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Combinatorial Set Theory and its Applications to Topology[1]

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Combinatorial set theory - although a beautiful subject in its own right - has become increasingly important in recent years as a "bag of tricks" for solving problems in a variety of mathematical fields. Outside of logic, this phenomenon has been most evident in point-set topology, although infinite abelian group theory is not far behind, and there have been important applications to measure theory and functional analysis as well. Although I presume many Japanese logicians have heard of forcing, Martin's Axiom, and $\diamond$, I suspect most are unaware of the more sophisticated techniques of combinatorial set theory that have recently become important. In this survey, I shall mention these techniques and give references for those who wish to pursue them further. As illustrations of the applicability of combinatorial set theory, I shall also mention a variety of topological problems that these techniques have solved. However I shall emphasize the set theory, since it is applicable to many fields of mathematics.

Set theory flourished in the 1930's, as is evident to any reader of Fundamenta Mathematicae. In the 1940's and 50's, the subject stagnated. Its increasing formalization, perhaps brought on by Gödel's proof of the consistency of the continuum hypothesis or the general emphasis in logic on proof theory at that point, tended to isolate it from the rest of mathematics, while formal developments failed to improve Gödel. Cohen's invention of forcing in 1963 gave the field a fresh impetus, and vigorous activity has continued unabated for twenty years. The early 1960's also saw the full development of the partition calculus by its Hungarian practitioners, as well as the beginning of the modern theory of large cardinals. By the late 1960's and early 1970's, combinatorial set theory burst upon the topological scene with the invention of Martin's Axiom and $\diamondsuit$, so let me pause now to be specific about the set-theoretic techniques and topological applications current at that point in time.

1. Partition Calculus.

This subject is concerned with generalizations and variations of Ramsey's theorem, and is known for its $\kappa \longrightarrow (\lambda)^n_m$ notation. I believe most logicians are somewhat acquainted with this area so I won't pause for definitions. The best sources (in ascending order of comprehensiveness) are [K_1], [Wi] and [EHMR]. There were a number of clever applications of patition calculus by Hajnal and Juhász to various problems concerning cardinal invariants of topological spaces. For example one calculates the cardinality of Hausdorff spaces in terms of the supremum of cardinalities of discrete subspaces, or in terms of numbers of disjoint open sets and cardinalities of local bases, etc. The best sources here are the two books of Juhász, [Ju_1], [Ju_2].

2. Combinatorial Principles in the Constructible Universe L.

Jensen's $\diamondsuit$ and its more sophisticated relations are powerful induction principles that enable one to e.g. handle

$2^{\aleph_1}$ objects in ω_1 steps. Recall $\diamondsuit$ says that there are sets $\{A_\alpha\}_{\alpha<\omega_1}$, $A_\alpha \subseteq \alpha$, such that for each $A \subseteq \omega_1$, $\{\alpha : A \cap \alpha = A_\alpha\}$ is uncountable, indeed stationary. Jensen invented it to isolate the combinatorics in L that yield a Souslin tree (for this proof, see e.g. [K_2]), and it has had many other applications. The most typical topological one is Ostazewski's construction from $\diamondsuit$ of a countably compact perfectly normal space which is not compact, via an inductive construction of the topology. Good places to read about this are [R_2] and [V].

3. Combinatorial Properties of Sets of Natural Numbers.

A typical such principle is P(c): Suppose $\{A_\alpha\}_{\alpha<\lambda}$, $\lambda < 2^{\aleph_0}$, are infinite subsets of ω such that each finite collection of A_α's has infinite intersection. Then there is an infinite $A \subseteq \omega$ such that for all α, $A - A_\alpha$ is finite.

There are many other such principles, but this is the most useful and powerful. It enables diagonal constructions which normally work for countably many steps, to be extended for $< c$ steps. For example P(c) implies the product of fewer than $2^{\aleph_0}$ sequentially compact spaces is sequentially compact (Booth), and it implies that sets of reals of power less than $2^{\aleph_0}$ have the property that in the subspace topology they inherit from the reals, every subset is an F_σ (Rothberger). There are several sources where this material is dealt with, e.g. [vD], [T_1], [T_2], [R_2], [K_2].

