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# DEFINABILITY IN $L^p$-SPACES

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This report consists mainly in the excerpts from the references [6]~[10]. It is divided into three parts; the introduction, the proof-theoretical background and the mathematical development.

I.G.Takeuti in his book [5] defined a conservative extension of Peano arithmetic (with finite predicate types) in which he developed differential and integral calculus. In his theory, the reals are defined as the Dedekind cuts of the rationals, and the basic logic of the theory is classical. The whole universe is restricted to the "arithmetically definable" world, where an object is said to be arithmetically definable if the quantifiers are restricted to the rationals.

Since the union of an arithmetically definable sequence of reals is again an arithmetically definable real, it is obvious that a bounded increasing sequence of reals is convergent. The intermediate value theorem for a continuous function, the mean value theorem and all other theorems in calculus can be stated and proved in the classical manner.

Now, we believe that much part of modern analysis can be naturally interpreted in a modest extension of Peano arithmetic, whose logic is classical and which has no peculiar principles specifically designed for such an interpretation. It is our intention to execute such a program to see (1) what formal system is necessary for this purpose, (2) which characterization of a mathematical concept is to be taken and (3) how the mathematical objects look like in our formulation.

We employ "definable arithmetic" with "definitions by definable induction" (which we abbreviate to DDI) as the basis of our machinery. Along this line, we have worked on a few areas of analysis as examples.

Here we shall explain how to interpret the abstract theory of integration in the definable setting.

II. Definition 1. 1) Type. There are two sorts of atomic types; one for the rationals and the other for the elements of a space. The compound (predicate) types are difined as usual.

2) The language consists of the symbols of the arithmetically definable theory of the reals augmented by the following.

a) Variables for all types.

b) The special symbols X,L,J and eq.

c) Predicate symbols for definitions by definable induction (DDI), $I_0, I_1, I_2, \cdots$

3) An expression in our language is said to be "definable" if the quantifiers in it are restricted to those of atomic types.

4) Terms, formulas, abstracts and sequents are defined as usual. The abstracts are restricted to the definable ones.

5) The logical system $L$ is the predicate calculus of our language augmented by the comprehension rule applied to our definable abstracts.

6) The sets of axioms.

$A$: the set of axioms of arithmetic, where the mathematical induction and the equality axiom are formulated in terms of the higher universal quantifiers.

$B$: Axioms on the abstract theory of Daniell integral, $X=(X,L,J)$, where X represents a space, L a family of elementary functions (from X to the reals) and J the integral on L. X is a set with eq as the equivalence relation.

$C$: DDI. Let $K_i(m,\vec{\phi},\vec{\psi},\Phi)$ be a definable formula which does not contain any of $I_i$, $I_{i+1}$, ...., where m, $\vec{\phi}$, $\vec{\psi}$, $\Phi$ exhaust all the free variables $K_i$ may contain, the variables in $\vec{\phi}$ are of atomic types and $\Phi$ is of appropriate type.

$$\forall m \forall \vec{\phi} \forall \vec{\psi} (I_i(m,\vec{\phi},\vec{\psi}) \equiv K_i(m,\vec{\phi},\vec{\psi},I_i[m])),$$

where $I_i[m]$ abbreviates

$$\{n,\vec{x}\}(n<m \wedge I_i(n,\vec{x},\vec{\psi})).$$

It is an expirical fact that the mathematical axioms (such as those in $B$ above) are in the prenex universal form with respect to higher types. What is normally assumed to exist is regarded as a parameter in our approach

7) A sequent $\Gamma\rightarrow\Delta$ of our language is said to be a theorem of $T$ if

$$A, B, C, \Gamma\rightarrow\Delta$$

is provable in the system $L$.

8) Definable instantiation.

Let G be one of the axioms in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. Since G is universal with respect to higher types, it must be of the form

$$\forall\psi_1\cdots\forall\psi_n F(\psi_1,\ldots,\psi_n),\ n\geq 0,$$

where $\psi_1,\ldots,\psi_n$ are of higher types, and $F(\psi_1,\ldots,\psi_n)$ is definable. Let $J_1,\ldots,J_n$ be any abstracts of appropriate types which do not contain higher type free variables. Then

$$F'\colon\ \forall\phi_1\cdots\forall\phi_m\ F(J_1',\ldots,J_n')$$

will be called a definable instantiation of G, where $J_i'$ is obtained from $J_i$ by replacing all the free variables (of atomic types) which do not occur in $F(\psi_1,\ldots,\psi_n)$ by appropriate bound variables, $\phi_1,\ldots,\phi_m$.

Henceforth $\mathcal{A}'$ will stand for the set of all definable instantiations of $\mathcal{A}$, and $\mathcal{A}^*$ will stand for a finite sequence from $\mathcal{A}'$. Similarly for $\mathcal{B}$ and $\mathcal{C}$.

