t_o \geq 0$. This simple notion of stability is too weak a condition to be useful in the numerical analysis of the Navier-Stokes problem. However, the following stronger condition provided the basis, in [1], for extending error estimates which are local in time to ones which are global in time.

Definition 2. The solution u of problem (1) is said to be exponentially stable if there exist numbers $\delta, T > 0$ such that every perturbation w, with $w_o \in J$ and $\|w_o\| < \delta$, satisfies $\|w(t_o + T)\| \leq \frac{1}{2}\|w_o\|$.

If u satisfies the conditions of either of these definitions with $\delta = \infty$, we say u is unconditionally stable.

An example of an exponentially stable flow is provided by simple axially symmetric Taylor cells occurring in flow between rotating coaxial cylinders. The situation is one in which, if the data are steady, there exist multiple steady solutions. If the difference between two such solutions is considered as a perturbation, it certainly will not decay. Thus Taylor cells are not unconditionally stable. Further, there generally exist even small perturbations whose decay in the L^2-norm is not monotonic. However, the cells are certainly stable in some sense, and intuitive considerations of linearization suggest that the decay of small perturbations is exponential.

Our development of a stability theory is based on several lemmas asserting the continuous dependence of solutions on their initial values. Below, c is a generic constant depending only on Ω.

Lemma 1. For every perturbation w of u there holds

$$\|w(t)\|^2 + \int_{t_o}^{t} \|\nabla w\|^2 d\tau \le \|w(t_o)\|^2 e^{cM^4(t-t_o)} ,$$

for $t \ge t_o$.

Lemma 2. For every $T > 0$, there exists a number $\delta > 0$ such that every perturbation w of u, with $w_o \in J_1$ and $\|\nabla w_o\| < \delta$, satisfies

$$\|\nabla w(t)\|^2 + \int_{t_o}^{t} \|\tilde{\Delta} w\|^2 d\tau \le \|\nabla w(t_o)\|^2 e^{c(1+M^4)(t-t_o)} ,$$

for $t_o \le t \le t_o + T$.

Lemma 3. For every $T > 0$, there exists numbers $\rho, B > 0$ such that every perturbation w of u, with $\|w(t_o)\| < \rho$, satisfies

$$\|\nabla w(t_o+T)\| \le B\|w(t_o)\| .$$

To prove these lemmas, one begins by writing the perturbation equation

$$(3) \qquad w_t - \Delta w + w\cdot\nabla w + u\cdot\nabla w + w\cdot\nabla u = -\nabla q ,$$

for the difference $w = v - u$ of the solutions of (2) and (1). Multiplying (3) by w and integrating leads to Lemma 1. Multiplying (3) by $\tilde{\Delta} w$ and integrating leads to Lemma 2. In both cases, the constants c arise from use of Sobolev's inequality. Lemma 3 is obtained by combining Lemmas 1 and 2. All these lemmas need somewhat more precise statements if v is understood only as a "weak solution."

Using the preceding lemmas, we can establish the equivalence of various definitions of stability. We prove the following simple theorem to give the flavor of our arguments.

Theorem 1. The stability condition of Definition 1 is equivalent to the following: For every $\varepsilon > 0$, there exists a number $\delta > 0$ such that every perturbation w, with $w_o \in \mathbf{J}_1$ and $\|\nabla w_o\| < \delta$, satisfies $\sup_{[t_o,\infty)} \|\nabla w\| < \varepsilon$.

Proof. First we check that the condition of Definition 1 implies that of Theorem 1. According to Lemma 2, one may guarantee that $\|\nabla w(t)\|$ is small, for $t_o \leq t \leq t_o + 1$, by taking $\|\nabla w_o\|$ small. Mindful of Poincaré's inequality $\|w_o\| \leq c\|\nabla w_o\|$, we see that if $\|\nabla w_o\|$ is taken small, then the condition of Definition 1 ensures that $\|w(t)\|$ is small for all $t \geq t_o$, and hence Lemma 3 ensures that $\|\nabla w(t)\|$ is small for all $t \geq t_o + 1$. Thus the condition of Theorem 1 is satisified.