4. Trees.

Combinatorial set theory has long considered Aronszajn, Souslin, and Kurepa trees and generalizations of them. A tree of height ω_1 with countable levels is Aronszajn if it has no uncountable branches, is Souslin if it has no uncountable branches and no uncountable antichains, and is Kurepa if it has more than $\aleph_1$ uncountable branches. Aronszajn trees exist in ZFC; the existence of Souslin trees is consistent and independent; the existence of Kurepa trees is consistent, while their non-existence is equiconsistent with the existence of an inaccessible cardinal. See [K_2] or [To] for all this. Interesting topological applications include that if there is a Souslin tree, there is a regular hereditarily separable non-Lindelöf space [R_1], and that the existence of a Lindelöf ω_1-metrizable space of cardinality $\geq \aleph_2$ is equivalent to the existence of a Kurepa tree with no Aronszajn subtree [JW].

5. Forcing.

Until the late 1970's there were only a few applications of forcing to topology; the lack of suitable texts made it difficult to learn the subject. Some early noteworthy examples include of course Souslin's Hypothesis - if one wishes to consider this as topology, the author's work on the normal Moore space problem [T_1], and some results on hereditarily Lindelöf and hereditarily separable spaces by Hajnal and Juhász, e.g. [HJ].

6. Martin's Axiom.

Topologists were pleased with Martin's Axiom because they could enjoy part of the power of forcing without actually learning it. Recall that Martin's Axiom asserts that if $\mathcal{P}$ is a partial order which satisfies the countable chain condition (i.e. every collection of pairwise incompatible elements is countable), and $\mathcal{D}$ is a collection of $< 2^{\aleph_0}$ dense subsets of $\mathcal{P}$, then there is a filter on $\mathcal{P}$ meeting each element of $\mathcal{D}$.

There are literally hundreds of applications of Martin's Axiom to topology. Some can be found in [R_1], [K_2], [T_3], and the all-encompassing [Fr]. To mention some noteworthy ones, MA $\longrightarrow$ P(c) (Booth, see [R_2]); MA + $2^{\aleph_0} > \aleph_1 \longrightarrow$ Souslin's Hypothesis (Solovay-Tennenbaum, see [R_2]); MA + $2^{\aleph_0} > \aleph_1 \longrightarrow$ countably compact perfectly normal spaces are compact (Weiss [W] or see [V]).

Also worthy of mention are some older techniques such as the next two:

7. Stationary sets.

It can be argued that "stationary" is to ω_1 as "infinite" is to ω. The concept thus appears frequently in examples and proofs in topology, especially Fodor's Theorem, which asserts that a function which is regressive on a stationary set S is constant on a stationary $S' \subseteq S$. Fleissner brought the concept into prominence in connection with the normal Moore space problem - see e.g. the exposition in [T_4].

8. Independent families; almost disjoint families; Δ-systems.

There are several useful properties that a collection of subsets of a cardinal κ can have; the three I have singled out are frequently useful in set theory and topology. All three are discussed in [K_2]. A family $\mathcal{A}$ of subsets of κ is independent if for every finite $\mathcal{B} \subseteq \mathcal{A}$, $|\cap\{A_\varepsilon : A \in \mathcal{B}\}| = \kappa$, where A_ε is either A or $\kappa - A$. It can be shown that for every κ, there is a family of 2^κ independent subsets of κ. This is essentially equivalent to the fact that the product of 2^κ copies of the discrete space with 2 points has a dense set of power κ. Hausdorff used independent sets to show there are $2^{2^{\aleph_0}}$ ultra-filters on ω.