9) Systems.

$M$ is obtained from $L$ by suppressing all the variables of higher types.

$P$ is the system $M$ augmented by the following.

1°. Rule of inference: mathematical induction applied to the formulas of $M$.

2°. Initial sequents: formulas of $\mathcal{A}'$ and $\mathcal{C}'$.

Theorem 1. Let $\Gamma\to\Delta$ be a sequent which expresses an elementary theorem of integration. Then it is a theorem of $T$, that is,

$$\mathcal{A},\mathcal{B},\mathcal{C},\Gamma\to\Delta$$

is provable in $L$, hence without cuts.

The proof of this theorem is our major task.

Theorem 2. $P$ is consistent.

The basis of the proof is the accessibility of the order of the system $(\Pi,\prec)$ defined below. Let $(\Lambda,<)$ be an accessible (ordered) structure.

(1) 1° The symbol 0 is a connected element of $\Pi$.

2°. If $\alpha\in\Lambda$ and $\beta\in\Pi$, then $(\alpha,\beta)$ is a connected element of $\Pi$.

3°. Suppose $\alpha_1,\ldots,\alpha_n$ are connected elements of $\Pi$, $n\geq 2$. Then $\alpha_1\#\ldots\#\alpha_n$ is a non-connected element of $\Pi$.

(2) We define the orders $\prec$ and $\prec'$ for $\Pi$.

(2.1) If $\beta$ is not 0, then $0\prec\beta$ and $0\prec'\beta$

(2.2) # is interpreted as the natural sum for both $\prec$ and $\prec'$.

(2.3) $(\alpha,\beta)\prec'(\gamma,\delta)$ if $\alpha\prec\gamma$, or $\alpha=\gamma$ and $\beta\prec\delta$.

(2.4) $(\alpha,\beta)\prec(\gamma,\delta)$ if one of the following holds.

(2.4.1) $(\alpha,\beta)\prec'(\gamma,\delta)$ and $\beta\prec(\gamma,\delta)$.

(2.4.2) $(\alpha,\beta)\preceq\delta$.

Here we need a $\Lambda$ whose order type is $\exp(\omega, \exp(\omega,2))$.

Define $\omega^{\sim}=\{j^{\sim};\ j\in\omega\}$ and $K_i=\{(j,i);\ j\in\omega\cup\omega^{\sim}\}\cup\{\infty_i\}$, where $K_i$ is ordered so that $j<j^{\sim}<j+1<\infty_i$ for every $j$ in $\omega$. The order type of $K_i$ is $\exp(\omega,2)$. The scales we need are $r(I_i;A)$(the rank of $I_i$ in A) and $n(A)$(the norm of A). $r(I_i;A)$ is defined to be an element of $K_i$, and $n(A)$ is defined to be an element of $\exp(\omega,\exp(\omega,2))$. Notice that in our particular case, we need not worry about comprehensions, since there are none.

Theorem 3. (Relative soundness) The theory $T$ is sound relative to definable instantiations of $B$.

Proof. Suppose $B'$ is consistent (with $P$). Then $\{A',B',C'\}$ is consistent with $M$, and hence $\{A,B,C\}$ is consistent with $L$. But $T$ is a consequence of $A$, $B$ and $C$ (in $L$), and hence is consistent (since the cut elimination of $L$ can be proved with $\exp(\omega,3)$).

Note. In defining subsets, relations and functions, it is always required that these notions be closed with respect to equivalent objects.

Definition 2. Two concepts $\Sigma$ and $\Sigma'$ are said to be "mutually definably interpretable" if there are definable $\Theta^*$ and $\Xi^*$ such that

$$\Sigma(\Theta)\to\Sigma'(\Xi^*(\Theta)) \text{ and } \Sigma'(\Xi)\to\Sigma(\Theta^*(\Xi))$$

are both theorems of $T$, where $\Theta$ and $\Xi$ are parameters.

Proposition 1. 1) The definability property and the subset property are both preserved under the basic set-theoretical operations.

2) The definability property is preserved under the following operations on the reals and the functions; $a\phi$, $\Sigma$, $\Pi$, max, min, limsup, liminf, lim, the absolute value, $\phi^+$ and $\phi^-$.

III. The proof of Theorem 1.

Proposition 2. The immediate (mathematical) consequences of $\mathcal{B}$ are the theorems of $\mathcal{T}$.

We shall henceforth state the propositions simply as mathematical statements, although they should be read as "the theorems of $\mathcal{T}$".