Next we check that the condition of Theorem 1 implies that of Definition 1. According to Lemma 1, one may guarantee that $\|w(t)\|$ is small, for $t_o \leq t \leq t_o + 1$, by taking $\|w_o\|$ small. But then, $\|\nabla w(t_o+1)\|$ is also small, according to Lemma 3. Hence the condition of Theorem 1, considered with starting time $t_o + 1$, implies $\|\nabla w(t)\|$ is small for $t \geq t_o + 1$. Thus, remembering Poincaré's inequality, $\|w(t)\|$

is small for $t \geq t_o + 1$. This completes the proof.

The next theorem is more complicated, but proved by a similar type of argument.

Theorem 2. The stability condition of Definition 2 is equivalent to any one of the following conditions:

(i) There exist numbers $\delta, T > 0$ such that every perturbation w, with $w_o \in J_1$ and $\|\nabla w_o\| < \delta$, satisfies

$$\|w(t_o+T)\| \leq \frac{1}{2}\|w_o\| .$$

(ii) There exist numbers $\delta, \alpha, A > 0$ such that every perturbation w, with $w_o \in J$ and $\|w_o\| < \delta$, satisfies

$$\|w(t)\| \leq A e^{-\alpha(t-t_o)}\|w_o\| , \quad \text{for all} \quad t \geq t_o .$$

(iii) There exist numbers $\delta, \alpha, A > 0$ such that every perturbation w, with $w_o \in J_1$ and $\|\nabla w_o\| < \delta$, satisifies

$$\|\nabla w(t)\| \leq A e^{-\alpha(t-t_o)}\|\nabla w_o\| , \quad \text{for all} \quad t \geq t_o .$$

Much of the existing theory of hydrodynamic stability rests upon the "principle of linearized stability". This is a general assertion that in determining the stability of a solution u it suffices to consider the linearized perturbation equation

$$\bar{w}_t - \Delta\bar{w} + u\cdot\nabla\bar{w} + \bar{w}\cdot\nabla u = -\nabla q , \tag{4}$$

in place of the full nonlinear perturbation equation (3). In the following theorem we give a precise statement of the principle of linearized stability appropriate in the general context of the nonstationary problem. The proof is a direct and simple one, entirely bypassing spectral methods, as indeed one must in the nonstationary case.

Theorem 3. The solution u of problem (1) is exponentially stable if and only if there exist numbers $\alpha, A > 0$, such that every solution $\bar{w}$ of the linearized perturbation equation (4) satisfies

$$(5) \qquad \|\bar{w}(t)\| \le A\,e^{-\alpha(t-t_o)}\|\bar{w}_o\| , \quad \text{for} \quad t \ge t_o .$$

Proof. Let $\psi = \bar{w} - w$, where $\bar{w}$ and w are solutions of (4) and (3), respectively, satisfying $\bar{w}(t_o) = w(t_o) = w_o$. Subtracting (3) from (4) gives

$$\psi_t - \Delta\psi + u\cdot\nabla\psi + \psi\cdot\nabla u - w\cdot\nabla w = -\nabla q ,$$

for some scalar function q. Multiplying by ψ and integrating, this leads to

$$\frac{d}{dt}\|\psi\|^2 + \|\nabla\psi\|^2 \le c\|\nabla u\|^4\|\psi\|^2 + c\|\nabla w\|^4 .$$

Using Gronwall's inequality now yields

$$\begin{aligned} \|\psi(t_o+T)\|^2 &\le c\,e^{cM^4T}\int_{t_o}^{t_o+T}\|\nabla w\|^4 d\tau \\ &\le c\,e^{cM^4T}\sup_{[t_o,t_o,T]}\|\nabla w\|^2\int_{t_o}^{t_o+T}\|\nabla w\|^2 d\tau , \end{aligned}$$

for any fixed $T > 0$. Thus, if $\|\nabla w_o\|$ is sufficiently small, depending on T, Lemmas 2 and 1 imply

$$(6) \qquad \|\psi(t_o+T)\|^2 \le c\,e^{c(M_3^4+1)T}\|\nabla w_o\|^2\|w_o\|^2 .$$

Now suppose the condition of Theorem 3 holds. Choose T above such that (5) implies

$$\|\bar{w}(t_o+T)\| \le \frac{1}{4}\|w_o\| \; .$$

Then, also, provided $\|\nabla w_o\|$ is sufficiently small, (6) implies

$$\|\psi(t_o+T)\| \le \frac{1}{4}\|w_o\| \; .$$

Combining these gives

$$\|w(t_o+T)\| \le \|\bar{w}(t_o+T)\| + \|\psi(t_o+T)\| \le \frac{1}{2}\|w_o\| \; ,$$

showing that condition (i) of Theorem 2 is satisfied, implying the exponential stability of u .