A family $\mathcal{A}$ of subsets of κ is almost disjoint if any two have intersection of power $< \kappa$. The question of the cardinality of almost disjoint families is not so simple: see [B_1]. Such families can arise from and are useful in constructing Hausdorff spaces, since e.g. convergent sequences have almost disjoint ranges. See e.g. [T_1].

The simplest case of a Δ-system involves finite subsets of ω_1. A collection $\mathcal{A}$ of such subsets is said to form a Δ-system with root r if $A \cap A' = r$ for all distinct $A, A' \in \mathcal{A}$. The Δ-system lemma asserts that given uncountably many finite subsets of ω_1, uncountably many form a Δ-system. This lemma is frequently useful in reducing finite support problems to finite problems, for example showing that if every finite product of countable chain condition topological spaces satisfies the countable chain condition, so does every product.

During the 1970's, point-set topologists - who now called themselves "set-theoretic topologists" - busied themselves applying the above techniques, but with very few exceptions avoided learning forcing. The little bit of logic involved was too much for them. Fortunately, Kunen's book is now changing that situation. The set-theorists however continued to develop new techniques and that is what I shall discuss now. There is not much new in items 1, 3, 4, 7, 8 above, but 2, 5 and 6 have had many exciting developments, as has 9: large cardinals.

Let me first consider 2). Jensen developed a whole series of powerful combinatorial propositions in L. The best source for Jensen's work in L is Devlin's forthcoming [D_2], of which [D_1] can be considered a very early version. The most difficult of Jensen's combinatorial principles involve morasses, which in their simplest form are devices for constructing objects of size $\aleph_2$ in $\aleph_1$ steps, using countable subobjects. The canonical example of such an object is a Kurepa tree, and indeed a morass can be thought of as a Kurepa tree with additional structure. There are only a few people in the world who are comfortable with morasses, but there have recently been great simplifications of the theory, due to Velleman [Ve] and to Shelah and Stanley [SS], who find forcing axioms equivalent to the existence of morasses, axioms that are much easier to work with, although still difficult.

We should also mention Jensen's profound Covering Lemma, which says that either

a) for every uncountable set X of ordinals, there is an uncountable constructible Y such that $X \subseteq Y$ and $|X| = |Y|$ or b) "$0^{\#}$ exists" (which implies the consistency of the existence of inaccessible cardinals).

A generalization due to Jensen and Dodd [DJ] uses the Core Model K instead of L, and gets the consistency of (many) measurable cardinals in b). An important topological application due to Fleissner is that the normal Moore space conjecture implies the consistency of the existence of measurable cardinals, and hence that its consistency cannot be proved [F].

The theory of forcing has advanced significantly in the past several years, culminating in the development of iteration axioms much stronger than Martin's Axiom. The principal figures here have been Shelah and Baumgartner. Iterated forcing is simply repeated forcing. It is useful for proving the consistency of universal statements, because one takes care of each object in question, one after another. E.g. if one wants to show - as did Shelah [S] - that there is no P-point in $\beta N - N$, one systematically forces to kill all possible candidates for P-points. Among the difficulties is to have control over the end result of the iteration, knowing the individual steps. For example, are cardinals preserved? There have been hundreds of sophisticated applications of iterated forcing techniques in the past few years; the best introduction is [B_2]. The most promising new technique in iterated forcing is that of proper forcing, introduced by Shelah.

Definition. $P_\kappa\lambda = \{X \subseteq \lambda : |X| < \kappa\}$. $\mathcal{C} \subseteq P_\kappa\lambda$ is closed if every chain in $\mathcal{C}$ of length $< \kappa$ has an upper bound in $\mathcal{C}$. $\mathcal{C}$ is unbounded if $(\forall X \in P_\kappa\lambda)(\exists Y \in \mathcal{C})(X \subseteq Y)$. $\mathcal{S} \subseteq P_\kappa\lambda$ is stationary if it meets every closed unbounded set. A partial order is proper if forcing with it preserves stationary subsets of $P_{\aleph_1}\lambda$, for all λ.