Definition 3. nls(E,χ): $ss(X,E) \wedge \chi \subset L \wedge \forall n(\chi(n) \leq \chi(n+1)) \wedge \forall x \in E \forall r > 0 \exists n(\chi(n,x) > r) \wedge \lim J(\chi(n)) \in R$

(E is a null set by χ.)

$$ae(x,P,E,\chi):\quad nls(E,\chi) \wedge \forall x \notin E\, P(x)$$

$$itg(f,\Phi,E,\chi):\quad \Phi \subset L \wedge ae(x, f(x)=\Sigma\Phi(n,x), E, \chi) \wedge \Sigma J(|\Phi(n)|) \in R$$

(f is integrable with respect to Φ, E and χ.)

$$J^1(f,\Phi,E,\chi):\quad \limsup\{\Sigma\{J(\Phi(i));\ i \leq m\};\ m=1,2,\ldots\}$$

This may be abbreviated to $J^1(f)$, or even to J(f). Notice that the definiens above is an extended real.

($J^1(f)$ is the Daniell integral of f with respect to Φ, E and χ if f is integrable.)

Proposition 3. 1) $J^1$ is definable, and is independent of the parameters when itg(f,Φ,E,χ) is assumed.

2) The properties for (L,J) (in $\mathcal{B}$) hold for (itg,$J^1$).

Proposition 4. All the basic properties of the Daniell integral, such as Fatou's lemma and Lebesgue's dominated convergence theorem, hold.

Let us state Fatou's lemma as an example.

(Fatou's lemma) There are definable Ψ*, E* and χ* such that

$$\forall n\ \mathrm{itg}(F(n),\Theta(n),\Lambda(n),\Xi(n)),$$

$$\forall n\ \mathrm{ae}(x,F(n,x)\geq 0,\Lambda(n),\Xi(n)),$$

$$\mathrm{liminf}J(F(n))<\infty$$

$$\rightarrow \mathrm{ae}(x,\mathrm{liminf}F(n,x)<\infty,E^*,\chi^*)$$

$$\wedge[\mathrm{ae}(x,g(x)=\mathrm{liminf}(F(n,x),E^*,\chi^*)$$

$$\vdash \mathrm{itg}(g,\Psi^*,E^*,\chi^*)\wedge J(g)\leq \mathrm{liminf}J(F(n))].$$

These objects can be easily obtained through the course of the mathematical proof.

[A1] $\forall\phi(L(\phi)\vdash L(1\wedge\phi))$

We shall work in the theory $T$ with $\mathcal{B}_0=\mathcal{B}+\{[A1]\}$.

Definition 4. 1) $\mathrm{mbl}(f,\Theta,\Lambda,\Xi)$:

$$\forall\phi\in L(+)\mathrm{itg}(\mathrm{mid}(-\phi,f,\phi),\Theta(\phi),\Lambda(\phi),\Xi(\phi))$$

(f is measurable with respect to the parameters $\Theta$, $\Lambda$ and $\Xi$; the parameters may be abbreviated to a single letter W, or even omitted altogether.)

Note. mbl is not a definable notion.

[A2] $\rho\subset L(+)\wedge\forall x\forall r>0\exists m\forall n\geq m(\rho(n,x)>r)$.

We shall work under the assumption [A2].

2) $\mu(f,W)$: $\mathrm{limsup}J^1(\mathrm{mid}(-\rho(n),f,\rho(n)),W(\rho))$, where W is a sequence of appropriate parameters.

Proposition 5. All the basic properties of measurable functions are satisfied by (mbl,$\mu$). $\mu$ does not depend on the parameters, presuming that mbl is assumed for a function f. Let us consider the proposition that

$$(\dagger)\ a\in R,\ mbl(f)\rightarrow mbl(af)$$

as an example. In proving this, we must demonstrate a more refined result than just ($\dagger$). That is, ($\dagger$) automatically implies that there are definable $\Phi^*$ amd $W^*$ (with appropriate parameters) such that

$$\phi\in L(\dagger)\rightarrow\forall n(\Phi^*(n)\in L(\dagger))$$

and

$$\phi\in L(\dagger),\ \forall n\ itg(mid(-\Phi^*(n),f,\Phi^*(n)),\ W(\Phi^*(n)))$$
$$\rightarrow itg(mid(-\phi,af,\phi),\ W^*(\phi))$$

From these, and by the (definable) comprehension rule, we obtain

$$mbl(f,W)\rightarrow mbl(af,W^*)$$

In order to formulate the theory of products of functions and the Fubini theorem, we need an alternative farmily of elementary functions which yields the same family of integrable functions.

Definition 5. We shall use $\underset{\sim}{n}$ to denote a natural number in a specific context.

sqn($\underset{\sim}{n}$): "$\underset{\sim}{n}$ is a finite sequence of distinct rationals, say $(r_1,\ldots,r_\ell)$, arranged in the natural, increasing order."

lg($\underset{\sim}{n}$): the length of $\underset{\sim}{n}$; that is, the $\ell$ above.