To show that exponential stability implies linearized stability, we argue similarly, starting again with (6). This completes the proof.

In [1], we proved the following result as a consequence of Theorem 3. <u>If, for some given initial value</u> a , <u>the solution</u> u <u>of problem (1) is exponentially stable and satisfies</u> $\sup_{[0,\infty)} \|\nabla u\| < \infty$, <u>then so are all other solutions</u> $\tilde{u}$ <u>starting with initial values</u> $\tilde{a}$ <u>near</u> a . Here near is meant in the sense that $\|\nabla(\tilde{a} - a)\|$ should be sufficiently small. Concerning the behaviour of exponentially stable solutions as $t \to \infty$, we proved the following result.

<u>Theorem 4</u>. <u>If</u> u <u>is exponentially stable and</u> f <u>is time periodic with period</u> T , <u>there exists a time periodic solution</u> u_∞ , <u>with period</u> $T_\infty = kT$ <u>for some integer</u> k , <u>such that</u>

$$\|u(t) - u_\infty(t)\|_1 = O(e^{-\alpha t}) \; , \text{ as } \; t \to \infty \; .$$

<u>If</u> f <u>is time independent, then</u> u_∞ <u>is a steady state solution</u>.

4. Quasi-exponential Stability

Before giving a formal definition, let us point to some physical examples to explain what we mean by "quasi-exponential stability."

A simple example occurs in the Taylor experiment. At certain rotational speeds of the cylinders, the convection cells loose rotational symmetry, taking on a wavy appearance in the angular variable. Clearly, if the boundary values and forces are rotationally symmetric, a small angular shift in the pattern of waves will constitute an admissible perturbation with no tendency to decay. However, the same reasoning that leads one to believe simple Taylor cells are exponentially stable leads to the conclusion that wavy Taylor cells are "quasi-exponentially stable modulo spatial rotations", meaning that there is a fixed length of time during which the difference between a slightly disturbed flow and a slightly rotated image of the original undisturbed flow will decay to half the size of the initial perturbation, and further that the required rotation should be less than a fixed constant times the size of the initial perturbation.

Similarly, we consider "quasi-exponential stability modulo time shifts", provided the forces and boundary values are time independent. An important example is von-Kármán vortex shedding behind a cylinder. Another example is provided again by wavy Taylor cells, if the waves are precessing about the axis of symmetry. For such motions, perturbations which consist initially of the difference in the states of the motion at two slightly different times will have no tendency to decay. Yet all small perturbations may be expected to decay modulo slight shifts in the time phase.

Quasi-exponential stability is also considered modulo both time shifts and spatial rotations simultaneously. An example occurs in the Taylor experiment, when at certain rotational speeds of the cylinders wavy cells are observed to undergo a further time periodic oscillation, odd and even numbered cells alternately expanding and contracting. Though these cells are sometimes referred to as doubly time periodic, it seems clear that the second time periodicity is possible only because the first one is equivalent to a spatial periodicity. Our formal definition of quasi-exponential stability covers all the various possibilities at once.

Below, ϕ will represent the angular variable about an axis of symmetry common to both Ω and f, if there is one. For simplicity, we will write $u = u(\phi,t)$, suppressing in our notation the usually nontrivial dependence of u on the other spatial variables. The symbol ω will also denote an angle about the axis of symmetry, thought of as a rotation. If f and Ω do not possess a common axis of symmetry, it will be understood that $\omega = 0$. Further, for any Ω, if f is time independent we will consider time shifts denoted by s. If f is not time independent, it will be understood that $s = 0$.