The Proper Forcing AXiom (PFA) is Martin's Axiom with the countable chain condition replaced by "proper". It is not difficult to show that both countable chain condition and countably closed partial orders are proper. What does require effort is to show that proper is preserved by iterations -see e.g. [D_3]. Most applications of proper forcing so far have in fact used finite mixed iterations of countable chain condition and countably closed partial orders.

Theorem (Baumgartner-Shelah). Con(there is a supercompact cardinal) $\longrightarrow$ Con(PFA + $2^{\aleph_0} = \aleph_2$).

For a proof, see e.g. [D_3]. It is not known whether $2^{\aleph_0}$ can be greater than $\aleph_2$ here. The best source to learn about PFA is Baumgartner's article [B_3] in the Handbook of Set-theoretic topology. (See also [D_3] and [S].) This forthcoming compendium of more than 20 survey articles will set the tone for set-theoretic topology for years to come. (There are three additional articles of set-theoretic interest in this volume: [K_3], [M], [To].) Some typical applications of PFA are

1) there is a Boolean algebra of cardinality $2^{\aleph_0}$ not embeddable in $\mathcal{P}(\omega)$/(finite sets),

2) every real-valued function from an uncountable set

of reals is monotonic on an uncountable set.

3) every T_3 hereditarily separable space is hereditarily Lindelöf.

Actually, these particular ones do not need large cardinals. PFA + $2^{\aleph_0} = \aleph_2$ does however imply

4) there are no Kurepa trees,

so in fact it does have large cardinal strength. For more details, see [B_3] where many other topological and set-theoretic applications are given.

Finally, I wish to talk briefly on the subject of large cardinals. It has become clear in the past few years that these cardinals cannot be ignored, since they have consequences for small cardinals such as $\aleph_1$ and $2^{\aleph_0}$. This has become evident not only in topology, but in measure theory and the structure theory of ideals [BTW] as well. The best reference on large cardinals is the survey [KM], although [J] often needs to be consulted for details. In general - and speaking loosely - large cardinals have the property that if a proposition φ holds for all $\lambda < \kappa$, then $\varphi(\kappa)$ holds; and for "very" large cardinals, φ holds for all $\lambda \geq \kappa$. (A typical large cardinal is a weakly compact one; a typical very large cardinal is a supercompact one.) To make these cardinals relevant to "ordinary" mathematics, one forces to collapse them, say to $\aleph_1$ or $\aleph_2$, or else adds κ many subsets of some smaller cardinal, e.g. $\aleph_0$, and argues that enough power of the large cardinal remains to get e.g. $(\forall\lambda < \aleph_2)\varphi(\lambda) \longrightarrow (\forall\lambda)\varphi(\lambda)$. For example, in the proof of Con(PFA + $2^{\aleph_0} = \aleph_2$), one only has to find generic sets for partial orders of cardinality less than the supercompact (which becomes $\aleph_2$), so that the proof is much like that for MA. Putting it another way, a typical iterated forcing argument allows one to take care of objects of cardinality $< \kappa$ in κ stages; if κ is very large, one argues by reflection that objects of unbounded cardinality have been taken care of as well. Thus large cardinals plus iterated forcing can be used to prove the consistency of universal statements without cardinal bounds. For a typical topological example, see [TW], where it is shown that

Theorem Con(there is a supercompact) $\longrightarrow$ Con(every normal Moore space is metrizable).

I should also touch briefly on two other areas of combinatorial set theory: the structure theory of ideals and the Axiom of Determinancy. The former is an outgrowth of the study of saturated ideals. As yet its topological applications are few; a noteworthy one is to a problem of Katetov concerning topological spaces without isolated points on which every real-valued function is continuous at some point. See [KST]. The only topological application of AD can also be found there, as Woodin's inner model for ZFC is used.

I have treated the subject of combinatorial set theory and its applications to topology all too briefly, but this is after all supposed to be a concise survey rather than a book. Let me close by expressing the hope that there will be more Japanese work in these areas in the future.

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