$\underset{\sim}{n}(k)$: $r_k$ if $1\leq k\leq\ell$.

$S_0(\alpha,\underset{\sim}{n},\Phi,E,\chi)$: $\forall x\exists! k\leq lg(\underset{\sim}{n})(\alpha(x)=r_k)\wedge itg(\alpha,\Phi,E,\chi)$

($\alpha$ is a rational-valued simple function with respect to $\underset{\sim}{n}$, $\Phi$, E, $\chi$.)

$K(\alpha,\underset{\sim}{n},k)$: the characteristic function of D, where $D\equiv\{x; \alpha(x)=r_k\}$.

$J_0(\alpha,\underset{\sim}{n},\Phi,E,\chi)$: $\Sigma\{r_k J^1(K(\alpha,\underset{\sim}{n},k),\Phi,E,\chi); k\leq lg(\underset{\sim}{n})\}$

We may omit or abbreviate $\Phi$, E, $\chi$, and even $\underset{\sim}{n}$ when the circumstances allow us to.

Proposition 6. 1) The functions and the predicates defined above are definable. In particular, $J_0$ is arithmetically definable. $S_0$ and $J_0$ have the intended properties.

2) We may assume $S_0$ as the class of elementary functions over the coefficient set Q in developing the theory of integration in our definable system.

Definition 6. 1) Let $\mathcal{J}_2$ be the theory $\mathcal{J}$ with the axiom sets of two integration spaces $\mathcal{X}=(X,L_1,J_1)$ and $\mathcal{Y}=(Y,L_2,J_2)$ in the place of $\mathcal{B}$. The properties which are claimed subsequently are the theorems of $\mathcal{J}_2$.

2) $Z\equiv X\times Y=\{(x,y); x\in X, y\in Y\}$

$(x,y)=(u,v)$: $x=u\wedge y=v$

3) $S_1$: the $S_0$ for $L_1$ in $\mathcal{X}$

$S_2$: the $S_0$ for $L_2$ in $\mathcal{Y}$

$I_1$: the $J_0$ for $S_1$

$I_2$: the $J_0$ for $S_2$

4) $S(m,\alpha,\xi,\beta,\eta)$: $\forall k\leq m(S_1(\alpha(k),\xi(k))\wedge S_2(\beta(k),\eta(k)))$, where $\xi(k)$ and $\eta(k)$ each stands for four parameters.

5) $\pi(m,\alpha,\beta,z)$: $\Sigma\{\alpha(k,x)\beta(k,y);\ k\leq m\}$, where $z=(x,y)$.

6) $I(m,\alpha,\beta)$: $I_2^{\ 1}(\{y\}I_1(\tau_y))$, where $\tau_y\equiv\{u\}\pi(m,\alpha,\beta,u,y)$.

Proposition 7. 1) The functions and the predicates defined above are definable.

2) $(S,I)$ satisfies the axioms on integration over the coefficient set Q.

Proposition 8. (Fubini)

$S^1(f,W)\rightarrow I_2^{\ 1}(\{y\}I_1^{\ 1}(f_y))=I_1^{\ 1}(\{x\}I_2^{\ 1}(f_x))=I^1(f,W)\in R$,

where $f_y\equiv\{x\}f(x,y)$ and the unwritten parameters are definable in W.

Note. We do not take the "quotient" modulo null set.

The mathematical proof goes through.

Definition 7. 1) Let $\mathcal{D}$ be the axiom set $\mathcal{B}$ modified as follows.

(a) The primitive symbols $\delta$, $J_0^{\ +}$, $J_0^{\ -}$ and $\iota$ are added.

(b) The condition $\forall\phi\in L(+)(J(\phi)\geq 0)$ is eliminated.

(c) The continuity property 4° is replaced by a stronger one:

4' $\forall\varepsilon>0\forall\phi\in L(\|\phi\|<\delta(\varepsilon)\vdash|J(\phi)|<\varepsilon)$,

where $\|\phi\|=\sup\{|\phi(x)|;\ x\in X\}$.

(d) The axioms on $J_0^{\ +}$, $J_0^{\ -}$ and $\iota$ are added.

$\forall\phi\in L(+)$;

$J_0^{\ +}(\phi)$, $J_0^{\ -}(\phi)$, $\iota(\phi)\in R$,

$$J_0^+(\phi)=\sup J(\psi);\ 0\leq\psi\leq\phi,\ \psi\in L(+)\},$$
$$J_0^-(\phi)=-\inf\{J(\psi);\ 0\leq\psi\leq\phi,\ \psi\in L(+)\},$$
$$\imath(\phi)=\sup\{J_0^+(\psi)-J_0^-(\psi);\ 0\leq\psi\leq\phi,\ \psi\in L(+)\}.$$

(Here the sup in the right hand side is not a formal object, but the entire equation represents a relation which determines the property of $J_0^+(\phi)$(or $\imath(\phi)$). Similarly for inf.)

