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On the Uniformization of Analytic Sets with the Countable Sections and Related Results

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Introduction. In a letter of Hadamard to Borel [1], Hadamard discussed an effective choice of an element from a given Borel set. Then Luzin [8] introduced the general problem of uniformization, and announced several results. One of these is that every analytic set is uniformized by the defference of two analytic sets. In 1978, Steel and Martin gave an example of an analytic set which can not be uniformized by the defference of two analytic sets(Cf. Mochovakis [10]).

In this paper, we will state some results concerning uniformization of analytic and Borel sets with special properties and enumerability of Borel, analytic and co-analytic sets. Our main aim is to give the positions in the σ-algebra generated by the analytic sets. We use the recursion theoretic methods, or Effective Descriptive Set Theory. We assume familiarity with [12,14], and use the notations of them. In our proofs we shall often use the following uniformization theorem which was first proved by Kondô in the classical case:

Number Uniformization Theorem (Kondô [3], Kreisel [7]). *Every Π^1_1 set in ${}^\omega\omega \times {}^\omega\omega$ can be made uniform by a set of the same class.*

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§1. Our main tools in this paper are Harrison's Effective Perfect Set Theorem [2] and

Theorem 1. There is an Π^1_1 relation $C(\alpha,n)$ and a mapping $(\alpha,n) \longmapsto \delta(\alpha,n) \in {}^\omega\omega$ defined on the set C such that

$$\beta \in \Delta^1_1(\alpha) \Leftrightarrow \exists n[C(\alpha,n) \,\&\, \beta = \delta(\alpha,n)]$$

and the relation "$\beta = \delta(\alpha,n)$" is Δ^1_1 uniformly on $(\alpha,n) \in C$. Conversly there is a mapping $(\alpha,\beta) \mapsto \delta^{-1}(\alpha,\beta) \in \omega$ defined on the set $\{(\alpha,\beta) \mid \beta \in \Delta^1_1(\alpha)\}$ such that

$$\beta \in \Delta^1_1(\alpha) \Rightarrow \delta(\alpha, \delta^{-1}(\alpha,\beta)) = \beta$$

and the relation "$n = \delta^{-1}(\alpha,\beta)$" is Δ^1_1 uniformly on $\beta \in \Delta^1_1(\alpha)$.

The first part of Theorem 1 is due to Kechris [3].

§2. We can extend Steel-Martin result (Cf. Introduction) as follows

Theorem 2. There is an Σ^1_1 set in ${}^\omega\omega \times {}^\omega\omega$ which can not uniform by an $(\Sigma^1_1)_{\rho\sigma}$ set.

This shows that Watanabe's result [17], which is an improvement of Luzin [10], (Cf. Tanaka [15]) is the best possible.

For the analytic sets with the countable sections, we have

Theorem 3. Every Σ^1_1 set in ${}^\omega\omega \times {}^\omega\omega$ with the countable sections can be uniform by an $(\Sigma^1_1)_\rho$ set.

For the negative side as Theorem 1 we have

Theorem 4. There is an Σ^1_1 set in ${}^\omega\omega \times {}^\omega\omega$ with the countable sections which can not be uniform by an analytic set.

Corollary 5. There is an Σ^1_1 set in ${}^\omega\omega \times {}^\omega\omega$ with the countable sections which can not be uniform by a co-analytic set.

§3. Luzin [10] showed that every Borel(analytic) set with the countable sections is the union of countably many Borel(analytic) curves(Cf.Kondô [7] and Tanaka [16]). We can prove

Theorem 6. Let B be an Δ^1_1 set in ${}^\omega\omega \times {}^\omega\omega$ such that if $B^{\langle\alpha\rangle}$ is non-empty then $B^{\langle\alpha\rangle}$ is denummerable. Then there is an Δ^1_1 set B* in $\omega \times {}^\omega\omega \times {}^\omega\omega$ such that

(i) $B^{*\langle n\rangle}$s are Δ^1_1 pairewise disjoint uniformizators of B,

(ii) $B(\alpha,\beta) \Leftrightarrow \exists n B^*(n,\alpha,\beta)$.

Corollary 7. Let A be an Σ^1_1 set in ${}^\omega\omega \times {}^\omega\omega$ such that if $A^{\langle\alpha\rangle}$ is non-empty then $A^{\langle\alpha\rangle}$ is denummerable. Then there is an Σ^1_1 set A* in $\omega \times {}^\omega\omega \times {}^\omega\omega$ such that

(i) $A^{*\langle n\rangle}$s are Σ^1_1 pairewise disjoint curves,

(ii) $A(\alpha,\beta) \Leftrightarrow \exists n A^*(n,\alpha,\beta)$.

§4. Sierpiński [14] showed that every denummerable G_δ set is effectively denummerable. We have

Theorem 8. Let E be a denummerable Δ^1_1 set of reals. Then the members of E can be enumerated by an Δ^1_1 mapping.

Kondô [5] showed that every denummerable analytic set is effectively denummerable. Sampei [12] and Tanaka [15] showed that every denummerable Σ^1_1 set of reals can be enumerated by an Δ^1_2 mapping. We can prove

Theorem 9. Let E be a denummerable Σ^1_1 set of reals. Then the members of E can be enumerated by an $(\Sigma^1_1)_{\rho\sigma} \cap (\Sigma^1_1)_{\rho\sigma c}$ mapping.

Theorem 10. There is a denummrable Σ^1_1 set of reals whose members can not be enumerated by an Σ^1_1 mapping.

Corollary 11. There is a denummerable Σ^1_1 set of reals whose members can not be enumerated by an Π^1_1 mapping.

For the denummerable Π^1_1 set of reals we have

Theorem 12. Let E be a denummerable Π^1_1 set of reals. Then the members of E can be enumerated by an $(\Delta^1_2)_{\rho\sigma}$ mapping.

This is an answer of a problem of Kondô [6].

In the neare future, we will publish the detaild proofs of the above theorems elsewhere.

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