Definition 3. We say u is quasi-exponentially stable if there are numbers $\delta, T, B > 0$ such that for every perturbation w, with $w_0 \in \mathbf{J}$ and $\|w_0\| < \delta$, there exist a time shift s and a spatial rotation ω satisfying

(7) $$|s| + |\omega| \le B\|w_o\| ,$$

(8) $$\|(v-\tilde{u})(t_o+T)\| \le \frac{1}{2}\|w_o\| ,$$

where v *is the solution of the perturbed problem (2) corresponding to the perturbation* w , *and* $\tilde{u}(x,t) = u(\phi+\omega, t+s)$.

In [1], the theory of quasi-exponential stability has been developed like that of exponential stability, beginning with an analogue of Theorem 2 giving a number of equivalent definitions. We omit these here, but would like to state the corresponding principle of linearized stability. To understand the modification needed in Theorem 3, note that if f is independent of time, and/or Ω and f possess a common axis of rotational symmetry with the corresponding angular variable ϕ , then the derivatives u_t and/or u_ϕ are necessarily solutions of the linearized perturbation equation (4).

Theorem 5. The solution u *of problem (1) is quasi-exponentially stable if and only if there exist numbers* $\alpha, A, B > 0$, *such that every solution* $\bar{w}(t)$ *of the linearized perturbation equation (4) satisfies*

(9) $$\|\bar{w}(t) - \sigma u_t(t) - \rho u_\phi(t)\| \le A e^{-\alpha(t-t_o)} \|\bar{w}(t_o)\|$$

for $t \ge t_o + 1$, *where* σ *and* ρ *are scalar multipliers satisfying*

(10) $$\max(|\sigma|,|\rho|) \le B\|\bar{w}_o(t)\| .$$

Nonzero multipliers σ *and* ρ *are required in (9) if and only if nonzero time shifts* s *and nontrivial rotations* ω , *respectively, are required in (8).*

Corresponding to Theorem 4, we have the following result on the behaviour of quasi-exponentially stable solutions as $t \to \infty$.

Theorem 6. Suppose that u is quasi-exponentially stable and that f is time independent. Then

$$\|(u-u_\infty)(t)\|_1 = O(e^{-\alpha t}) \text{ , as } t \to \infty \text{ ,}$$

where u_∞ is a solution of the Navier-Stokes equations, also corresponding to f, with the following property: There exists an angle ω such that the function $u_\infty(\phi+\omega t,t)$ is time periodic. One may take $\omega = 0$ if rotations are not required in (8). If time shifts are not required in (8), there exists ω such that $u_\infty(\phi+\omega t,t)$ is time independent.

5. Contractive Stability to a Tolerance.

We think that many unstable flows possess the following type of partial stability, which provided a basis for error estimates to a tolerance in our study of the numerical analysis of the Navier-Stokes problem.

Definition 3. The solution u of problem (1) is said to be "contractively J_1-stable to a tolerance" if there exist positive numbers ρ, δ, A and T, with $\rho < \delta$, such that for every perturbation w of u satisfying $\|\nabla w_o\| < \delta$, there holds

$$\|w(t_o+T)\| < \rho \text{ , } \sup_{[t_o,t_o+T]} \|\nabla w\| \le A \text{ .}$$

To understand this concept in terms of an example, imagine that

Ω is a section of pipe or tubing and let smooth boundary values be prescribed for a flow entering across an upstream section and exiting across a downstream section. Adjusting the rate of flow and the length of the pipe, one may expect to observe incipient turbulence in a flow which is yet, in some sense, stable to larger disturbances. Small perturbations in the nearly uniform upstream flow begin to grow. However, before they grow very large they pass out of Ω across the downstream boundary. Yet, their effect may not decay to zero. Even as they pass downstream they influence the upstream flow; the flow is analytic after all. Their effect might be likened to the introduction of new perturbations upstream, which in their turn will grow, pass downstream, and again create new perturbations upstream. If a larger disturbance is introduced, its effect will decay to the same ambient level of minor disturbances. Another type of example occurs in von-Kármán vortex shedding, if there are slight instabilities in the vortices. Still another occurs in the Taylor experiment, when wavy cells appear with slightly turbulent cores.

Reference

1. Heywood, J.G., and Rannacher, R., Finite element approximation of the nonstationary Navier-Stokes problem, Part II: Stability of solutions and error estimates uniform in time.