Let $\mathcal{J}'$ be the theory obtained from $\mathcal{J}$ by replacing $\mathcal{B}$ by $\mathcal{D}$. $\mathcal{J}'$ will be called the theory of signed integral and J will be called a signed integral on L.

2) $J^+(\phi)$: $J_0^+(\phi^+)-J_0^+(\phi^-)$

$J^-(\phi)$: $J_0^-(\phi^+)-J_0^-(\phi^-)$

3) For any two integrals (in the original sense), we say $I_1$ and $I_2$ are compatible by K if

$$\forall\phi\in L(+)(K(\phi)\in R\wedge K(\phi)=\sup\{I_1(\psi)-I_2(\psi);\ 0\leq\psi\leq\phi,\ \psi\in L(+)\}).$$

In such a case, define $I_1\wedge I_2$ to be $I_1-K$ (on L(+)).

4) If $I_1\wedge I_2\equiv 0$ on L(+), we say $I_1$ and $I_2$ are mutually singular (with respect to K).

Proposition 9. 1) $J^+$ and $J^-$ are compatible by $\imath$.

2) $J^+$ and $J^-$ are mutually singular.

3) $J=J^+-J^-$ is the unique decomposition of J by mutually singular integrals.

By virtue of the proposition above, $(J^+,J^-)$ can be regarded as the Jordan decomposition of J.

Definition 8. $L(p;\ f,W_1,W_2)$: $\mathrm{mbl}(f,W_2)\wedge[(1\leq p<\infty \wedge \mathrm{itg}(\exp(|f|,p),W_1))\vee(p=\infty\wedge\exists r\ \mathrm{ae}(x,|f(x)|\leq r,\ W_1'))]$, where $W_1$ and $W_2$ stand for finite sequences of parameters and $W_1'$ is a subset of $W_1$.

$$\text{ess sup}(f,W_1')\text{:}\quad \inf\{r;\ ae(x,|f(x)|\le r,W_1')\}$$

$$\text{norm}(p,f,W_1,t)\text{:}\quad [1\le p<\infty \wedge t<\exp(J^1(\exp(|f|,p),W_1),1/p)]$$
$$\vee[p=\infty\wedge t<\text{ess sup}(f,W_1')]$$

We write norm(p; f,$W_1$), norm(p; f) or norm(f) for {t}norm(p; f,$W_1$,t). Notice that norm(p; f,$W_1$) is definable.

In the following the propositions are meant to be provable in $J$.

Proposition 10. ({f,$W_1$,$W_2$}L(p,f,$W_1$,$W_2$), {f,$W_1$,t}norm(p, f,$W_1$; t)) satisfies the conditions on $L^p$-spaces, and the basic properties such as Hölder's inequality and the Riesz-Fischer theorem hold (uniformly in p).

Since the construction in the proof of the Riesz-Fischer theorem is the basis of all the subsequent constructions, we work it as an example.

Proposition 11. (Riesz-Fischer)

$$\forall k\ L(p;\ F(k),V_1(k),V_2(k)),$$
$$\lim\{\text{norm}(p;\ F(k)-F(\ell));\ k,\ell\}=0$$
$$\to L(p;\ f^*,W_1^*,W_2^*)\wedge \lim\ \text{norm}(p;\ f^*-F(k))=0,$$

for a definable f*.

Proof. As applications of DDI, we define $\nu$ and G as follows.

$\nu(1)=\min(\ell,\forall m\geq\ell(\mathrm{norm}(F(m)-F(\ell)\leq \exp(2,-1)))$,

$\nu(n+1)=\min(\ell,\ell>\nu(n)\wedge\forall m\geq\ell(\mathrm{norm}(F(m)-F(\ell))\leq\exp(2,-(n+1))))$,

$G(1)=F(\nu(1))$,

$G(n+1)=F(\nu(n+1))-F(\nu(n))$.

Then $\{\nu(n)\}_n$ is increasing,

$$\mathrm{norm}(F(m)-F(\nu(n)))\leq\exp(2,-n)$$

if $m\geq\nu(n)$, $G(n)\in L(p)$ for each n and

$$\Sigma\mathrm{norm}(G(n))\leq\mathrm{norm}(F(\nu(1)))+\Sigma\exp(2,-(n-1))$$
$$=\mathrm{norm}(F(\nu(1)))+1.$$

Thus, $\Sigma G(n)$ is absolutely convergent "almost everywhere". Now define f* by

$$f^*(x)=\begin{cases}\Sigma G(n,x)=\lim F(\nu(n),x) & \text{if } \Sigma|G(n,x)|<\infty\\ 0 & \text{otherwise}\end{cases}$$

This f* will do.

Definition 9. $\mathrm{lnfl}(p;\ T)$: $\forall f\forall W_1\forall W_2$
$(L(p;\ f,W_1,W_2)\vdash T(f)\in R)$
$\wedge\forall k\forall F\forall V_1\forall V_2\forall a(\forall i\leq k L(p;\ F(i),V_1(i),V_2(i))$
$\wedge a\in R\vdash[T(aF(1))=aT(F(1))$
$\wedge T(\Sigma[F(i);\ i\leq k])$
$=\Sigma\{T(F(i));\ i\leq k\}])$

$\mathrm{bdf}(p;\ T,a)$: $a>0\wedge\forall f\forall W_1\forall W_2$
$(L(p;\ f,W_1,W_2)\vdash|T(f)|\leq a\ \mathrm{norm}(f))$

$\mathrm{blf}(p;\ T,a)$: $\mathrm{lnfl}(p;\ T)\wedge\mathrm{bdf}(p;\ T,a)$

$\mathrm{cntf}(p;\ T,\delta)$: $\forall f\forall W_1\forall W_2\forall\varepsilon>0$
$[\delta(\varepsilon)>0\wedge(L(p;\ f,W_1,W_2)\wedge\mathrm{norm}(f)<\delta(\varepsilon)\vdash|T(f)|<\varepsilon)]$

$\mathrm{nrm}(p;\ T,b)$: $\mathrm{bdf}(p;\ T,b)\wedge\forall a(\mathrm{bdf}(p;\ T,a)\vdash b\leq a)$

The representation theorem for the bounded linear functionals on $L^2$ assumes the following form.

Proposition 11. $\mathrm{blf}(2;\ T,a),\ L(2;\ h,Z_1,Z_2),\ T(h)\neq 0,$

$\mathrm{nrm}(2;\ T,b),\ \forall n(L(2;\ G(n),V_1(n),V_2(n))$

$\wedge T(G(n))=\exp(b,2)$

$\wedge \mathrm{norm}(G(n+1))\leq \mathrm{norm}(G(n))),$

$\forall f\forall W_1\forall W_2(L(2;\ f,W_1,W_2)\wedge T(f)=\exp(b,2)$

$\vdash \exists n(\mathrm{norm}(G(n))\leq \mathrm{norm}(f)))$

$\rightarrow\ L(2;\ g^*,U_1{}^*,U_2{}^*)\wedge T(g^*)=\exp(b,2)$

$\wedge\forall f\forall W_1\forall W_2(L(2;\ f,W_1,W_2)\wedge T(f)=\exp(b,2)$

$\vdash[\mathrm{norm}(g^*)\leq \mathrm{norm}(f)$

$\wedge(\mathrm{norm}(g^*)=\mathrm{norm}(f)$

$\vdash \mathrm{ae}(x,f(x)=g^*(x),E^*,\chi^*))$

$\wedge\forall f\forall W_1\forall W_2(L(2;\ f,W_1,W_2)\vdash T(f)=J(fg^*,W^*))])$

for some definable $g^*$, $U_1{}^*$, $U_2{}^*$, $E^*$, $\chi^*$ and $W^*$.

Proof. Put $c=\lim \mathrm{norm}G(n)$. Then $c\geq 0$ and $c$ satisfies that

$$c=\inf\{\mathrm{norm}(f);\ L(2;\ f,W_1,W_2)\}.$$

Also, $0<b\leq c$, and hence $c>0$.

$$\lim\{\mathrm{norm}(G(k)-G(\ell));\ k,\ell\}=0,$$

and hence, by virtue of the Riesz-Fischer theorem, there are $g^*$, $U_1{}^*$ and $U_2{}^*$ such that $L(2;\ g^*,U_1{}^*,U_2{}^*)$ and $\lim\ \mathrm{norm}(g^*-G(k))=0$. This $g^*$ will do.

[Assumption] Let $\chi=(X,L,J)$ denote a (real) integration space. We place further conditions on $(X,L)$ as listed below.

$$\eta\subset L(+)\wedge X=\smile X_n,$$

where $X_n=\{x;\ \eta(n,x)>0\}$, and $\eta$ is a parameter.

$$\forall\phi,\ \psi\in L(\phi\psi\in L\wedge(\psi\neq 0\vdash\phi/\psi\in L)),$$

where $\psi\neq 0$ means $\forall x(\psi(x)\neq 0)$.

Definition 10. abscnt(J,I; Ω):

$$\forall E\forall\chi(\mathrm{nls}(I;\ E,\chi)\vdash\mathrm{nls}(J;\ E,\Omega(\chi)))$$

$$\mathrm{litg}(I;\ h,\Xi):\ \ \mathrm{mp}(h,X,R)\wedge\forall\phi\in L\ \mathrm{itg}(I;\ h\phi,\Xi(\phi))$$

$$\Gamma(J,I;\ h,\Xi):\ \ h\geq 0\wedge\mathrm{litg}(I;\ h,\Xi)$$

$$\wedge\forall\phi\in L(J(\phi)=I(\phi h))$$

$$\mathrm{nrm}(I;\ J,a):\ \ \mathrm{nrm}(2;\ J,a),$$

where J is regarded as a linear functional on L(2; K) and nrm(2) is taken with respect to K, K being J+I.

Proposition 12. (Radon-Nikodym) Let J and I be as in our [Assumption]. Then (a) and (b) below are "mutually definably interpretable", provided that nrm(I; J,$a_0$) is auusmed.

(a) Γ(J,I; h,Ξ).

(b) abscnt(J,I; Ω).

The h in (a) is unique up to the addition of an I-null function.

Proof. We work on (b)→(a) as an example. It suffices to deal with the case where 1 is integrable both for J and I. Define K to be J+I. Due to the assumption nrm(I; J,$a_0$) and Proposition 11, there is a definable g* such that J(f)=K(fg*) for f∈L(2; K). Define D={x; g*(x)≥1}, and

$$h(x)=\begin{cases}\Sigma\exp(g^*(x),k) & \text{if } x\notin D,\\ 0 & \text{if } x\in D.\end{cases}$$

This h will do.

As a consequence of the Radon-Nikodym theorem, we obtain the general cases of the Riesz-representation theorem.

As a special topic, we shall present a sufficient condition for the differentiablility of linear functionals of a certain kind (under our [Assumption]). It is a modified version of the implication "B→C" in [3].

Definition 11. $K(h,U,V)$:

$$\mathrm{itg}(h,U)\wedge \mathrm{ae}(x,0\leq h(x)\leq 1,V)$$

$\mathrm{dsj}(F)$: $\forall x\forall i\forall j(i\neq j\vdash F(i,x)F(i,y)=0)$

$\mathrm{cad}(T)$: $\forall F\forall U\forall W(\forall i(\mathrm{itg}(F(i),W(i)))$

$$\wedge \mathrm{dsj}(F)\wedge \mathrm{itg}(\{x\}\Sigma\{F(i,x);\ i=1,2,\ldots\},U)$$

$$\vdash T(\{x\}\Sigma\{F(i,x);\ i=1,2,\ldots\})$$

$$=\Sigma\{T(F(i));\ i=1,2,\ldots\})$$

[Assumption] In the following, we assume

$\mathrm{blf}(1;\ T,K)\wedge \mathrm{cad}(T)$.

Recall that $\mathrm{norm}(1;\ f)=J(|f|)$.

$\Lambda(\rho,\sigma)$: $\forall n K(\sigma_1(n),\sigma_2(n),\sigma_3(n))$

$$\wedge\forall n(\rho(\sigma_1(n))>0)$$

$$\wedge\forall x\forall\varepsilon>0\exists n(\sigma_1(n,x)\neq 0\wedge\rho(\sigma_1(n))<\varepsilon)$$

$\mathrm{sstm}(h,\rho,\sigma,\nu)$: "$\nu(h)$ is a sequence of natural numbers"

$$\wedge\forall x\forall\varepsilon>0(h(x)\neq 0\vdash\exists m(\sigma_1(\nu(h,m),x)\neq 0$$

$$\wedge\ \rho(\sigma_1(\nu(h,m)))<\varepsilon))$$

$B(\rho,\sigma,\tau)$: $\forall h\forall U\forall V\forall\nu\forall r>0(K(h,U,V)$

$$\wedge J(h)>0\wedge \mathrm{sstm}(h,\rho,\sigma,\nu)$$

$$\vdash \forall i \exists m(\tau(h,r,i)=\nu(h,m))$$
$$\wedge dsj(\{i\}\sigma_1(\tau(h,r,i)))\wedge J(h)$$
$$=J(h\{x\}\Sigma\{\sigma_1(\tau(h,r,i),x);\ i=1,2,\ldots\})$$
$$\wedge\Sigma\{J(\sigma_1(\tau(h,r,i)));\ i=1,2,\ldots\}<J(h)+r)$$

$u\Delta(T,\rho,\sigma;\ x,\ell)$:

$$\lim\ \sup\{T(\sigma_1(n))/J(\sigma_1(n));$$
$$n=1,2,\ldots,\sigma_1(n,x)\neq 0,$$
$$\rho(\sigma_1(n))<1/\ell\}$$

$l\Delta(T,\rho,\sigma;\ x,\ell)$: $\lim\ \inf\{T(\sigma_1(n))/J(\sigma_1(n));$

$$n=1,2,\ldots,\sigma_1(n,x)\neq 0,$$
$$\rho(\sigma_1(n))<1/\ell\}$$

$uD(T,\rho,\sigma;\ x)$: $\lim\{u\Delta(T,\rho,\sigma;\ x,\ell);\ \ell=1,2,\ldots\}$

$lD(T,\rho,\sigma;\ x)$: $\lim\{l\Delta(T,\rho,\sigma;\ x,\ell);\ \ell=1,2,\ldots\}$

We shall abbreviate $\{x\}uD(T,\rho,\sigma;\ x)$ to uD. Similarly for lD.

$C(T,\rho,\sigma,W_1,W_2,W,\theta,g)$: $mbl(uD,W_1)\wedge mbl(lD,W_2)$

$$\wedge L(\infty;\ g,W)\wedge ae(x,uD(x)=lD(x)=g(x),\theta)$$
$$\wedge\forall f\forall U(itg(f,U)\vdash T(f)=J(f(uD))$$
$$=J(f(lD))=J(fg))$$

Proposition 13. $b\in R$, $h\in K$,

$$\forall x(h(x)\neq 0\vdash uD(x)\geq b)\rightarrow T(h)\geq bJ(h)$$

is "mutually definably interpretable" from Λ and B.

The mathematical proofs of this proposition and the next one are more or less due to [4]. The author also owes to Mamoru Kanda for his comments in this regard.

Proof. It suffices to consider the case where $J(h)>0$, since $u\Delta(x,\ell)$ is decreasing with respect to $\ell$, $uD(x)\geq b$ (where $h(x)\neq 0$) implies

$$h(x)>0\rightarrow\forall\ell\exists n(\sigma_1(n,x)>0\wedge\rho(\sigma_1(n))<1/\ell\wedge T(\sigma_1(n))\geq bJ(\sigma_1(n))).$$

Define $\nu^*$ by:

$$\nu^*(1)=\min(n,G(n)),$$

$$\nu^*(m+1)=\min(n,n\geq\nu^*(m)\wedge G(n)),$$

where $G(n)$ stands for

$$\exists x\exists\ell(h(x)\neq 0\wedge\sigma_1(n,x)>0$$
$$\wedge\rho(\sigma_1(n))<1/\ell\wedge T(\sigma_1(n))\leq bJ(\sigma_1(n))).$$

Then $sstm(h,\rho,\sigma,\nu^*)$ follows. The condition B applied to this h and $\nu:\nu^*$ yields

$$\forall r>0[\forall i\exists m(\tau(h,r,i)=\nu^*(h,m))$$
$$\wedge dsj(\{i\}\sigma_1(\tau(h,r,i)))$$
$$\wedge J(h)=J(h\Sigma\omega(i))\wedge\Sigma J(\omega(i))<J(h)+r].$$

$f_0\equiv\Sigma\omega(i)\in K$, $J(f_0)=\Sigma J(\omega(i))$ and $J(h)\leq J(f_0)$. "$h,f_0\in K$" implies that $hf_0$, $h-hf_0$, $f_0-hf_0\in K$. From these facts and the complete additivity of T, we can easily obtain, successively,

$$T(h)=T(hf_0)>T(f_0)-Kr$$
$$=\Sigma T(\omega(i))-Kr\geq b\Sigma J(\omega(i))-Kr$$
$$=bJ(f_0)\geq bJ(h)-Kr;$$

that is, $\forall r>0$ $(T(h)\geq bJ(h)-Kr)$, from which follows $T(h)\geq bJ(h)$.

Proposition 14. $C(T,\rho,\sigma,W_1,W_2,W,\theta,g)$ is "mutually definable interpretable" from r and B.

Proof. $\{x\}u\Delta(x,\ell)$ is measurable if and only if $A(s)=\{x;\ u\Delta(x,\ell)>s\}$ for every rational s. But

$$A(s)=\smile\{\{x;\ \sigma_1(n,x)\neq 0\};\ \rho(\sigma_1(n))<1/\ell$$

$\wedge T(\sigma_1(n))/J(\sigma_1(n))>s\}$

and $\sigma_1(n)$ is measurable.

By the Riesz-representation theorem for L(1), there is a definable $g^*\in L(\infty)$ such that

$$T(f)=J(fg^*) \text{ for every } f\in L(1).$$

So, it suffices to show $g^*=uD=1D$ almost everywhere. This follows from the proposition above.

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